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Radu Balescu and the search for a stochastic description of turbulent transport in plasmas

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Abstract

An idea that the late Prof. Radu Balescu often pondered during his long and distinguished scientific career was the possibility of constructing simple stochastic or probabilistic models able to capture the basic features of the complex dynamics of turbulent transport in magnetically confined plasmas. In particular, the application of the continuous-time random walk (CTRW) concept to this task was one of his favorites. In the last few years prior to his death, we also became interested in applying (variations of the standard) CTRW to these problems. In our case, it was the natural way to move beyond the simple paradigms based on sandpile constructs that we had been previously studying. This common interest fueled an intense electronic correspondence between Prof. Balescu and us that started in 2004 and was only interrupted by his unexpected death in June 2006. In this paper, we pay tribute to his memory by reviewing some of these exciting concepts that interested him so much and by sketching the problems and ideas that we discussed so frequently during these two years. Regretfully, he will no longer be here to help us solve them.

1 Introduction

Professor Emeritus Radu Balescu unexpectedly passed away on June 1, 2006 during a trip to Romania at the age of 73 years. He hardly needs any introduction. Born in Romania, he spent most of his scientific life in Belgium, at the Free University in Brussels. Prof. Balescu was known world-wide thanks to his many remarkable contributions to the fields of statistical mechanics and plasma physics, some of which bear his name like the famous *Lenard-Balescu collision operator* [1]. In recognition to his long scientific career, he became the first recipient of the Hannes Alfvèn Prize awarded by the Plasma Physics Division of the European Physical Society in 2000.

At the moment of his untimely death he was actively pursuing several ideas that had interested him for many years. One of them, which is specially close to us as well, was an old favorite of his: "*the possibility of a model of anomalous transport problems in a turbulent plasma by means of a purely stochastic process*" [2]. The simplest example of the application of an stochastic model to bypass the complexity of the microscopic reality is the use of a Langevin equation to describe the motion of a single Brownian particle [3]. In this case, the complex background dynamics are substituted by an *"effective" stochastic force* with prescribed statistical properties. An alternative to prescribing an stochastic force is to prescribe instead the motion of the particle in a probabilistic fashion. This is the idea behind the **continuous-time random walk** (CTRW) concept, introduced by Elliott W. Montroll in the 60s [4]. In 1995, Prof. Balescu was one of the first plasma physicists to propose a construct based on the CTRW idea as a model for the sub-diffusive transport of guiding centers in regions with stochastic magnetic fields [2]. His views on this approach are described at length in the last book he published in 2005 [5]. It is an extremely clear piece of work that deserves a very careful reading and that beautifully illustrates his exploration of CTRWs as a possible alternative to more standard transport frameworks.

During the last decade, we also had explored similar alternative avenues in an attempt to understand and capture the essential dynamics of turbulent transport in plasmas. This work, which we did along the years together with many other collaborators, successively explored the use of cellular automata [6, 7, 8], fractional differential equations [9, 10, 11] and, of course, CTRWs [12, 13]. As a result of this confluence of interests, it was only natural that Prof. Balescu and us initiated an intense electronic correspondence in 2004 regarding the theory of inhomogeneous/nonlinear CTRWs and its application to problems relevant to magnetically confined plasmas. He was even kind enough to devote the last chapter of his new book to discuss some of our research on the topic [5]. As a result of his interactions with us, he became convinced of the intimate relationship that exists between CTRWs and fractional differential equations (FDE), since the latter provide the natural fluid limit of most CTRWs [14, 15, 16]. From that moment on, he became extremely interested in finding out if these FDE models could be endowed with some physical basis by formally connecting them to (some simplified form of) the microscopic dynamics. Specially, for the case in which the CTRW/FDE models exhibit superdiffusive behavior which, in his own words, *".. remains a challenging open conundrum."* [5]. As a result, he started to investigate enthusiastically this connection.

