A note on the BRST cohomology of
the extended antifield formalism

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Abstract
The relevance of the BRST cohomology of the extended antifield
formalism is briefly discussed along with standard homological tools
needed for its computation.

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The lectures given at the school *Q.F.T., Supersymmetry and Superstrings*, in Călimănești, Romania in April 1998 were intitled *Classical and quantum aspects of the Batalin-Vilkovisky formalism*. They covered the following material:

- **Lesson 1:** Algebraic structure of gauge symmetries


- **Lesson 2:** Locality in field theory


- **Lesson 3:** Consistency conditions on anomalies. Non renormalization theorems.


Useful review references connected to the material covered here are [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Details on specific topics, reflecting the point of view of the author, can be found in [11, 12, 13, 14, 15, 16, 17, 18] and in the literature cited therein.

The purpose of this note is to discuss briefly the BRST cohomology of the extended antifield formalism, to give some details on exact couples and spectral sequences, and to apply these concepts in the problem at hand.
1 Introduction

1.1 Classical theory

The Batalin-Vilkovisky formalism [19, 20] allows one to formulate the BRST differential [21, 22, 23, 24] controlling the gauge symmetries under renormalization for generic gauge theories. The formalism can be extended so as to include (non linear) global symmetries (see [12, 13] and references therein), which is achieved by coupling the BRST cohomology classes in negative ghost numbers with constant ghosts. A further extension including the BRST cohomology classes in all the ghost numbers can be constructed [16]. Some features of this extension are:

- it allows one to take into account in a systematic way all higher order cohomological constraints due to the antibracket maps [14],

- it is the appropriate formalism to prove stability independently of power counting restrictions, also called renormalizability in the modern sense [25], in the case of generic gauge theories,

- an appropriate BRST differential can be constructed on the classical and the quantum level, even in the case of anomalous theories [17].

Let us briefly summarize the results of [16] needed in the following. The extended formalism is obtained by first computing a basis for the local BRST cohomology classes. This basis contains as a subset those classes that can be obtained from the solution $S$ of the master equation by differentiation of $S$ with respect to so-called essential coupling constants. The additional classes completing the basis are then coupled with the help of new independent coupling constants. This action can then be extended by terms of higher orders in the new couplings in such a way that, if we denote by $\xi^A$ all the couplings corresponding to the independent BRST cohomology classes, the corresponding action $S$ satisfies the extended master equation

$$\frac{1}{2}(S, S) + \Delta_c S = 0. \quad (1.1)$$

The BRST differential associated to the solution of the extended master equation is

$$\bar{s} = (S, \cdot) + \Delta^L_c, \quad (1.2)$$
where $\Delta_c = \frac{\partial R}{\partial \xi^A} f^A$, while $\Delta^L_c = (-)^A f^A \frac{\partial L}{\partial \xi^A}$, with $f^A$ depending (at least quadratically) on the couplings $\xi$ alone. Both antiderivations satisfy $(\Delta_c)^2 = 0 = (\Delta^L_c)^2$. Since there is no dependence on the fields and the antifields, $\Delta^L_c(A, B) = (\Delta^L_c A, B) + (-)^{A+1}(A, \Delta^L_c B)$, with the appropriate version holding for the right derivation $\Delta_c$. The local BRST cohomology classes contain the generators of all the generalized non trivial symmetries of the theory in negative ghost number, the generalized observables in ghost number zero, and the anomalies (and anomalies for anomalies) in positive ghost number. This is the reason why the extended master equation encodes the invariance of the original action under all the non trivial gauge and glocal symmetries, their commutator algebra as well as the antibracket algebra of all the local BRST cohomology classes.

