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The large scale two-dimensional stationary vortex in a magnetized plasma

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Abstract

We present results of a study of the differential equation governing the stationary states of the two-dimensional planetary atmosphere and magnetized plasma (within the Charney Hasegawa Mima model). We compare the results to the experimental observations on a large scale vortical flow observed in the Hyper-I device. The role of the effective Larmor radius is emphasized and the asymptotic stationary states are characterized in terms of vorticity and of density profiles close to adiabaticity.

1 Introduction

The complex processes of the plasma dynamics have imposed the development of powerful statistical methods, able to provide better understanding, as well as tools for concrete applications. In this spirit the works of R. Balescu represent an excellent reference, for their systematic structure and for clarity [1], [2], [3]. In particular the statistical properties of the test particle positions in plasma have led to a quantitative description of anomalous transport in confined plasma [4]. These works also represent sources for the development of new approaches, in particular in the study of processes found at the boundary between the physics of plasma turbulence and plasma structures.

In most experimental situations plasma is in a turbulent state and may be described as a large statistical system. There are however situations where the plasma exhibits very regular flow pattern or coherent structures, in a quasi-stationary state. Recently it has been observed in a plasma column in magnetic field the formation of a large-scale regular vortical flow pattern which attains a stationary state. This state has been obtained in the "High Density Plasma Experiment" device (*Hyper-I*) [5] and the flow has been measured in detail. Since this experiment is well documented and plasma parameters have been accurately measured, it may be taken as a reference case for the application of the theory of asymptotically stationary two-dimensional flows.

In the absence of dissipation the model of ion drift waves in a two-dimensional geometry transversal to a strong magnetic field reduces to a set of two differential equations for the electrostatic potential ϕ (or streamfunction ψ for the ion fluid velocity) and the density n. From this, a further simplification of adiabatic density response leads to the Charney-Hasegawa-Mima (CHM) equation [6], [7]. Numerical simulations show that the stationary states reached in relaxation are very regular and persist for a long time period [8], [9], [10],

[11]. Most of these studies have concentrated on the identification of small vortices, on the scales of few Larmor radius, ρ_s. The prototype is the dipolar vortex (modon) Larichev-Reznik with an effective diameter of few Larmor radii. However one should expect that *large* scale organized flows can form, with the dynamics being governed by the ion drift wave model, in particular the CHM equation. The notations are as follows $\rho_s = c_s/\Omega_{ci}$, where $c_s = \sqrt{T_e/m_i}$ is the sound speed (in m/s), T_e is the electron temperature (in eV), $\Omega_{ci} = eB/m_i$ is the ion cyclotron frequency (in s^{-1}), e is the electric charge (in C), B is the magnetic field (in T) and m_i is the ion mass (in kg).

In the ion drift wave model the Larmor gyration of ions plays the essential role. In the unperturbed state it establishes an intrinsic space scale (the ion sound Larmor radius, ρ_s) which excludes the scale invariance of the theory. And, of equal importance, the unperturbed state consists of a continuum of vorticity, given by the Larmor gyrofrequency, Ω_{ci}. When there are perturbations which vanish at large distance, the total vorticity must smoothly match the constant vorticity of the asymptotically far regions. Any vortical flow must be seen as an excitation developing on this background. We will call this background a *condensate of vorticity.*

It is known that the asymptotic stationary states of the ideal Euler fluid attained by relaxation from turbulent states are described by the *sinh*-Poisson equation. For the Charney-Hasegawa-Mima equation it appeared to be much more difficult to find the equation governing the asymptotic stationary states [12], [13].

There is a deep reason for this. The presence of the condensate of vorticity and of the intrinsic finite length not only removes the space-scale invariance (typical for the ideal Euler fluid) but also induces a finite range of interaction between the plasma elements. To understand this we have to remind that in the Euler fluid case the vorticity is connected with the streamfunction by simply the Laplacean operator and the inverse appears as the *logarithmic* interaction, *i.e.* a long range interaction. For the ion drift theory the vorticity of any physically interesting flow is actually placed upon the background of the Larmor gyration, the condensate of vorticity. Only the part of the perturbed vorticity which remains after substracting the background is directly invloved in physical processes and this modifies the operator: instead of the Laplace operator we now have the Helmholtz operator,

$$\Delta \rightarrow \Delta - k^2$$

with k the inverse of the length scale. This operator leads to a finite range interaction. The origin of this short range interaction can be traced back to the presence of the condensate which makes that any perturbation cannot propagate instantaneously, but becomes effectively heavy. This is however only part of the problem.

Before going further we specify our theoretical framework. We have developed a field theoretical model for the point-like vortices with short range interaction, based on Chern-Simons action for the gauge field minimally coupled to a nonlinear matter field, in $SU(2)$ algebra. It is possible to derive the energy as a functional that becomes extremum on a subset of stationary states and presents particular properties. The general characterization of this family of states is their *self-duality*, which here means that the energy functional becomes minimum because the square terms are all vanishing, leaving as lower bound a quantity with topological meaning. A very detailed account of the derivation is in Refs. [16], [19].

The result is a set of equations parametrized by the solutions of the Laplacean equation in two-dimensions.

The simplest of these equations is

$$\Delta\psi + \frac{1}{2p^2}\sinh\psi\left(\cosh\psi - p\right) = 0 \tag{1}$$

(where p is a positive constant). We have proved that this equation can explain known results: the scatterplots of (ψ, ω) = (streamfunction, vorticity) obtained in experiments [20] and the scatterplots obtained in numerical simulations [12] are very similar to the nonlinear term of Eq.(1). We have also shown that this equation is able to describe the large scale vortical flows (*typhoons*) of the planetary atmosphere (see [17]), reproducing the main characteristics of the flow and determining quantitative elements in the range which is consistent to the real situation.

We now add a short comment, mentioning the second part of the difficulty of identifying the form of the equation at asymptotic stationarity.

In order to obtain self-duality (here expressed as the vanishing of the square terms in the energy functional) within the non-Abelian theory one has to take the potential of self-interaction of the scalar matter field (which actually represents by its modulus square the density of point-like vortices) with a factor consisting of the square of this complex scalar field, which generates a new minimum, this time at zero scalar field. The other two minima are at the cyclotron frequency $\pm\Omega_{ci}$. This new *symmetric* minimum induces a class of solutions (distributions of flow vorticity) which are not topological. Although there is a Higgs generation of mass (= the short range of the interaction between point vortices) as in the Abelian-Higgs case of superfluids, the class of vortices is richer and a clear relationship between the vorticity and the streamfunction could not be established.

Only the formalism of field theory for the model of point-like vortices interacting via short range potential can provide the form of the equation (1).

In this work we report numerical investigation of this equation, carried out with the objective to reproduce the experimental data of the large vortex flow of the NIFS experiment. We have considered that this large scale organized flow cannot be simply assimilated to a vortex of the CHM equation, as it is Larichev-Reznik modon. In consequence the velocity is considered a factor that can influence the dynamics, in the way which has been established for ion drift waves in sheared flows. We have also taken into account the effect of the strong density gradient since it can modify, combined with the translational velocity, the space scale of the problem, replacing the Larmor radius by an effective Larmor radius depending on these parameters. Since in the theoretical model for the asymptotic stationary states there is only this parameter which is present, the determination (with the help of the experiment) of this effective space scale is an essential prerequisite.

The results are summarised here.