To attack this difficult problem, both him and us agreed to start from a simplified approach. In his own words again: *"A tractable starting point is provided by a "semi-dynamical approach", based on a V-Langevin equation: an equation of motion of Newtonian (or Hamiltonian) type for a tracer particle moving in presence of a random potential."* [17]. That is, the starting point was to be the continuity equation for tracers advected by an incompressible turbulent flow with prescribed statistical properties:

$$\frac{\partial n}{\partial t} + \mathbf{V} \cdot \nabla n = 0. \tag{1}$$

Traditionally, a quasi-linear renormalization procedure is used to derive the standard diffusive equation, with effective diffusivity $D \sim V_c^2 \tau_D$ (V_c is the characteristic fluctuating velocity and τ_D the velocity decorrelation time), from Eq. 1 [18]. The basic assumptions made in the process are the *locality* and the *lack of memory* of the flow. Since FDEs are precisely designed to keep both non-local and memory effects, both him and us started to try, in parallel but pursuing different approaches, different strategies that attempted to avoid making these assumptions. In this way, we expected that FDEs could be derived as well. In our case, we completed successfully the generalization by applying functional integration techniques to the fluctuating particle trajectories (assumed self-similar). Under appropriate assumptions, this path allowed us to connect Eq. 1 to the FDE models [19]. Prof. Balescu, on the other hand, followed his own approach. He explored the possibility of introducing a non-local extension of an approximation similar to the Corrsin factorization assumption of standard turbulent theory [17]. In his last communication with us, he told us that he had finally succeeded in connecting Eq. 1 with the FDE models in this manner. He was going to present his new findings in a workshop on anomalous transport, in which some of us would also participate, to be held in Bad Honnef, Germany, in July 2006. Regretfully, his unexpected death impeded it and we will probably never know now what his solution to this problem was.

In this paper, we pay tribute to his memory by reviewing the exciting world of CTRWs and FDEs. These models have found application in numerous problems beyond the magnetic plasma confinement situations that Prof. Balescu was interested in, including many problems in the earth and physical sciences [15, 16]. We will start by reviewing the fundamentals of CTRWs in Sec. 2, using as an illustration the beautiful application that Prof. Balescu proposed to the problem of subdiffusion in stochastic magnetic fields [2]. Then, in Sec. 3, we will discuss the basic ingredient needed to construct CTRW models: Lévy distributions. In Sec. 4, we will briefly introduce fractional differential operators and show that CTRWs are connected to evolution equations that contain fractional differencial operators in the limit of long distances and times (the so-called "*fluid limit*"). Finally, in Sec. 5, we will discuss the fundamental aspects of the renormalization problem that connects Eq. 1 to FDEs and give some hints about how it can be solved. For further study in any of these topics, we enthusiastically recommend the readers to explore Prof. Balescu's last book [5], the bibliography given at the end of this paper and all references within them.

2 Continuous-time random walks

The CTRW is a generalization of the standard random walk [4]. In its simplest (separable) form, it describes the motion of an arbitrary number of particles (or walkers), each of which waits at its current position $\mathbf{r}'$ for a lapse of time Δt (a **waiting-time**) before taking a step of size $\Delta\mathbf{r}$ (a **step-size**) and moving to $\mathbf{r} = \mathbf{r}' + \Delta\mathbf{r}$. After arriving at the updated location, a new waiting-time is chosen and the process is repeated over and over. Assuming that the system is invariant under time and space translations, Δt and $\Delta\mathbf{r}$ are drawn by each walker from two prescribed probability density functions (pdf), $p(\Delta\mathbf{r})$ and $\psi(\Delta t)$ which contain all the dynamical information of the system.