The cohomology of $\tilde{s}$ in the space $F$ of $\xi$ dependent local functionals in the fields, the antifields and their derivatives is isomorphic to the cohomology of

$$s_{\Delta_e} = [\Delta_e, \cdot]$$

in the space $G$ of graded right derivations $\lambda = \frac{\partial R}{\partial \xi^A} \lambda^A$, with $\lambda^A$ a function of $\xi$ alone, $[\cdot, \cdot]$ being the graded commutator for graded right derivations,

$$H(\tilde{s}, F) \simeq H(s_{\Delta_e}, G).$$

If $\mu$ is a $s_{\Delta_e}$ cocycle, the corresponding $\tilde{s}$ cocycle is given by $\mu S = \frac{\partial R \tilde{s}}{\partial \xi^B} \mu^B$.

### 1.2 Quantum theory

In the standard version of the BRST-Zinn-Justin-Batalin-Vilkovisky set-up, there are two main issues to be considered: stability and anomalies.

The problem of stability is the question if to every local BRST cohomology class in ghost number 0, there corresponds an independent coupling of the action. The extended formalism solves this problem trivially by construction, since all cohomology classes have been coupled with independent couplings. The non trivial part of the formalism is the proof of the existence of the extended master equation and the associated differential, which allows to control the symmetries after the extension. Of course, it will be often convenient in practice not to couple all the local BRST cohomology classes but only a subset needed to guarantee that the theory is stable.
In the standard set-up, the question of anomalies is mostly reduced to the question of the local BRST cohomology in ghost number 1 and to a discussion of the coefficients of these cohomology classes. In the presence of anomalies, there is no differential on the quantum level associated to the anomalously broken Zinn-Justin equation for the effective action. In the extended formalism however, because all the local BRST cohomology classes in positive ghost numbers have been coupled to the solution of the master equation, such a differential exists [16]. Indeed, the quantum action principle [26, 27, 28] applied to (1.1) gives

\[ \frac{1}{2} (\Gamma, \Gamma) + \Delta_c \Gamma = \bar{h} A \circ \Gamma, \]  

where \( \Gamma \) is the renormalized generating functional for 1PI vertices associated to the solution \( S \) of the extended master equation and the local functional \( A \) is an element of \( F \) in ghost number 1. Using the result (1.4) on the cohomology of \( \bar{s} \), one can show [16, 17] that, through the addition of local counterterms, (1.5) can be written as

\[ \frac{1}{2} (\Gamma^\infty, \Gamma^\infty) + \Delta^\infty \Gamma^\infty = 0, \]  

where \( \Gamma^\infty \) is associated to the action \( S^\infty = S - \sum_{k=1}^{\infty} \bar{h}^k \Sigma_k \) containing local finite BRST breaking counterterms \( \Sigma_k \) and \( \Delta^\infty = \Delta_c + \bar{h} \Delta_1 + \bar{h}^2 \Delta_2 + \ldots \) satisfies \((\Delta^\infty)^2 = 0\). The associated quantum BRST differential is

\[ s^q = (\Gamma^\infty, \cdot) + (\Delta^\infty)^L. \]  

In the limit \( \bar{h} \) going to zero, we recover both the classical extended master equation (1.1) and the classical differential \( \bar{s} \).

In the extended antifield formalism, the anomalous Zinn-Justin equation can thus be written as a functional differential equation for the renormalized effective action. The derivations \( \Delta_1, \Delta_2, \ldots \) are guaranteed to exist due to the quantum action principles. They satisfy a priori cohomological restrictions due to the fact that the differential \( \Delta^\infty \) is a formal deformation with deformation parameter \( \bar{h} \) of the differential \( \Delta_c \).

For instance, the derivation \( \Delta_1 \) is a cocycle of \( s_{\Delta_c} \) in ghost number 1, because \([\Delta_c, \Delta_1] = 0\). This cocycle can be assumed to be non trivial, because otherwise, \( \Delta_1 \) could have been absorbed by the counterterm \( \Sigma_1 \). Hence, non
trivial anomalies, which correspond in this formalism to non trivial deforma-
tions of \( \Delta_c \), are controled by \( H^1(s_{\Delta_c}, G) \).