The main parameters of the vortical flow obtained by numerical solution of the equation (1) are in good agreement with the experimental values of the Hyper-I device reported in [5].

1. The vorticity has a profile which is close to the experimental one and the magnitude of the vorticity is close to the ion Larmor gyrofrequency.

2. The tangential velocity has a profile which is similar to the experimental one. The absolute magnitude is smaller than the ion sound speed.

3. The radial velocity has a magnitude comparable to the experimental one, but the spatial ($2D$) distribution is more complex, without exhibiting a constant orientation of the flow toward the center of the plasma.

4. The plasma density is obtained from the application of the Ertel 's theorem; for our numerical values of the vorticity which in the center are very close to Ω_{ci} we obtain very low plasma density close to the center of the hole.

2 Parameters of plasma in the Hyper-I experiment

For the Hyper-I device (NIFS), we have indications on the values of the parameters from the experiment. the gas is Helium ($m_i = \mu m_p = 4m_p$). The space domain investigated experimentally is of the order $[-8.5, 8.5] \times [-8.5, 8.5]$ in *centimeters* (Fig.2 of ref.[5]).

The magnetic field at the axis of the machine was in the range

$$B \sim 875...1250 \ (Gs) \tag{2}$$

The radial electric field in the transition zone : hole - bulk plasma

$$E_r = 40 \ (V/cm) \tag{3}$$

The ion gyrofrequency is

$$\Omega_{ci} = 9.58 \times 10^3 Z \frac{B}{\mu} \ \left(s^{-1}\right) \tag{4}$$

where $\mu = 4$, B is measured in (Gs) and Z is the effective ion charge in the experimental condition and we assume $Z = 1$. For $B = 1000\,(Gs)$ we have $\Omega_{ci} = 2.395 \times 10^6\,(s^{-1})$.

The measured sound velocity

$$c_s = 3 \times 10^4 \ (m/s) \tag{5}$$

and the *measured* maximum azimuthal velocity was of the same order

$$v_{\theta\max}^{phy} \sim c_s = 3 \times 10^4 \ (m/s) \tag{6}$$

Knowing c_s we have the constraint on the two parameters ρ_s and Ω_{ci}

$$c_s = \rho_s \Omega_{ci} \tag{7}$$

The variation from $v_\theta = 0$ (in the center) to $v_{\theta\max}$ (at $r = a$) takes place on a distance

$$a^{phy} \sim 3 \times 10^{-2} \ (m) \tag{8}$$

In a large volume with small gradients or no flow we could take these as physical units, fora process which is driven by the ion fluid

$$\rho_i = \frac{c_s}{\Omega_{ci}} = \frac{3 \times 10^4}{2.395 \times 10^6} = 1.25 \times 10^{-2}\,(m) \tag{9}$$

$$\Omega_{ci} = 2.395 \times 10^{6} \left(s^{-1}\right) \tag{10}$$

$$c_s = 3 \times 10^{4} \ (m/s) \tag{11}$$

In the present case they will be used below for the identification of the appropriate range to start the numerical integration of the differential equation (1) and to obtain the appropriate physical units.

3 Numerical range for searching solutions

The fundament of the analytical model is the ion drift waves in a strong magnetic field, in the simplest two-dimensional geometry. Assuming uniform density and temperature, the unperturbed state is a continuum of ions performing gyromotion, without any other motion. Two parameters are sufficient to describe this state: the frequency of the cyclotron gyration Ω_{ci} and the Larmor radius ρ_s. They are combined into a third physically useful parameter, the sound speed c_s.

The CHM equation describes the effect of excited ion fluid motion over this background. The intrinsic space scale of the model is the Larmor radius ρ_s. The alternative description of this system consists of a discrete set of point-like vortices interacting via the short range potential (the Stewart-Morikawa model).

On this basis it was possible to develop a field theory model as a continuum version of the model of discrete vortices. Two elements are essential: the existence of a condensate of vorticity (a constant value of the vorticity of the fluid motion at asymptotically large distance on the plane, where any other perturbation has vanished) and the short range of the potential. The field theory at self-duality reduces to a differential equation for the streamfunction. There is only one physical parameter in this equation, the space scale. As we have said, the original framework indicates unequivocally that this length is the Larmor radius ρ_s, the intrinsic scale of the CHM equation, equivalently the short space range of the potential of the interacting discrete vortices.

Physical states described by the CHM equation are inevitably more complex than this basic framework. In particular the existence of a gradient of density and of a global translational motion of the plasma modifies the equation.

It has been shown that in certain circumstances the modification induced by these two physical elements: a drift velocity and a translational velocity are represented in the theory to a rescaling of the Larmor radius.

We discuss in the following the arguments leading to an effective redefinition of the elementary space scale in ion drift wave, using essentially the works of Horton and collaborators.

In studies of the planetary atmosphere (especially of the Red Spot of Jupiter) Petviashvili revealed the possible role of the scalar of KdV nonlinearity, of lower differential degree than the term of convection of the vorticity (or vector nonlinearity). This term may become more important at large spatial scales and at later times (Mikhailovskya), being therefore interesting for stationarity obtained from relaxation of the fluid turbulent states. In plasma physics Lakhin et al. , Makino et al., Horton et al. and Spatschek et al. have clarified the effect of the temperature gradient and of the second space derivatives of the density in the equation, showing that the equation can be reduced at stationarity to the Flierl-Petviashvili form. The existence of solitary vortices could not be proved. However, it has been shown that monopolar vortices with non-solitonic nature (they are slowly decaying by radiation emission) are possible when there is a higher space variation of the density (or, equivalently, the space variation of the drift velocity $v_d(x,y)$) and the space variation of the temperature. The vortices are not exponentially localised.

Using multiple space and time scale analysis Spatschek et al. have shown that the vortical flow should be seen at scales of the order

$$\frac{\rho_s}{\varepsilon}$$

where ε is the drift wave parameter

$$\varepsilon = \frac{\rho_s}{L_n}$$

The derivation of the equation for a sheared flow is made by Su, Horton, Morrison.

The spatial scale after assuming a drift wave ordering practically is an effective (or renormalized) Larmor radius

$$\frac{1}{\rho_s^2} \rightarrow \frac{1}{\rho_s^{eff}} \equiv \frac{1}{\rho_s^2}\left(1-\frac{v_d}{u}\right)$$

We note here that the oredering adopted in that reference, $u \simeq v_d$ seems consistent with the results of the experiments at NIFS. This is because it has been observed a rotation velocity in the vortical flow of the order of the sound velocity $c_s \simeq 4 \times 10^4 \ (m/s)$. The drift velocity is in the experiment measured close to the region of higher gradient of the density (the transition region from the hole to the higher density)

$$v_d = \frac{\rho_s c_s}{L_n} \tag{12}$$

It is observed that the decrease of the density takes place on few Larmor radii

$$L_n \sim \rho_s$$

Then from Eq.(12)

$$v_d \sim c_s$$

The fact that from the experimental observation we have

$$u \sim c_s$$

means that

$$1-\frac{v_d}{u} \ll 1$$

and the effective Larmor radius ρ_s^{eff} may be considerably greater than ρ_s.