Therefore, to apply the CTRW construct to any problem, we simply need to choose these two pdfs in a manner that captures the fundamental microscopic physics of the problem. That, of course, is the difficult part. We will discuss at length in Sec. 3 which pdf choices, among the infinite number available, appear to make the most physical sense. But for the time being, it is sufficient to realize that, once the pdfs are known, the time evolution of the density of walkers $n(\mathbf{r}, t)$can be described by the following generalized master equation (GME):

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = \int_0^t dt'\phi(t-t')\left[\int d\mathbf{r}' p(\mathbf{r}-\mathbf{r}')n(\mathbf{r}',t') - n(\mathbf{r},t')\right], \tag{2}$$

that simply states that the total number of particles is conserved [20]. Indeed, the first (positive) term within brackets counts how many walkers move from $\mathbf{r}'$ to $\mathbf{r}$ by performing a jump of appropriate length. The second (negative) term in brackets, counts how many walkers leave the position $\mathbf{r}$. Their sum gives the local rate of change in the number of walkers (or more precisely, their density). Two additional facts must be kept in mind regarding the GME:

1. The function ϕ is usually called the **memory function**. Its Laplace transform is related to the waiting-time pdf through the relationship: $\phi(s) = s\psi(s)/(1-\psi(s))$. Note that, were the triggerings of successive jumps uncorrelated (i.e., if there is no memory in the process), the generation of the waiting-times must then be a **Poisson process**, which requires that $\psi(\Delta t) = \tau_o^{-1}\exp(-\Delta t/\tau_0)$, being τ_0 the mean waiting time. Then, use of the previous relation shows that $\phi(t) = \tau_0^{-1}\delta(t)$. The GME reduces then to a more standard Markovian master equation:

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = \frac{1}{\tau_o}\left[\int d\mathbf{r}' p(\mathbf{r}-\mathbf{r}')n(\mathbf{r}',t') - n(\mathbf{r},t)\right]. \tag{3}$$

   Clearly, the past history of the system no longer plays any role once the time integral has disappeared.

2. The second choice to make refers to the pdf of step-sizes, p. The most usual choice, in the absence of motion bias, is the **Gaussian law**: $p(\mathbf{\Delta r}) = (2\pi\sigma)^{-n/2}\exp(-|\mathbf{\Delta r}|^2/2\sigma^2)$. This choice, together with the exponential waiting-time pdf, allows to connect the CTRW with a very familiar equation: the **classical diffusive equation**. Indeed, it is only needed to rewrite Eq. 3 as:

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = \frac{1}{\tau_0}\left[\int d\mathbf{\Delta r}' p(\mathbf{\Delta r})n(\mathbf{r}-\mathbf{\Delta r},t') - n(\mathbf{r},t)\right], \tag{4}$$

   carry out a simple Taylor expansion in around $\mathbf{r}$, use the form of $p(\mathbf{\Delta r})$ and keep only the lowest order to obtain:

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = D\nabla^2 n(\mathbf{r},t), \qquad D = \sigma^2/\tau_0. \tag{5}$$

   The diffusivity, as expected, is then obtained as the quotient between the average squared displacement, σ^2, and the average waiting-time, τ_0. σ gives then the magnitude of the mean step-size.

The power of the CTRW formalism lies in the fact that $p(\mathbf{\Delta r})$ and $\psi(\Delta t)$ need not be respectively Gaussian and exponential, becoming thus suitable to model systems beyond those with local, memory-less dynamics. Other pdfs can be chosen, which extends the range of applicability of CTRW models to include non-local and non-Markovian situations. These situations often occur in practice and reveal themselves, for instance, when measuring characteristic transport exponents of passive quantities or tracers by some flow. A well known fact about the classical diffusive equation is that it implies the following scaling for the mean tracer square displacement:

$$\left\langle|\Delta\mathbf{r}|^2\right\rangle \sim Dt. \tag{6}$$

In many practical cases, it is however found to scale as t^{2H}, with $H \neq 1/2$ [15, 16]. When $H < 1/2$ one typically speaks of **subdiffusion**. When $1/2 < H < 1$, of **superdiffusion**.

