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Development and exploitation of a spectral code for magnetohydrodynamics

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Abstract

The study of plasma and conductive fluid motions in presence of electro-magnetic fields is based on the magnetohydrodynamic balance equations. Depending on several parameters such as the kinetic and magnetic Reynolds numbers, as well as the magnetic Prandtl number, a very wide class of phenomena can be described by these equations. For instance, almost the same formalism may be used to investigate fairly different systems such as astrophysical media, planetary cores, fusion plasmas and liquid metal flows. This has motivated the development of a unified numerical tool capable of describing these phenomena, at least in the limit of the available computational power. We briefly describe the strategy adopted in developing this code as well as various examples in which it has been exploited.

1 Introduction

The career of Radu Balescu has been mostly devoted to the theoretical investigation of various phenomena in plasma physics. Nevertheless, even if he sometimes regarded the output of numerical simulations with some suspicion, he often showed a real interests in the development of numerical tools. When works on the numerical investigation of hydrodynamic turbulence have been initiated in his group in the early nineties, Radu Balescu regularly insisted on the importance of MHD phenomena in fusion plasmas and suggested to extend the numerical analysis to magnetohydrodynamics (MHD). His advices have been both helpful and supportive. Today, the study of MHD has been significantly developed in our research unit and numerical simulation is now one of our major research tools.

This paper describes one of the tools that have been developed in our group to simulate plasmas or fluids when they can be treated as continuous media. The TURBO code is designed to solve numerically the equations for an incompressible fluid in a three dimensional geometry with periodic boundary conditions in the three directions. In that respect, this code is limited to the investigation of rather academic problems. Nevertheless, it has proved to be a valuable tool to explore various MHD phenomena. The paper is organized as follows. The TURBO code is briefly described in Section 2. Applications to MHD turbulence are discussed in Section 3. Extension of the code to investigate MHD effects in liquid metal flows are presented in Section 4. The code has also been used to analyze charge particle trajectories in turbulent electromagnetic fields (Section 5). Future possible extensions of the code are briefly discussed in the last Section.

2 The turbo code

The MHD equations that can be solved using the TURBO code are the following balance equations for the velocity field u_i, the magnetic field b_i and a set of passive scalars c_α:

$$\partial_t u_i = -\partial_j(u_i u_j) - \partial_i p + \nu\nabla^2 u_i + f_i + f_i^{\mathrm{Lorentz}} \quad (1)$$

$$\partial_t b_i = -\partial_j(b_i u_j - u_i b_j) + \eta\nabla^2 b_i \quad (2)$$

$$\partial_t c_\alpha = -\partial_j(c_\alpha u_j) + \kappa_\alpha\nabla^2 c_\alpha + \sigma_\alpha(\{c_\beta\}) \quad (3)$$

where ν is the kinematic viscosity and η is the magnetic diffusivity. Each passive scalar is characterized by a diffusion coefficient κ_α. Summation is assumed over repeated Latin style indices, but not over Greek style indices which correspond to passive scalar species. There is the possibility to include source or sink terms or even chemistry terms in the scalar equations through the quantities $\sigma_\alpha(\{c_\beta\})$. The Lorentz force $\vec{f}^{\mathrm{Lorentz}} = \vec{j} \times \vec{b}$ (where $\vec{j}$ is the current density) will take different expressions depending on the range of parameters. It is also possible to add other forcing terms through f_i such as a random stirring force with a prescribed spectrum as well as a deterministic force that can depend on both space and time. The velocity field is assumed to be divergence free ($\partial_i u_i = 0$; incompressible flow), so that no state equation is needed for the pressure p. The role of the pressure is simply to ensure that the velocity field remains divergence free, which implies the following Poisson equation:

$$\nabla^2 p = -\partial_i\partial_j(u_i u_j) + \partial_i f_i + \partial_i f_i^{\mathrm{Lorentz}} \quad (4)$$