There are two difficulties which cannot be solved by a simple scale analysis. They would possibly be solved by a major extension of the field theoretical model which has led us to the differential equation. The drift velocity is varying across the plasma section and there is no unique factor which enhances ρ_s . This means that there is a space varying scale of interaction in the point-like vortex model (Stewart-Morikawa), and accordingly, the Chern-Simons part of the Lagrangean should enter with a coupling constant κ with space variation. This space variation must be determined self-consistently when we provide physical information. Instead there is no need to consider a change of the condensate of vorticity which remains Ω_{ci}. We will take a constant effective Larmor radius.

The field theoretical model has not assumed a variation of the space scale. This should be introduced in the expression of the Lagrangean density by a space dependence of the coefficient of the Chern-Simons term. In this case it is not clear if the self-duality condition can be attained. Before this problem is clarified we may try to use the suggestion of the plasma theory that the manifestation of the density gradient and plasma flow is a modified space scale, a rescaled Larmor radius. Taking a constant scale will not change anything from the equation. But all physical variables (superscript *phy*) will now be scaled to the new space scale, the *effective* Larmor radius. The Larmor gyration frequency is the same as before.

The problem, especially when we intend to describe a concrete case (like the vortical motion in the device Hyper-I at NIFS), is that the gradient of the density is in general not constant across the section of the plasma and that the flow has curved streamlines and these elements cannot be captured into a simple rescaling of ρ_s. On the other hand we must recall that the essential of the CHM model is restrained to a space domain which is dominated by lengths of the order of ρ_s. When we know (from experiment) that the scales are much larger, we can try to use for the identification of the *effective* Larmor radius, a local slab approximation, with a constant density gradient (*i.e.* constant drift velocity v_d) and constant flow velocity u directed alond the same direction as v_d. The experimental measuremets provide a clear range of values for these two velocities, and our *representative* values are located in these ranges. After several numerical tests it appeared possible to take the *effective* Larmor radius as a parameter and scan the range of its values until a fit to the experimental values is obtained. We now explain how this is done in the numerical studies.

According to cited works, the effective Larmor radius (in physical units, m) is introduced as

$$\frac{1}{\left(\rho_s^{eff}\right)^2} = \frac{1}{\rho_s^2}\left(1 - \frac{v_d^{phy}}{u^{phy}}\right) \tag{13}$$

By definition

$$v_d^{phy} = \frac{\rho_s c_s}{L_n} \tag{14}$$

In the experiment it is seen that the decrease of the density toward the center of the hole is rather sharp , on few Larmor radii. We will take formally

$$L_n = g\rho_s \tag{15}$$

where g is a factor of few units. This implies

$$v_d^{phy} = \frac{c_s}{g} \tag{16}$$

From the numerical solution of the equation we obtain the tangential velocity of the vortical flow, v_θ. In terms of the physical space unit, ρ_s^{eff} and time unit, Ω_{ci}^{-1}, it becomes

$$u^{phy} \equiv v_\theta^{phy} = v_\theta\left(\rho_s^{eff}\Omega_{ci}\right) \tag{17}$$

Now we will introduce a factor for the modification of the Larmor radius

$$\rho_s^{eff} = q\rho_s \tag{18}$$

and we can rewrite Eq.(13)

$$\begin{aligned}\frac{1}{q^2} &= 1 - \frac{c_s/g}{v_\theta q c_s} \\ &= 1 - \frac{1}{g q v_\theta}\end{aligned}$$

From this equation we obtain

$$\frac{1}{q} = -\frac{1}{2 g v_\theta} + \sqrt{\frac{1}{4 g^2 v_\theta^2} + 1} \tag{19}$$

The experimental data (Fig.1 from [5]) suggests

$$g \simeq 2 \tag{20}$$

since the Larmor radius is $\rho_s \sim 1.25 \times 10^{-2}\,(m)$ and the decay of the density takes place in approximately 2.5 (cm). Then $L_n \simeq 2\rho_s$.

$$\frac{1}{q} = -\frac{1}{4 v_\theta} + \sqrt{\frac{1}{16 v_\theta^2} + 1} \tag{21}$$

The other parameter which is immediately confronted to the experiment is the position a^{phy} of the maximum of the tangential velocity

$$\begin{aligned}a^{phy} &= a \rho_s^{eff} \\ &= a q \rho_s\end{aligned}$$

From experiment it is known that this distance is approximately

$$\begin{aligned}a^{phy} &\sim 3\,(cm) \\ &= 2.4 \rho_s\end{aligned}$$

The second condition is then

$$2.4 = a q \tag{22}$$

The numerical solution will provide the two values

$$(v_\theta, a)$$

from which we determine q using Eq.(21) and compare with Eq.(22). The compatibility will indicate that the choice of *effective* space scale was adequate. At that moment we have to make comparison between the full set of numerical results and the experimental profiles.

In summary, the physical units which will allow us to map the numerical solution $(\psi, \omega, v_\theta, \text{etc.})$ to the physical quantities are

$$\text{unit of space } = \rho_s^{eff} = q \rho_s$$

where

$$\rho_s = 1.25 \times 10^{-2}\,(m)$$

$$\text{unit of vorticity } = \Omega_{ci} = 2.395 \times 10^6 \left(s^{-1}\right)$$

$$\text{unit of streamfunction } = \rho_s^{eff} \Omega_{ci}$$

$$\text{unit of velocity } = c_s = 3 \times 10^4\,(m/s)$$

The factor q remains to be determined from an optimum fit of the numerical and experimental results.

4 Numerical studies of the equation

4.1 Preliminaries to the numerical experiments

The structure of the function space representing the union of attractors for the various solutions of this equation appears to be very complex. This immediately translates into serious obstacles in the attempt to reach one of the presumed solution.

We use the code "**GIANT** A software package for the numerical solution of very large systems of highly nonlinear systems" written by U. Nowak and L. Weimann [23]. The code belongs to the numerical software library *CodeLib* of the **Konrad Zuse Zentrum fur Informationstechnik Berlin**. The meaning of the abbreviation is: GIANT = Global Inexact Affine Invariant Newton Techniques and corresponds to the implementation of the method proposed by Deuflhard (for many references see [23]).

All necessary description of the method, of the code and many studies of the numerical precision and computer efficiency are presented by Nowak and Weimann in the documentation of the code.

The boundary conditions are dependent on the value of p. The physical model imposes that the scalar function ψ remains nonzero at infinity for $p > 1$. This means that we must require that the boundary condition is one of the roots of the algebraic equation

$$\cosh \psi - p = 0 \tag{23}$$

which can give the vanishing of the physical vorticity at infinity. Then we impose

$$\begin{aligned} \text{boundary condition } \psi\left(r \to \infty\right) &= \psi_b^{(1,2)} \\ &= \ln\left(p \pm \sqrt{p^2 - 1}\right) \end{aligned} \tag{24}$$

The initial function is a symmetric profile taken as suggested by the monopolar solution of the Petviashvili and Pokhotelov. The existence of a monopolar very robust (possibly stable) solution is confirmed by the numerical studies carried out by Boyd and Tan for the Renormalized Long Wave equation, equivalent at stationarity with the Flierl-Petviashvili equation. They also offer an analytical expression for the monopole as a series of forty five terms. We take

$$\psi_{ini}\left(x, y\right) = A_{ini}\left[\sec h(kr)\right]^{4/3} + \psi_b \tag{25}$$

where

$$\begin{aligned} kr &= \sqrt{\frac{x^2 + y^2}{\delta^2}} \text{ , on } 0 \leq x, y \leq R_p \\ \delta &= \frac{x_{\max} - x_{\min}}{2}\frac{1}{d} \end{aligned} \tag{26}$$

The two parameters are : A_{ini} = the initial amplitude of the streamfunction after normalization to $\rho_i^2 \Omega_{ci}$, and d = a factor of order 3...6 such as δ is a fraction from the plasma radius R_p where the initial profile is approximately contained. The domain of integration is quadratic

$$\left[x_{\min}, x_{\max}\right] \times \left[y_{\min}, y_{\max}\right] \tag{27}$$

with the length of the sides $L \equiv x_{\max} - x_{\min}$ normalized to the unit if distance ρ_s^{eff}. The mesh is usually 51, 81 or 101 intervals on each side.