One example can be found in Prof. Balescu's own work: the motion of guiding centers in the presence of magnetic field fluctuations and of collisional diffusion in the parallel (to the average magnetic field) direction. The perpendicular (to the average magnetic field) mean squared displacement is found to scale, for the simple model he examined, as [2]:

$$\left\langle |\Delta r|^2 \right\rangle \simeq 4\ (\delta B)^2 \lambda_{\|}\ \chi_{\|}^{1/2}\ t^{1/2}, \qquad t >> 1, \tag{7}$$

where (δB) measures the relative amplitude of the magnetic fluctuations, $\lambda_{\|}$ is the parallel correlation length of the magnetic fluctuations and $\chi_{\|}$ is parallel collisional diffusion coefficient. Note that the transport exponent obtained from Eq. 7 is $H = 1/4$, which is clearly subdiffusive.

Prof. Balescu showed that the CTRW approach can provide an stochastic model for this problem. Indeed, it is sufficient to realize that any CTRW constructed with a Gaussian step-size pdf and a long-tailed waiting-time pdf that decays as (more details about the meaning of this distribution are given in the next section):

$$\psi(\Delta t) = \frac{\beta \tau_0^\beta}{\Gamma(1+\beta)} (\Delta t)^{-(1+\beta)} + \cdots, \qquad \beta \in (0,1),\ \Delta t \to \infty \tag{8}$$

exhibits a squared mean displacement scaling given by:

$$\left\langle |\Delta \mathbf{r}|^2 \right\rangle \sim \Gamma(1+\beta) \left(\frac{\sigma^2}{\tau_0^\beta} \right)\ t^\beta, \tag{9}$$

where $\Gamma(x)$ is Euler's Gamma function. Clearly, comparing this expression with Eq. 7, we can easily construct a one-dimensional CTRW model for the magnetic subdiffusion of guiding centers by choosing:

$$\beta = 1/2, \qquad \tau_0 = \lambda_{\|}^2 / 2\chi_{\|}, \quad \sigma^2 = \sqrt{2\pi}\beta^2 \lambda_{\|}^2. \tag{10}$$

These choices may seem rather arbitrary, but they can be justified heuristically by physical arguments [2, 5]. Note also that the choice for the characteristic perpendicular jump length σ gives dependencies on both the magnetic fluctuation amplitude and the parallel correlation length which are consistent with the famous Rechester-Rosenbluth formula for this process [21]. Use of these parameters in Eq. 2 provide us with a working CTRW model to study the magnetic subdiffusion of guiding centers.

The simple example just described illustrates on of the ways in which CTRW models can be used to retain 'physical memory effects, in any description of transport (we will also discuss a different possibility later). The choice of the non-exponential waiting-time pdf yields a non-trivial memory function in Eq. 2. In addition, non-local (spatial) effects can also be retained by choosing a non-Gaussian step-size pdf. This versatility of CTRWs was precisely what got us interested in them at the end of 2003. At the time, we were exploring different possibilities of going beyond the simple sandpile models that had been proposed in the mid-90s as the simplest paradigm of turbulent transport within a tokamak plasma [6, 7]. These sandpile models were very simple: they consisted of a chain of individual cells that were driven by intermittently adding sand grains from above. When the sand slope between any two successive cells overcame a prescribed threshold value (i.e., a critical sand slope), some amount of sand was removed from the unstable cell and transported to its next neighbor down the slope. From a plasma perspective, the chain of cells of the sandpile can be seen as a proxy for the tokamak radial direction, with each cell being say, a turbulent eddy attached to a different rational surface [23, 24]. The sandpile critical slope plays the role of the local threshold for instability that, if overcome, will drive a local turbulent flux (i.e., the amount of sand transported between successive cells) in order to bring the plasma profiles back to subcritical values.