In the standard way, once care has been taken of the trivial anomalies
through the counterterms \( \Sigma_k \), the remaining infinite and finite counterterms
are required to belong to \( H^0(\bar{s}, F) \simeq H^0(s_{\Delta_c}, G) \) in order to preserve (1.6)
to that order.

It is thus of interest to compute the cohomology of \( s_{\Delta_c} \).

2 BRST cohomology in the extended anti-field formalism

Let \( \lambda = \frac{\partial R}{\partial \xi} \lambda^A \) be a right derivation. We assume that the \( \lambda^A \) are formal
power series in \( \xi^A \). In the following, we provide this space with an obvious
filtration. It will however not have finite length, and for particular theories,
better filtrations have to be found in order to do a complete computation.
Since the techniques will be similar, it is nevertheless useful to show how
they work on this example.

2.1 Grading and filtration on the space of right deriva-
tions

Let \( N_\xi = \frac{\partial R}{\partial \xi} \xi^A \) be the operator counting the number of \( \xi \)'s. A general right
derivation admits the following decomposition according to the eigenvalues
of \( N_\xi \): \( \lambda = \lambda_{-1} + \lambda_0 + \lambda_1 + \ldots \), where \( [\lambda_p, N_\xi] = p\lambda_p \). Hence, \( G \) is a graded
space, \( G = \oplus_{p=1}^\infty G^p \). (It is actually a bigraded space, the other grading, for
which \( s_{\Delta_c} \) is homogeneous of degree 1 being the ghost number.)

The graded right commutator satisfies \( [[\lambda_m, \mu_n], N_\xi] = (m + n)[\lambda_m, \mu_n] \).
The decomposition of \( \Delta_c \) starts at eigenvalue 1: \( \Delta_c = \Delta_{c1} + \Delta_{c2} + \ldots \); the
 corresponding decomposition of \( s_{\Delta_c} \) being \( s_{\Delta_c} = [\Delta_{c1}, \cdot] + [\Delta_{c1}, \cdot] + \ldots \equiv s_1 + s_2 + \ldots \). It follows that the cocycle condition \( s_{\Delta_c} \lambda = 0 \) decomposes as

\[
\begin{align*}
  s_1 \lambda_{-1} &= 0, \\
  s_1 \lambda_0 + s_2 \lambda_{-1} &= 0, \\
  s_1 \lambda_1 + s_2 \lambda_0 + s_3 \lambda_{-1} &= 0, \\
  &\vdots 
\end{align*}
\]

(2.1)
while the coboundary condition $\lambda = s_{\Delta_c} \mu$ decomposes as

\[
\lambda_{-1} = 0, \\
\lambda_0 = s_1 \mu_{-1}, \\
\lambda_1 = s_1 \mu_0 + s_2 \mu_{-1}, \\
\lambda_2 = s_1 \mu_1 + s_2 \mu_0 + s_3 \mu_{-1}, \\
\vdots
\]

(2.2)

In order to construct the spectral sequence associated to this problem, we follow [29].

Consider the spaces $K_p$ of derivations having $N_\xi$ degree greater than $p$, i.e., $\lambda \in K_p$ if $\lambda = \lambda_p + \lambda_{p+1} + \ldots$. The space of all right derivations is $G = K_{-1}, K_{p+1} \subset K_p$ and $s_{\Delta_c} K_p \subset K_p$. The sequence of spaces $K_p$ is a decreasing filtration of $G$, with $K_p/K_{p+1} \simeq G^p$.