The parameter is chosen

$$p = 1$$

Another possible solution can be adopted as initial function, *i.e.* a single period of the double periodic elliptic Weierstrass function, found as an exact periodic solution of the FP equation [18]. This should be a valid configuration at large spatial scales, but requires a continuation to a decay type solution.

In the Appendix several possibilities are examined (within simple approximations) to reduce the spectrum of the parameters of the numerical integration. The conclusion at this moment can be formulated as follows: It is essential to retain the effect of the factor $\cosh\psi - 1$ at all higher orders. Otherwise, the truncations simply provide non-real solutions.

4.2 Results of the numerical experiments

The following solution has provided the values

$$\begin{aligned} \text{the maximum tangential velocity} &: \ v_{\theta\max} = 0.45 \\ \text{the radius where the maximum } v_\theta \text{ occurs} &: \ a = 1.5 \end{aligned}$$

From the first value we get using Eq.(21)

$$q = 1.7$$

and the RHS of the second condition, Eq.(22) is

$$aq = 1.5 \times 1.7 = 2.55$$

to be compared with the LHS of Eq.(22), 2.4. The match can be made even better. The solution has been obtained with accuracy of 0.983×10^{-5} in 242 steps of Newton iterations. The normalized range of spatial integration was $[-4.1, 4.1]$ from which we derive the physical spatial domain

$$\begin{aligned} L &= 8.2\rho_s^{eff} = 8.2 \times (q\rho_s) = 8.2 \times (1.7\rho_s) \\ &= 8.2 \times 21.2 \times 10^{-3}(m) = 17.38 \ (cm) \end{aligned}$$

This is consistent with the Fig.1 and Fig.2 of [5].

The vorticity is also consistent with the experimental data, in magnitude and in the spatial dependence (if we take into account the experimental error bars. It is more strongly localised than in the experiment and this may be explianed by the absence of viscosity or of inertial effect in our model.

The radial velocity is less than in experiments

$$|v_r| \sim 0.45 \times q \times c_s$$

and has a more complicated pattern, it is not always directed toward the center.

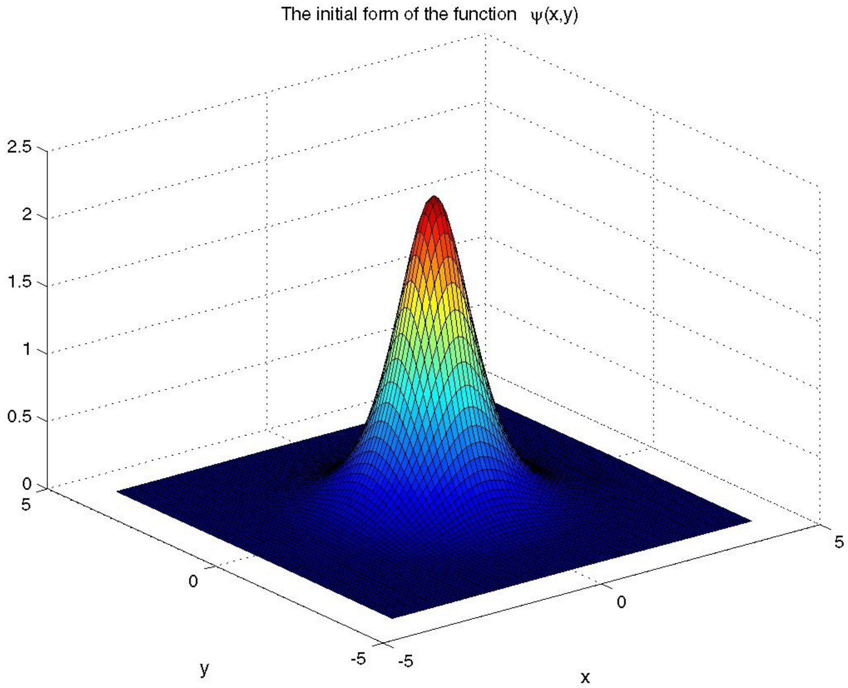

Figure 1: The initialization, $\psi_{ini}(x,y)$.

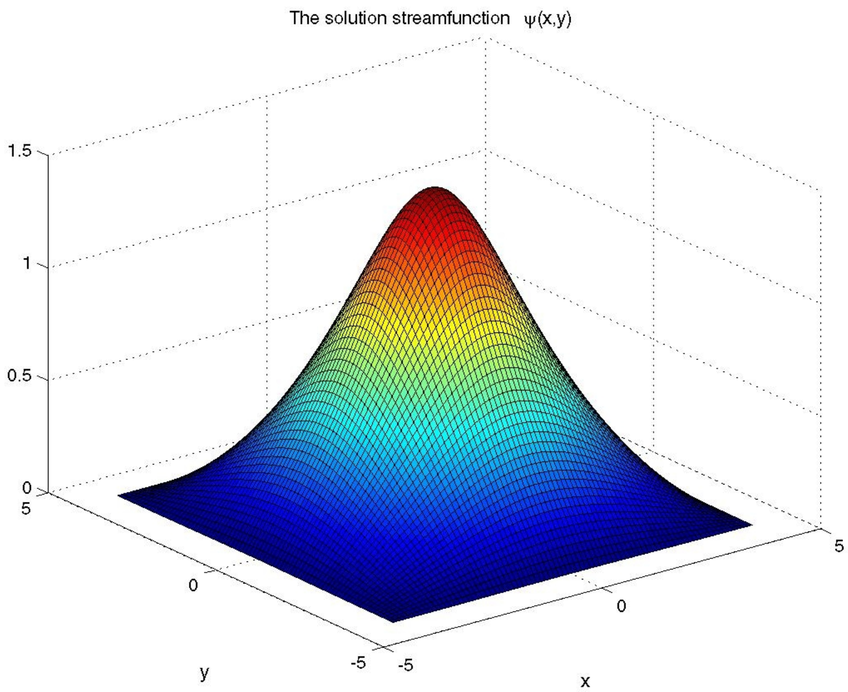

Figure 2: The solution, the streamfunction $\psi(x,y)$.