The magnetic field is, of course, also divergence free, $\partial_i b_i = 0$, but this does not imply any additional constraint in the code because TURBO is a pseudo-spectral code based on a Fourier expansion for all the variables. In such a formalism imposing $\partial_i b_i = 0$ is very easy and amounts to project each Fourier mode $\tilde{b}_i(\vec{k})$ in the plane perpendicular to the wave vector $\vec{k}$. The code is referred to as "pseudo-spectral" because the Fourier modes are advanced in time but the non linear terms are computed in real space using Fast Fourier Transform algorithms [1]. As a consequence, periodic boundary conditions are hard-coded in TURBO. The geometry is always a tri-dimensional rectangular box for which the length L_x, L_y and L_z are not necessarily the same and can be prescribed by the user. The solutions of the balance equations are then entirely determined by the initial conditions. Nevertheless, a fairly large variety of problems can be treated depending on these initial conditions, on the forcing terms, on the presence of passive scalars, on the source terms σ_α and on the values of the transport coefficients ν, η and κ_α.

The time stepping is base on a third order Runge-Kutta methods. Two different implementations can be used depending on the strategy adopted (truncation or phase shift) for removing aliasing errors generated when nonlinear terms have to be computed with a finite number of Fourier modes [2]. Finally, we mention that TURBO has been fully parallelized to take advantage of current large super-computers.

3 Simulation of MHD turbulence

Depending on the relative importance of nonlinear convective ($\partial_j(u_j u_i)$) and linear viscous ($\nu\nabla^2 u_i$) terms, the fluid can be either in the laminar or in the turbulent regime. The ratio between these two terms is characterized by the kinematic Reynolds number:

$$R_e = \frac{U\ L}{\nu} \tag{5}$$

where U and L are respectively the characteristic velocity and the characteristic length scale of the flow. High Reynolds number flows are expected to be turbulent. It is well known that the computational resources required to investigate a turbulent system increase rapidly with the Reynolds number. The magnetic Reynolds number, characterizing the ratio between nonlinear $\partial_j(b_i u_j - u_i b_j)$ and linear $\eta\nabla^2 b_i$ terms in the induction equation, may also strongly affect the behavior of the system:

$$R_m = \frac{U\ L}{\eta} \tag{6}$$

At moderate or large magnetic Reynolds numbers, the Lorentz force is expressed by

$$f_i^{\text{Lorentz}} = \partial_j(b_j b_i) \tag{7}$$

and the pressure actually contains both the hydrodynamic and the magnetic pressure contributions but remains solution of the same equation (4).

The TURBO code has been used for simulating a number of moderate kinetic and magnetic Reynolds number flows. First, homogeneous and isotropic (both decaying and forced) turbulence has been computed for fairly large resolutions (up to 512^3 grid points). These numerical experiments have been used to investigate the energy cascade in MHD turbulence. The study of the nonlinear interactions in Navier-Stokes turbulence has long been an active subject of research [3, 4, 5, 6]. The motivation for such studies is to improve the understanding of the physics of turbulence and, more specifically, of the mechanism(s) of energy transfer from the large, geometry dependent, structures to the small scales where dissipation into heat is observed. For MHD systems, we have found that, like in Navier-Stokes turbulence, the kinetic and magnetic energy cascades are essentially forward and local, although a non-local (in Fourier space) transfer of energy between the forced velocity scales and the small scale magnetic field has been observed [7]. This non-local transfer seems to be independent of the forcing mechanism since it has also been observed in previous studies [8] where different mechanical forces were used. No such non-local energy transfer is however observed in the decaying turbulence simulation [9].

The TURBO code has also been used to explore inhomogeneous turbulent MHD systems. Two examples are briefly discussed here. The first one is usually referred to as the Kolmogorov flow and is obtained by using an inhomogeneous unidirectional force $f_i = A\sin(ky)\delta_{i1}$. The properties of turbulence when considering more and more elongated computational boxes have been explored. It has been shown that, for the minimal computational cubic domain, the velocity statistics exhibits symmetries that are directly imposed by the forcing properties. However, for larger domains, the translational invariance in the streamwise direction appears to be broken and the turbulence statistics depends on the computational box aspect ratio (see Figure 1).