We have the short exact sequence:

\[
0 \longrightarrow \oplus_{p=-1} K_{p+1} \overset{i}{\longrightarrow} \oplus_{p=-1} K_p \overset{j}{\longrightarrow} \oplus_{p=-1} K_p/K_{p+1} \longrightarrow 0,
\]

(2.3)

where $\oplus_{p=-1} K_p/K_{p+1} \simeq \oplus_{p=-1} G^p$. The following diagram is exact at each corner:

\[
\begin{array}{ccc}
H(s_{\Delta_c}, \oplus_{p=-1} K_{p+1}) & \overset{i_0}{\longrightarrow} & H(s_{\Delta_c}, \oplus_{p=-1} K_p) \\
\uparrow k_0 & & \downarrow j_0 \\
E_0 & &
\end{array}
\]

(2.4)

where $E_0 = \oplus_{p=-1} K_p/K_{p+1} \simeq \oplus_{p=-1} G^p$. In this diagram, $H(s_{\Delta_c}, K_p)$ is defined by the cocycle condition $s_{\Delta_c}(\lambda_p + \lambda_{p+1} + \ldots) = 0$, and the coboundary condition $\lambda_p + \lambda_{p+1} + \ldots = s_{\Delta_c}(\mu_p + \mu_{p+1} + \ldots)$. The maps $i_0$ and $j_0$ are induced by $i$ and $j$, $i_0[\lambda_p + \lambda_{p+2} + \ldots] = [\lambda_{p+1} + \lambda_{p+2} + \ldots]$ and $j_0[\lambda_p + \lambda_{p+1} + \ldots] = [j(\lambda_p + \lambda_{p+1} + \ldots)] = [\lambda_p]$. They are well defined, because $i_0$ maps cocycles to cocycles and coboundaries to coboundaries, while $j(s_{\Delta_c}(\mu_p + \mu_{p+1} + \ldots)) \in K_{p+1}$. The map $k_0$ is defined by $k_0[\lambda_p] = [s_{\Delta_c} \lambda_p]$. It does not depend on the choice of representative for $[\lambda_p] \in K_p/K_{p+1}$, because $[s_{\Delta_c}(\lambda_{p+1} + \ldots)] = 0 \in H(s_{\Delta_c}, K_{p+1})$.

Let us check explicitly that this diagram is exact:

---

A diagram is said to be exact if the image of a map is equal to the kernel of the next map.
The representative chosen for a cocycle, because $s_{\Delta_c}(\lambda_p + \lambda_{p+1} + ...) = 0$ and $\lambda_p = 0$. This is the same than $i_0H(s, K_{p+1})$, which is given by $[\lambda_{p+1} + \lambda_{p+2} + ...]$, with $s_{\Delta_c}(\lambda_{p+1} + \lambda_{p+2} + ...) = 0$, the equivalence relation being the equivalence relation in $H(s, K_p)$ by definition of $i_0$.

- ker $k_0$ is given by elements $[\lambda_p]$ such that $[s_{\Delta_c}\lambda_p] = 0 \in H(s_{\Delta_c}, K_{p+1})$, i.e. such that $s_{\Delta_c}\lambda_p = s_{\Delta_c}(\mu_{p+1} + \mu_{p+2} + ...)$. By the identification $\lambda_{p+1} = -\mu_{p+1}, \lambda_{p+2} = -\mu_{p+2}, ...$, this is indeed the same than $j_0H(s_{\Delta_c}, K_p)$ given by $[\lambda_p]$ with $s_{\Delta_c}(\lambda_p + \lambda_{p+1} + ...) = 0$.

- ker $i_0$ is given by elements $[\lambda_{p+1} + \lambda_{p+2} + ...]$ such that $s_{\Delta_c}(\lambda_{p+1} + \lambda_{p+2} + ...) = 0$ and $\lambda_{p+1} + \lambda_{p+2} + ... = s_{\Delta_c}(\mu_p + \mu_{p+1} + ...)$, while $k_0[\mu_p]$ is given by $[\lambda_{p+1} + \lambda_{p+2} + ...]$ of the form $[s_{\Delta_c}\mu_p]$ so that $\lambda_{p+1} + \lambda_{p+2} + ... = s_{\Delta_c}\mu_p + s_{\Delta_c}(\mu_{p+1} + ...)$, which is indeed the same.