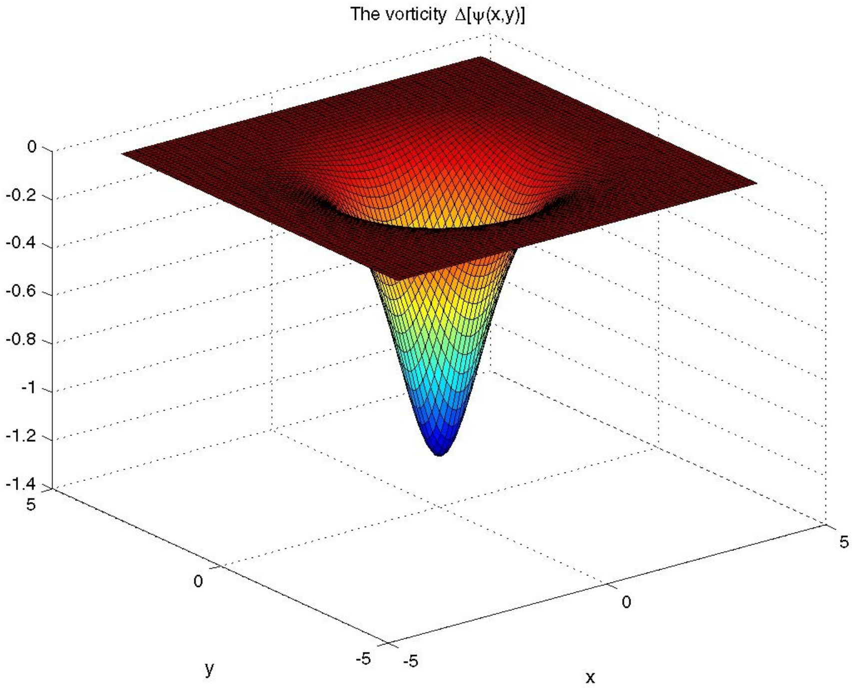

Figure 3: The vorticity $\omega(x, y)$. It is normalized with Ω_{ci}

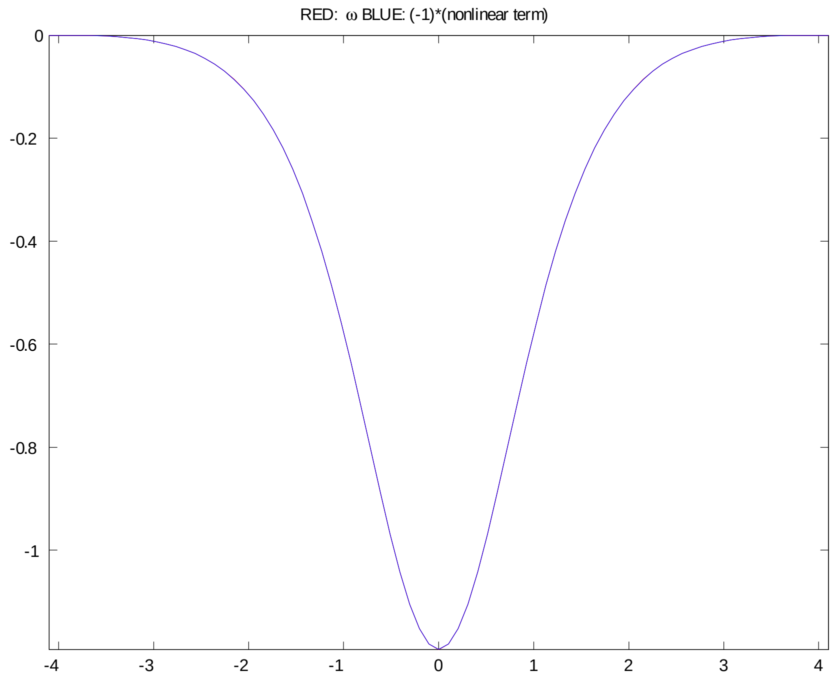

Figure 4: One of the test of quality of numerical solution; the vorticity profile is compared to the profile of the nonlinear term in the equation (the lines are indiscernable one of the other).

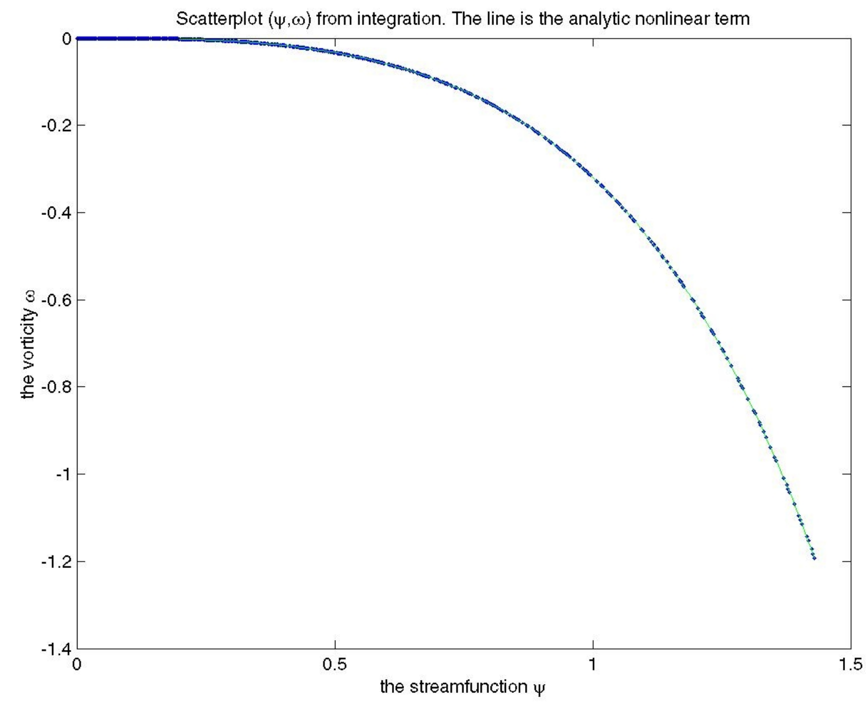

Figure 5: The scatterplot (ψ, ω). The continuum line is the analytic nonlinear term.

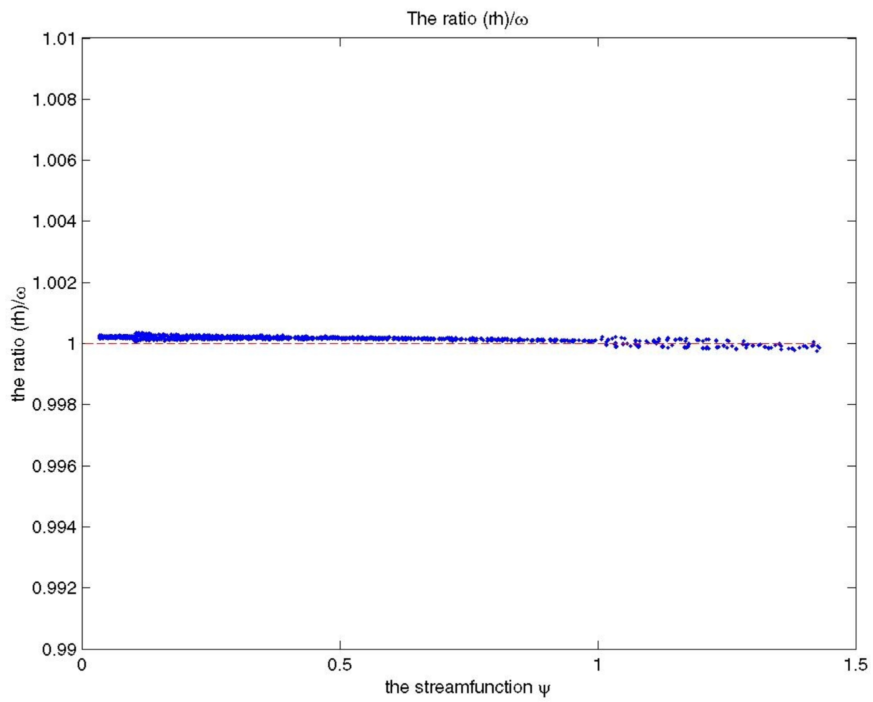

Figure 6: The test of quality of solution: the ratio of ω to the nonlinear term.