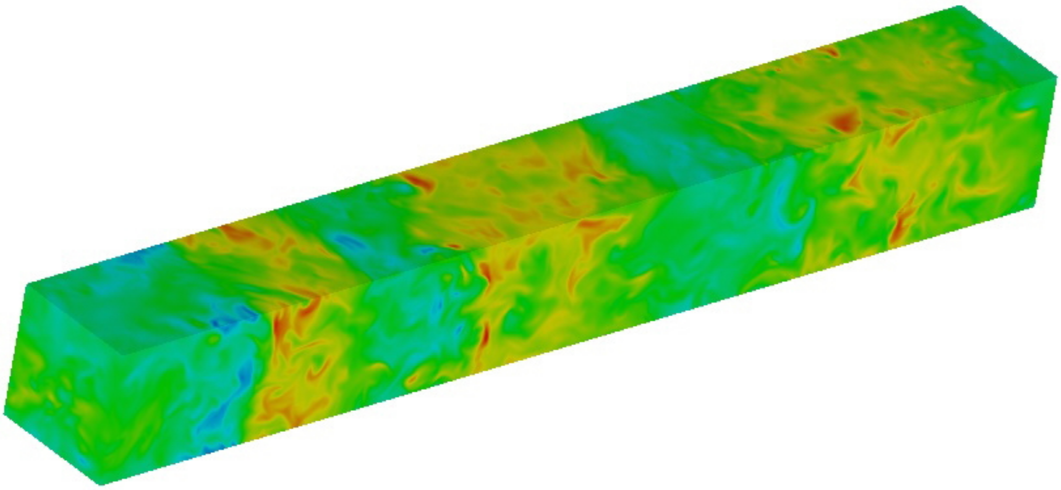


Figure 1: Turbulence intensities in the Kolmogorov flow. The largest size of the box corresponds to the x (streamwise) direction. This figure clearly shows that the translational invariance in the streamwise direction is broken.

Inhomogeneity can also be produced through the initial conditions. For instance, the evolution of a turbulent inhomogeneous system consisting of interacting layers with different turbulent intensities has been numerically computed. The simulation used $512 \times 1024 \times 512$ grid points (see Figure 2). This so-called shearless mixing layer computation allows the detailed investigation of the effect of inhomogeneities on the turbulence properties. In particular, the relation between the flux of energy and the gradient of the average turbulent intensity has been explored [10].

4 Simulation of liquid metal flows

The study of liquid metal flows in presence of intense external magnetic field is relevant for several problems faces in the tokamak blanket design. Liquid metals are usually characterized by a fairly high magnetic diffusivity or, equivalently, by a very low magnetic Prandtl number ($P_m = \nu/\eta$). Typically, the Prandtl number can be as low as 10^{-6}. As a consequence, the magnetic Reynolds number can also be very small, leading to the interesting limit $R_m \to 0$ in which the induction effects on the flow can be computed exactly. Indeed, keeping only the dominant terms in the equation for the magnetic fluctuations yields:

$$b_j^{ext} \partial_j u_i + \eta \Delta b_i \approx 0 \,. \tag{8}$$

where the flow is assumed to be submitted to an externally imposed constant magnetic field b_j^{ext}. This results in a closed expression for the Lorentz force which, assuming that b_j^{ext} is constant, can be written as follows:

$$f_i^{\text{Lorentz}} = -N \nabla^{-2} \partial_{\|}^2 u_i, \tag{9}$$

where N is the interaction parameter proportional to $|b^{ext}|^2$ while $\partial_{\|}$ represents the spatial derivative in the direction of the externally imposed magnetic field. The expression (9)