## 2.2 Exact couples and associated spectral sequence

To every exact couple $(A_0, B_0)$, i.e., exact diagram of the form

$$
\begin{array}{ccc}
A_0 & \xrightarrow{i_0} & A_0 \\
\downarrow k_0 & & \downarrow j_0 \\
B_0, & & \\
\end{array}
$$

(2.5)

one can associated a derived exact couple

$$
\begin{array}{ccc}
A_1 & \xrightarrow{i_1} & A_1 \\
\downarrow k_1 & & \downarrow j_1 \\
B_1, & & \\
\end{array}
$$

(2.6)

In this diagram, the spaces and maps are defined as follows: $A_1 = i_0A_0$; $B_1 = H(d_0, B_0)$, where $d_0 = j_0 \circ k_0$ ( $d_0^2 = 0$ because $k_0 \circ j_0 = 0$); for $a_1 = i_0a_0$, $i_1a_1 = i_1(i_0a_0) = i_0^2a_0$; $j_1a_1 = [j_0a_0]$ (this map is well defined: $j_0a_0$ is a cocycle, because $k_0 \circ j_0 = 0$, furthermore the map does not depend on the representative choosen for $a_0$, because if $i_0a_0 = 0$, $a_0 = k_0b_0$ for some $b_0$ and $j_1a_1 = [j_0 \circ k_0b_0] = 0$); $k_1[b_0] = k_0b_0$ ($k_0b_0 = i_0a_0$ for some $b_0$ because $d_0b_0 = j_0(k_0b_0) = 0$ implies $k_0b_0 = i_0a_0$, furthermore $k_0d_0c_0 = 0$ because $k_0 \circ j_0 = 0$).

Let us also check explicitly exactness of this diagram:
• ker $j_1$ is given by elements $a_1 = i_0 a_0$ such that $[j_0 a_0] = 0$, i.e., $j_0 a_0 = j_0 k_0 b_0$ and then $a_0 - k_0 b_0 = i_0 c_0$, implying that $a_1 = i_0^2 c_0$. $i_1 A_1$ is given by elements $a_1 = i_1 c_1 = i_0^2 c_0$. It follows that $\ker j_1 \subset i_1 A_1$, while the inverse inclusion follows from $j_0 \circ i_0 = 0$.

• ker $k_1$ is given by elements $[b_0]$ such that $k_0 b_0 = i_0 a_0 = 0$, i.e., such that $b_0 = j_0 c_0$, for some $c_0$, while im $j_1$ is given by elements $[b_0]$ such that $[b_0] = [j_0 e_0]$, i.e $b_0 = j_0 (c_0 + k_0 f_0)$. It follows that ker $k_1 = \im j_1$.

• ker $i_1$ is given by elements $a_1 = i_0 a_0$ such that $i_0(i_0 a_0) = 0$, i.e., $i_0 a_0 = k_0 b_0$ (which implies in particular $d_0 b_0 = 0$). im $k_1$ is given by elements $a_1 = i_0 a_0 = k_0 b_0$ for some $b_0$ with $d_0 b_0 = 0$, so both spaces are indeed the same.

Clearly, this construction can be iterated by taking as the starting exact couple the derived couple. We thus get a sequence of exact couples
\[ A_r \xrightarrow{i_r} A_r \]
\[ k_r \xleftarrow{j_r} B_r. \]
and the associate spectral sequence $(B_r, d_r)$, for $r = 0, 1, \ldots$, i.e., spaces $B_r$ and differentials $d_r$ satisfying $B_{r+1} = H(d_r, B_r)$.

### 2.3 Spectral sequence associated to the BRST cohomology of the extended antifield formalism

Let us now apply the general theory to the case of the exact couple (2.4) and give explicitly the differentials $d_r$ and the spaces $B_r$ (called $E_r$) in this case for $r = 0, 1, 2, 3$.