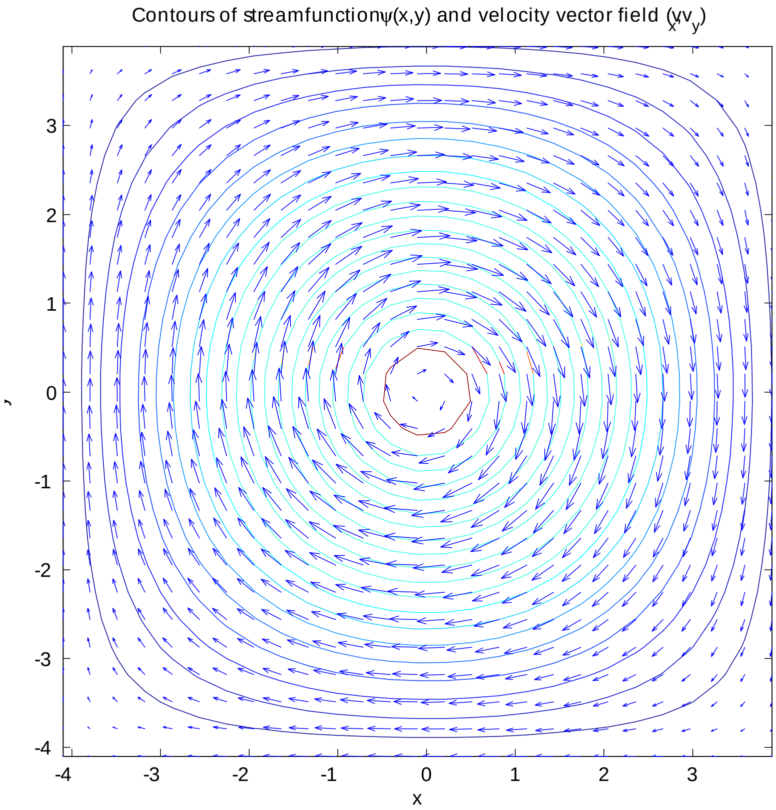

Figure 7: The contours of the streamfunction ψ and the velocity field.