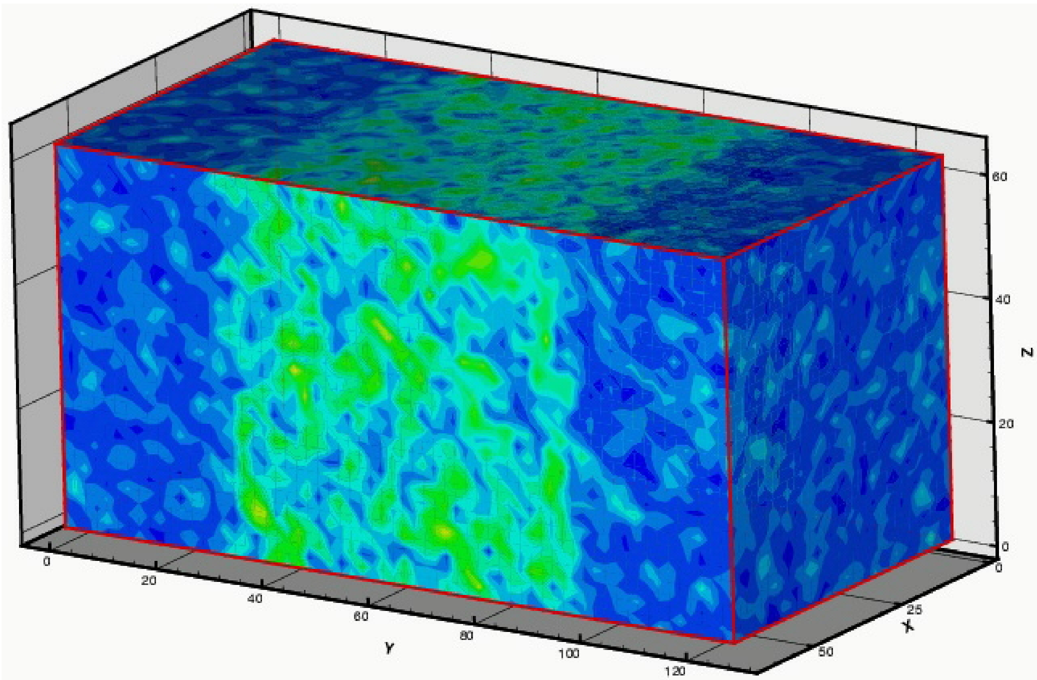
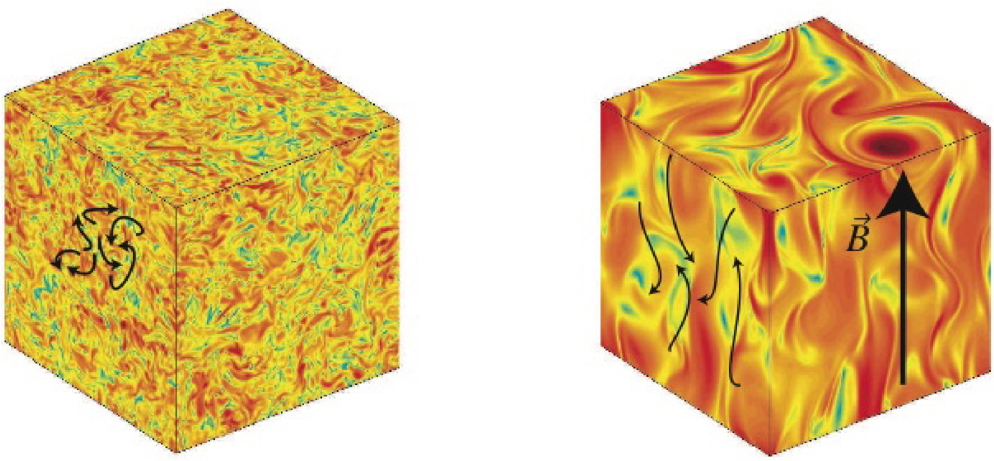


Figure 2: Numerical simulation of two interacting layers with different turbulent intensities. We only show the initial field. Dark regions correspond to low turbulence intensities.