We have $E_0 = \oplus_{p=-1} K_p/K_{p+1} \simeq \oplus_{p=-1} G^p$. The differential $d_0$ is defined by $d_0[\lambda_p]_0 = j_0[\delta_{\Delta} \lambda_p]$, where $[\delta_{\Delta} \lambda_p] \in H(s_{\Delta}, K_{p+1})$. It follows that $d_0[\lambda_p]_0 = [s_1 \lambda_p]$ This means that $E^1_r$ is defined by elements $[[\lambda_p]_0]_1$ with the cocycle condition
\[ s_1 \lambda_p = 0 \]
and the coboundary condition
\[ \lambda_p = s_1 \mu_{p-1}. \]
Because $s_1 = \frac{\partial}{\partial x^B} f^A_{BC} \xi^B \xi^C$, and $s_1^2 = 0$ implies that the $f^A_{BC}$ are the structure constants of a graded Lie algebra, this group is just a graded version of standard Lie algebra (Chevalley-Eilenberg) cohomology with representation space the adjoint representation.

Take now $[[\lambda_p]_0]_1 \in E^p_1$. The differential $d_1[[\lambda_p]_0]_1 = j_1k_1[[\lambda_p]_0]_1 = j_1k_0[\lambda_p]_0 = j_1[s_{\Delta_c} \lambda_p] = [j_0i_0^{-1}s_{\Delta_c} \lambda_p]_1$. This means that $[s_{\Delta_c} \lambda_p]$ has to be considered as an element of $H(s_{\Delta_c}, K_{p+2})$ so that $d_1[[\lambda_p]_0]_1 = [[s_2 \lambda_p]_0]_1$. Hence $E^p_2$ is defined by elements $[[[\lambda_p]_0]_1]_2$ with the cocycle condition

$$s_2 \lambda_p + s_1 \lambda_{p+1} = 0, \quad (2.10)$$

and the coboundary condition

$$\lambda_p = s_2 \mu_{p-2} + s_1 \mu_{p-1}, \quad (2.12)$$
$$0 = s_1 \mu_{p-2}. \quad (2.13)$$

We thus find that $E^p_2 = H^p(s_2, H(s_1))$.

The differential $d_2$ in $E^p_2$ is defined by $d_2[[[\lambda_p]_0]_1]_2 = j_2k_2[[[\lambda_p]_0]_1]_2 = j_2k_0[\lambda_p]_0 = [j_{1+1}^1k_0[\lambda_p]_0]_2 = [j_0i_0^{-1}i_1^{-1}k_0[\lambda_p]_0]_2$. In order to make sure that $k_0[\lambda_p]_0$ belongs to $i_1i_0H(s_{\Delta_c}, K_{p+1})$ we use $\lambda_p + \lambda_{p+1}$ as a representative for $[\lambda_p]_0$. It follows that $d_2[[[\lambda_p]_0]_1]_2 = [[[s_3 \lambda_p + s_2 \lambda_{p+1}]_0]_1]_2$. The cocycle condition for an element $[[[\lambda_p]_0]_1]_2 \in E^p_3$ is then given by

$$s_3 \lambda_p + s_2 \lambda_{p+1} = s_2 \mu_{p+1} + s_1 \mu_{p+2}, \quad (2.14)$$
$$s_2 \lambda_p + s_1 \lambda_{p+1} = 0, \quad (2.15)$$
$$s_1 \lambda_p = 0, \quad (2.16)$$

with $s_1 \mu_{p+2} = 0$. The redefinition $\lambda_{p+1} \rightarrow \lambda_{p+1} - \mu_{p+1}$ and $\lambda_{p+2} = -\mu_{p+2}$, then gives as cocycle condition

$$s_3 \lambda_p + s_2 \lambda_{p+1} + s_1 \lambda_{p+2} = 0, \quad (2.17)$$
$$s_2 \lambda_p + s_1 \lambda_{p+1} = 0, \quad (2.18)$$
$$s_1 \lambda_p = 0. \quad (2.19)$$