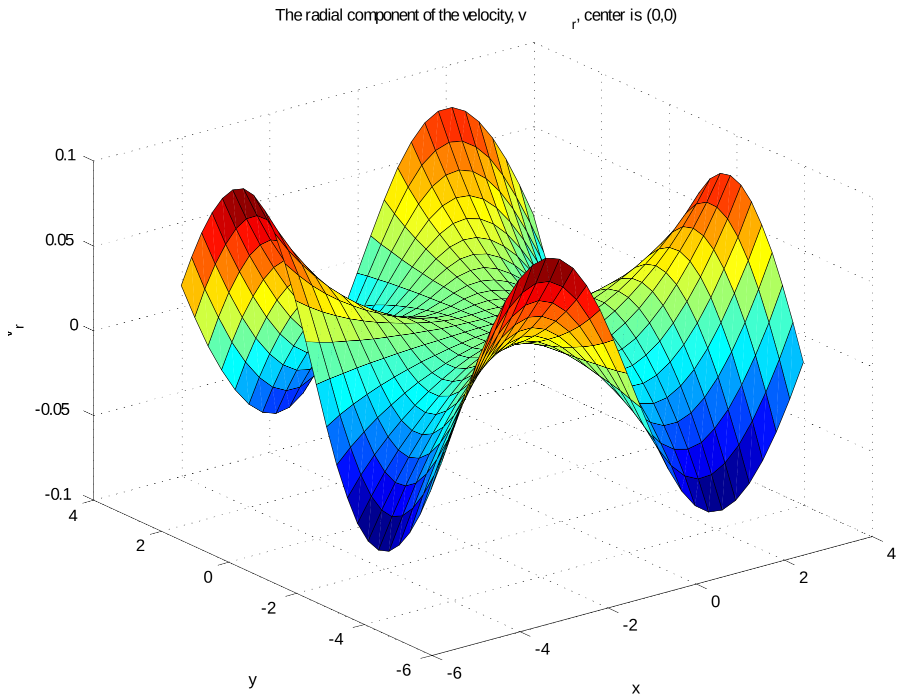

Figure 8: The radial component of the velocity, $v_r(x, y)$

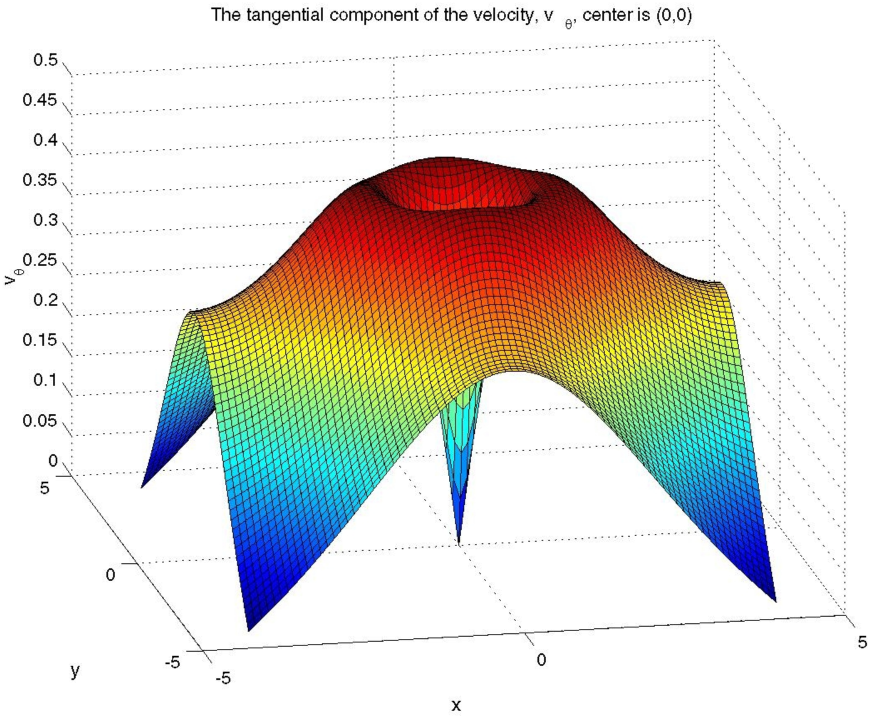

Figure 9: The tangential component of the velocity, $v_\theta(x, y)$

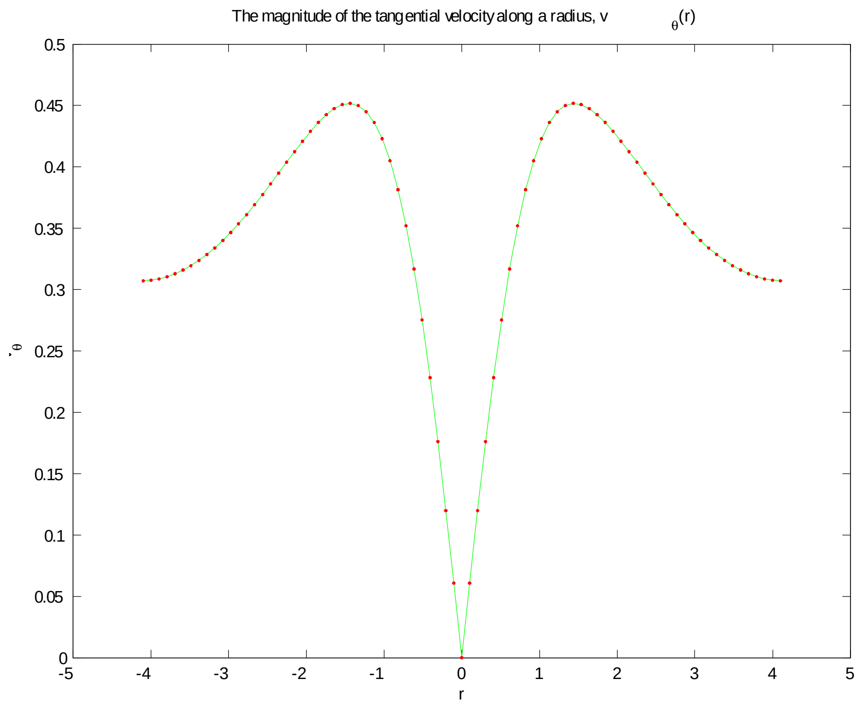

Figure 10: The profile of the magnitude of the tangential velocity.

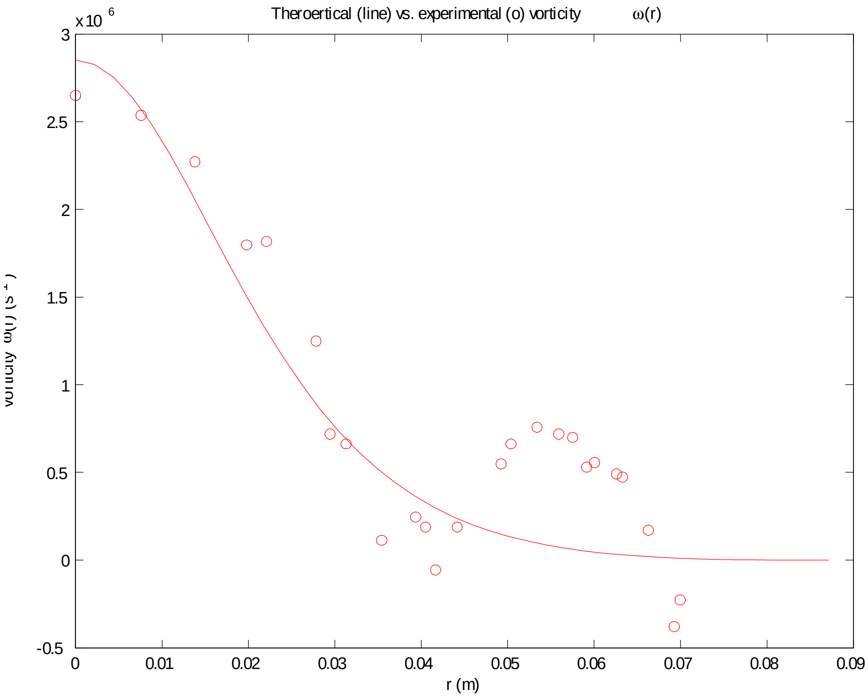

Figure 11: Comparison between the theoretical results (solution of the equation 1 and the experimental results (Fig.3 of ref.[5]): the vorticity.

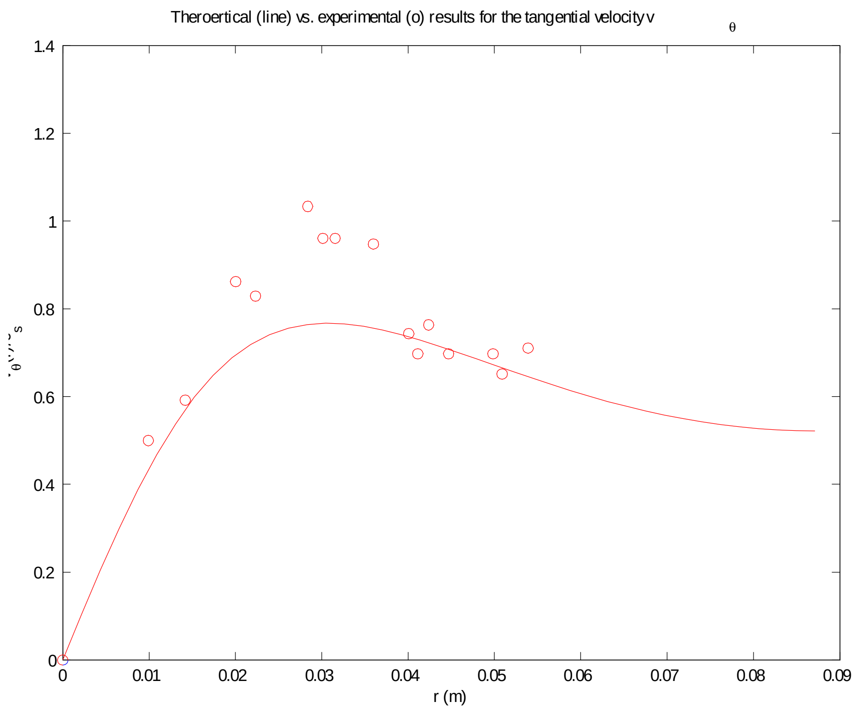

Figure 12: Comparison between the theoretical results (solution of the equation 1 and the experimental results (Fig.4 of ref.[5]): the velocity.

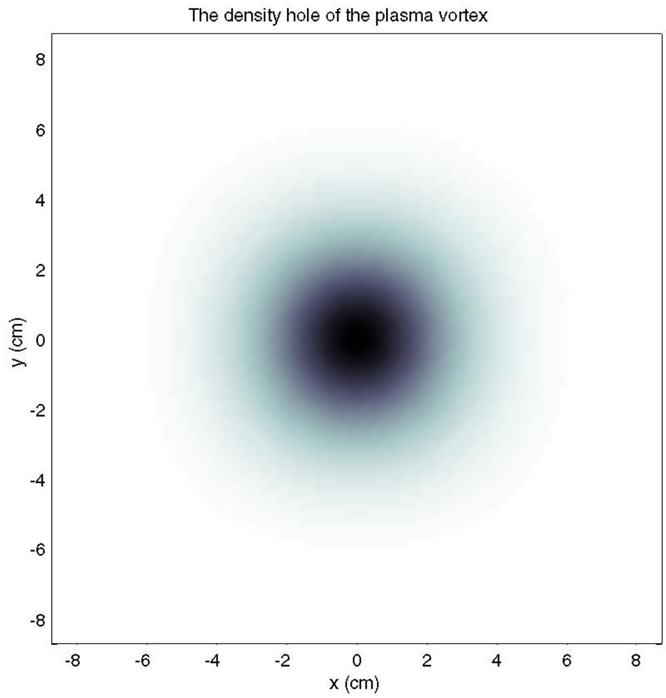

Figure 13: The density distribution as obtained by applying the Ertel theorem with the vorticity determined from numerical solution. (Compare with the Fig.1 of ref.[5])

4.3 The density variation in the plasma vortex

This can be applied for the case we study, the atmosphere or the plasma in the linear machine at NIFS. We have from numerical simulations that the vorticity is very peaked and it is always negative (since we have the streamfunction positive with maximum in the center)

$$\omega = -|\omega|$$

The Ertel's theorem is

$$\frac{d}{dt}\frac{\Omega_{ci} - |\omega|}{n} = 0$$

This suggests that, if the initial state is perfectly uniform and the streamlines are labelled by a single constant then

$$n(r) = const\,(\Omega_{ci} - |\omega(r)|)$$

Therefore in the central region, the only one where the vorticity ω is non-vanishingly small and negative, the density must be smaller than in the rest of the plasma or of the typhoon.