(a) (b)

Figure 3: The left figure represents energy density contours for an initial isotropic turbulent velocity field in a conducting fluid. The right figure represents the same quantity for the anisotropic state predicted by the QSA. The magnetic field is directed in the vertical direction.

is usually referred to as the quasi-static approximation (QSA). Hence, the only effect of the external magnetic field appears through a linear term similar to the viscous term. However, this effect is not a simple additional dissipation process. Indeed, it only affects flows with a spatial dependence along the external magnetic field. Secondly, the presence of the ∇^{-2} operator makes this term to affect the large scales as well as the small scales. The most striking effects of the external magnetic field is then to damp turbulence through Joule dissipation and to collapse the structures of the flow into a quasi two dimensional state [11, 12].

This last property is illustrated in Figure (3) for a simulation using TURBO for decaying homogeneous MHD turbulence in the limit of the QSA. The anisotropic velocity fields produced in presence of an external magnetic field are also studied in details to improve our understanding of the physical phenomena responsible for the energy transfer between different scales as well as between velocity modes with different angles with respect to b_j^{ext}.

5 Simulation of particle trajectories

The velocity and magnetic fields produced by TURBO have also been used to investigate the dynamics of charged test particles submitted to realistic turbulent electromagnetic fields. Indeed, despite the fact that the MHD equations solved by TURBO do not explicitly contain the electric field $\vec{e}$, the induction equation must be compatible with Faraday's law: $\partial_t \vec{b} = -\nabla \times \vec{e}$. Hence, the electric field can be reconstructed and is given by:

$$\vec{e} = -\vec{u} \times \vec{b} + \eta \nabla \times \vec{b} \qquad (10)$$

It is thus possible to derive from TURBO both the electric and the magnetic fields and to follow the motion of a set of charged particles submitted to the corresponding Lorentz force. The advantage of this approach is that the turbulent structures in the electromagnetic fields are generated self-consistently using the MHD equations. An alternative approach, that has been often used in the past, consists in assuming a given spectrum for both $\vec{e}$ and $\vec{b}$ and in generating synthetic fields with random phases. This procedure however produces fairly different fields as shown in Figure (4).

By randomizing the Fourier phases of a pseudospectral simulation of isotropic MHD turbulence at $Re \approx 300$ and tracing collisionless test particles in both the exact-MHD and phase-randomized fields, it has been found that the phase correlations enhance the acceleration efficiency during the first stage of the acceleration process [13]. A comparison of trajectories is shown in Figure (5). The helical motion of the test particle around the local magnetic field is not easily observed when the total trajectories are presented, but can be seen when they are considered with closer look.

6 Future developments

A number of extensions to the TURBO code are presently under investigation. For instance, it would be fairly simple to include the effect of a global rotation $\vec{\Omega}$ on the flow by introducing the Coriolis force ($-2\ \vec{\Omega} \times \vec{u}$) in the right hand side of the Navier-Stokes equation. Such an extension could be interesting when considering problems related to geo-dynamo generation and astrophysical objects.