The coboundary condition is $[[[\lambda_p]_0]_1]_2 = d_3[[[\mu_{p-3} + \mu_{p-2}]_0]_1]_2$, where $s_1 \mu_{p-3} = 0, s_2 \mu_{p-3} + s_2 \mu_{p-2} = 0$, hence $[[\lambda_p]_0]_1 = [[[s_3 \mu_{p-3} + s_2 \mu_{p-2}]_0]_1 + d_2[[\sigma_{p-2}]_0]_1$, 101
with \( s_1\sigma_{p-2} = 0 \) which gives

\[
\lambda_p = s_3\mu_{p-3} + s_2\mu_{p-2} + s_2\sigma_{p-2} + s_1\rho_{p-1},
\]

\[
0 = s_2\mu_{p-3} + s_1\mu_{p-2},
\]

\[
0 = s_1\mu_{p-3},
\]

\[
0 = s_1\sigma_{p-2}.
\]

The redefinition \( \mu_{p-2} \rightarrow \mu_{p-2} + \sigma_{p-2} \) and \( \rho_{p-1} = \mu_{p-1} \), then gives the coboundary condition

\[
\lambda_p = s_3\mu_{p-3} + s_2\mu_{p-2} + s_1\mu_{p-1},
\]

\[
0 = s_2\mu_{p-3} + s_1\mu_{p-2},
\]

\[
0 = s_1\mu_{p-3}.
\]

This construction can be continued in the same way for higher \( r \)'s.

The original problem was the computation of \( H(s_{\Delta_e}, G) = H(s_{\Delta_e}, K_{-1}) \).

From exactness of the couples (2.7), it follows that

\[
H(s_{\Delta_e}, K_{-1}) \simeq j_0 E_0^{-1} \oplus \ker j_0
\]

\[
\simeq \ker k_0(\subset E_0^{-1}) \oplus i_0 H(s_{\Delta_e}, K_0)
\]

\[
\simeq \ker k_0(\subset E_1^{-1}) \oplus \ker k_1(\subset E_1^0) \oplus i_1 i_0 H(s_{\Delta_e}, K_1)
\]

\[
\vdots
\]

\[
\simeq \oplus_{r=0}^R \ker k_r(\subset E_r^{-1}) \oplus i_R \ldots i_0 H(s_{\Delta_e}, K_R).
\]

Furthermore, \( E_0 \simeq E_1 \oplus F_0 \oplus d_0 F_0 \) and \( E_0^{-1} \simeq E_1^{-1} \oplus F_0^{-1} \). \( F_0 \) does not belong to \( \ker k_0 \) because \( d_0 F_0 \neq 0 \). Thus \( \ker k_0(\subset E_0^{-1}) \simeq \ker k_1(\subset E_1^{-1}) \).

Similarly, \( E_1 \simeq E_2 \oplus F_1 \oplus d_1 F_1 \) and \( (d_1 F_1)^{-1} = (d_1 F_1)^0 = 0 \). Again, \( d_1[F_1]_1 \neq 0 \) implies that \( F_1 \) does not belong to \( \ker k_1 \). This means that \( \ker k_1(\subset E_1^{-1}) \simeq \ker k_2(\subset E_2^{-1}) \) and \( \ker k_1(\subset E_1^0) \simeq \ker k_2(\subset E_2^0) \). Going on in the same way, we conclude that \( \ker k_r(\subset E_r^{-1}) \simeq \ker k_r(\subset E_r^0) \). We thus get

\[
H(s_{\Delta_e}, K_{-1}) \simeq \oplus_{r=0}^R \ker k_r(\subset E_r^{-1}) \oplus i_R \ldots i_0 H(s_{\Delta_e}, K_R).
\]

This construction is most useful if it would stop at some point. Indeed, suppose that \( K_R = 0 \). Because \( k_R[\ldots [\lambda_p]_0 \ldots]_R \) belongs to \( i_R \ldots i_0 H(s_{\Delta_e}, K_{p+R+1}) = 0 \), it follows that

\[
H(s_{\Delta_e}, K_{-1}) \simeq \oplus_{r=0}^{R-1} E_r^{-1}.
\]

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