This conclusion does not depend on the direction of the rotation of the plasma, in the following sense. As described in the NIFS experiment, the direction of the rotation of the plasma in the configuration with the central hole could be reversed, by inversing the sense of the magnetic field. However, we notice that in this case also the sign of the gyrofrequency Ω_{ci} is reversed, due to the change of the sign of the magnetic field. Therefore, there is an overall change of signs, for ω and for Ω_{ci}, but the relation remains the same, which means that the desnity is reduced in the region where the vorticity is concentrated.

5 Summary

We consider that the comparison with the experimental data is very encouraging, although much numerical work is still needed. It appears possible to develop a theoretical explanation for the large scale vortical flow of the Hyper-I device (and of other vortical flows) on the basis of the theory of the asymptotic stationary states of the ion drift fluid. This implies that the differential equation (1) plays a major role in describing structures at stationarity. A confirmation of this role means that the plasma evolves to that particular subset of states characterized by self-duality. This would be identical to the correspnding tendency of the ideal fluids toward states described by the *sinh*-Poisson equation.

Naturally, it is not possible to expect a complete explanation of the present case, since the model of stationary states is *non-dissipative*. It is clear that in Hyper-I (as underlined in [5]) the viscosity is an important element of the dynamics.

Nevertheless, the confirmation of the evolution of the plasma to the self-dual states would be a far reaching result.

6 Appendix. Preliminary search for the initial parameters of the integration

The case with $p \neq 1$

The space derivatives that must be estimated for the Laplacean. $\omega = \Delta\psi$ will imply

$$\omega = \Delta\psi = \frac{\delta^2\psi}{(\delta r)^2}$$

with

$$\delta r = \text{a fraction of } R_p$$

let us say, $\delta r = 10$. In this case, we should have numerically $\psi \sim (\delta r)^2$ which is very large. Introducing this value in the functions sinh and cosh would immediately lead to intractable problems. Then we would look for much smaller values of ψ, and compensate the large magnitude of $(\delta r)^2$ by other means, for example by the factor $1/(2p^2)$ of the nonlinear term. This means that we should take

$$p \sim \delta r$$

Since this value of p is high, the boundary condition for ψ implies

$$\psi_b = \ln\left(p - \sqrt{p^2 - 1}\right)$$

is also high

$$\psi_b \simeq -\ln(2p)$$

Suppose that the streamfunction above this boundary value is small

$$\left|\widetilde{\psi}\right| \ll |\psi_b|$$

Then we can treat perturbatively

$$\Delta\left(\psi_b+\widetilde{\psi}\right)+\frac{1}{2p^2}\sinh\left(\psi_b+\widetilde{\psi}\right)\left[\cosh\left(\psi_b+\widetilde{\psi}\right)-p\right]=0$$

$$\begin{aligned}
&\Delta\widetilde{\psi}+\frac{1}{2p^2}\left(\sinh\psi_b+\widetilde{\psi}\cosh\psi_b+\frac{\widetilde{\psi}^2}{2}\sinh\psi_b\right)\\
&\times\left(\cosh\psi_b-p+\widetilde{\psi}\sinh\psi_b+\frac{\widetilde{\psi}^2}{2}\cosh\psi_b\right)\\
&=0
\end{aligned}$$

Retaining the second order

$$\Delta\widetilde{\psi}+\frac{1}{2p^2}\left(\widetilde{\psi}\sinh^2\psi_b+\frac{3}{2}\widetilde{\psi}^2\sinh\psi_b\cosh\psi_b\right)=0$$

We recall that

$$\cosh\psi_b=p$$

and

$$\begin{aligned}
\sinh\psi_b&=\sqrt{\cosh^2\psi_b-1}\\
&=\sqrt{p^2-1}
\end{aligned}$$

Then

$$\begin{aligned}
\Delta\widetilde{\psi}+\frac{1}{2p^2}\left(p^2-1\right)\widetilde{\psi}+\frac{3}{2}\frac{1}{2p}\sqrt{p^2-1}\widetilde{\psi}^2&=0\\
\Delta\widetilde{\psi}+\frac{1}{2}\left(1-\frac{1}{p^2}\right)\widetilde{\psi}+\frac{3}{4}\sqrt{1-\frac{1}{p^2}}\widetilde{\psi}^2&=0
\end{aligned}$$

We note that the first two terms reduce the equation to the oscillator, if $p>1$. However, if $p<1$ the equation becomes that of the modified Bessel function,

$$K_0\left[\frac{1}{\rho_i}\frac{1}{2}\left(1-\frac{1}{p^2}\right)r\right]\equiv K_0\left(\frac{r}{\rho'}\right)$$

$$\rho'\equiv\frac{\rho_i}{\frac{1}{2}\left(1-1/p^2\right)}$$

The approximation has shown that the space decay of any solution takes play on a distance which is replaced with a larger one, for $p>1$. But for larger p the decay is on the Larmor radius again. On the other hand our approximation has been done for large $p>1$. The solution is then an oscillation, for small $\widetilde{\psi}$.

We also note that the presence of p^2 at the denominator is without effect for compensating the large magnitudes of $(\delta r)^2$ that appears in the Laplacean.

The case with $p = 1$

In this case we will assume that ψ is a small quantity compared with unity. The boundary condition is

$$\cosh \psi_b - 1 = 0$$
$$\psi_b = 0$$

Then

$$\Delta \psi + \frac{1}{2} \sinh \psi \left(\cosh \psi - 1 \right) = 0$$

is approximated as

$$\Delta \psi + \frac{1}{2} \psi \frac{\psi^2}{2} = 0$$
$$\Delta \psi + \frac{1}{4} \psi^3 = 0$$

In one dimension we have

$$\frac{d^2}{dx^2} \psi + \frac{1}{4} \psi^3 = 0$$

with solution

$$\psi = \frac{\alpha}{x}$$

$$\frac{d}{dx} \left(- \frac{\alpha}{x^2} \right) + \frac{1}{4} \frac{\alpha^3}{x^3} = 0$$
$$2\alpha \frac{1}{x^3} + \frac{1}{4} \frac{\alpha^3}{x^3} = 0$$

or

$$\alpha = \sqrt{-8}$$

therefore again imaginary.

On the other hand, for large ψ we have

$$\Delta \psi + \frac{1}{2} \exp \left(2\psi \right) = 0$$

which is the Liouville equation. This equation is equivalent to an approximative form (for large $|\psi|$) of the *sinh*-Poisson equation, with the meaning that any trace of the factor $\cosh \psi - p$ has disappeared. We cannot use such approximation, since the term $\cosh \psi - p$ represents the physical effect of the condensate of vorticity, the basic element of the model.

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