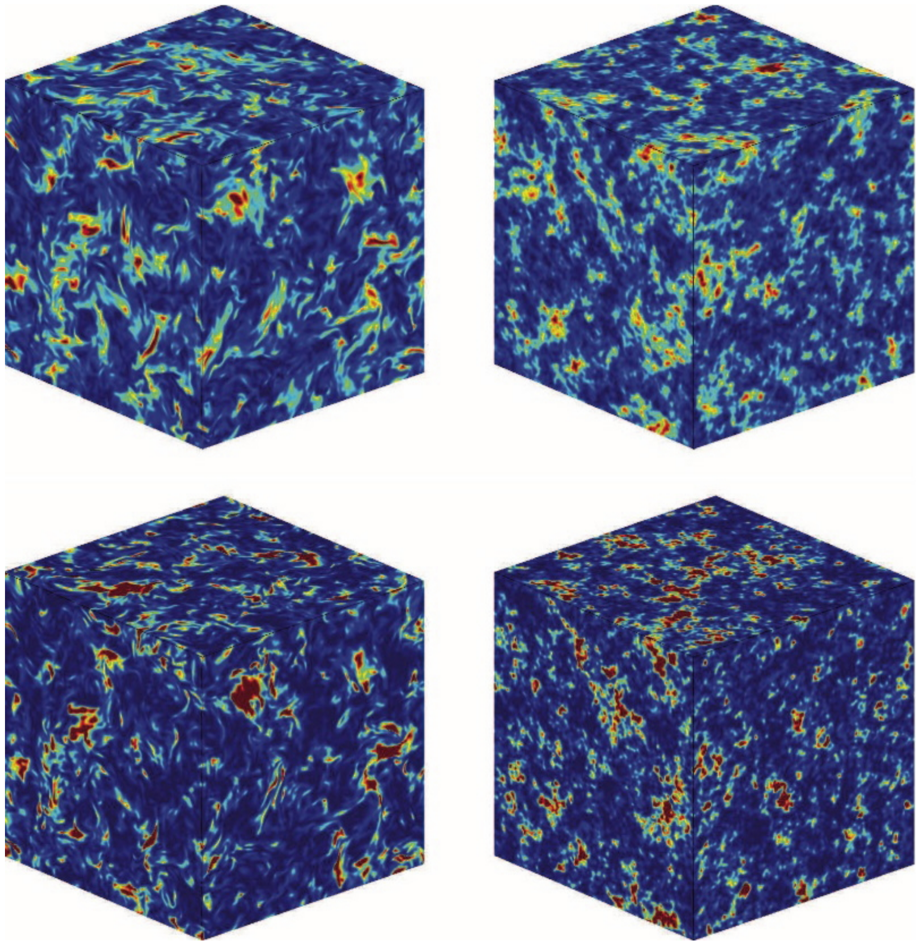


Figure 4: Magnetic (top) and electric (bottom) energy densities of the fields produced directly by TURBO (left) and fields produced by using the same spectra and random phases (right). Blue and red regions indicate, respectively, low and high values of these quantities.

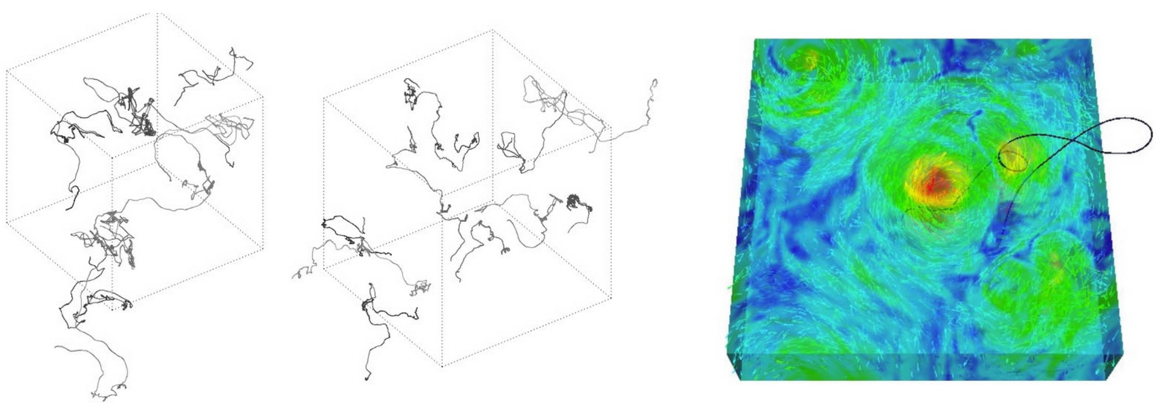


Figure 5: Real-space orbits in fields produced by TURBO (left) and fields with the same spectra but random phases (middle). Closer look at a test particle trajectory (right).

A much more ambitious project consists in extending the code to compressible fluids. Indeed, in that case, additional equation for the temperature and the density have to be included. Moreover, the pressure cannot be derived form the Poisson equation (4) and must be linked to the density and the temperature through a state equation. However, if fusion plasmas are considered, the ideal gas state equation can be used. Finally, we also mention the possibility to include radiation effect into the code. Again additional equation for the radiation energy and the momentum flux have to be solved (RMHD, Radiation Magnetohydrodynamics). Works in that direction are in progress.

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