The sandpile was, in spite of its sketchy character, extremely useful to identify some elements that should also play a role in plasma dynamics. In particular, it helped to make clear that transport can exhibit both non-local and memory effects when the system profiles are kept very close to its local threshold for instability via external forcing [22, 23]. Indeed, this situation makes possible that a chain of successive relaxations could take place, usually known as **avalanches**, whose maximum size is only limited by the number of cells in the system. Their existence implies that, if we are interested in characterizing transport only over time scales which are much longer than the time required for avalanches to traverse the system, transport through the system would behave essentially non-locally. On the other hand, information about the triggering of previous avalanches is stored in the system through the continuous carving of the profiles carried out by them, which affects the triggering of future avalanches and provides the system its "memory".

The relevance of this type of dynamics in the case of magnetically confined plasmas was soon confirmed by actual plasma turbulence simulations in which the system profiles were forced to evolve in the neighbourhood of their local instability thresholds [24, 25, 26, 27]. Some experimental evidence also was reported suggesting that this might be the dominant situation in some relevant tokamak regimes [28, 29, 30, 31, 32]. But the main lesson learned from these studies was that it might be necessary to go beyond descriptions of transport based on the classical diffusive equation and embrace instead models capable of including both non-locality and memory effects. That was our initial drive to investigate both CTRWs and FDEs.

As a proof of principle, and very much in the spirit of Prof. Balescu's previous work, we showed that a (nonlinear) CTRW can be constructed that captures much of the spirit of the sandpile and that exhibits phenomenology qualitatively similar to what is typically observed in tokamak experiments [12, 5]. The key is to construct a Markovian CTRW with an exponential waiting-time pdf but with a one-dimensional inhomogeneous step-size pdf like,

$$p(\Delta x; x', t) = \xi(x', t)\ p_1(\Delta x) + (1 - \xi(x', t))\ p_2(\Delta x), \tag{11}$$

where $\xi(x', t))$ is a suitable projector that contains the critical threshold condition (say, a critical gradient) and that allows to switch locally from a sub-critical to a super-critical transport channel. It may seem puzzling at first to note that we chose a Markovian CTRW in spite of the fact that the sandpile that it tries to model has a "memory" which is stored in its profiles. But memory effects are still captured within this CTRW through the threshold condition which is contained within the projector $\xi(x, t)$. This alternative scheme is thus quite different to that used in Prof. Balescu's CTRW, but closer to the physics of the system. It is another illustration of the breadth of possibilities that the CTRW framework offers. Finally, one must choose p_1 and p_2, the step-size pdfs respectively associated to sub-critical and super-critical transport, so that they capture the basic features of the microscopic physics of each channel. Since we expect that the system behaves normally when no instability is excited, we chose a Gaussian pdf for the sub-critical channel, so that subcritical transport is diffusive. But, to capture the non-local features characteristic of the sandpile avalanches, we made a different choice for p_2:

a **symmetric Lévy pdf**.

3 Lévy distributions

When implementing a CTRW model, one always has to make two choices: a waiting-time pdf and a step-size pdf. Since these choices must somehow reflect the cumulative effect of many microscopic processes (such as, for instance, molecular collisions), it seems physically well justified to use pdf forms that satisfy the *central limit theorem*. That is, limit distributions that are strictly stable with respect to the *sum of N independent and identically distributed (i.i.d.) random variables* [33]. These pdfs are known as the Lévy (or Lévy-Gnedenko) family of pdfs. They contain as a special case the Gaussian distribution, which becomes the only stable distribution if each of the random variables is also required to have a finite variance [33].

The Lévy family is defined in terms of three parameters. Its members are denoted by $L_{\alpha,\lambda,\sigma}(y)$. They can be defined in closed form in terms of their Fourier transform or characteristic function as ($0 < \alpha \leq 2$, $|\lambda| \leq 1$) [34]:

$$L_{\alpha,\lambda,\sigma}(k) = \exp\left[-\sigma^\alpha |k|^\alpha \left(1 - i\lambda \ \ \mathrm{sgn}(k) \ \tan\left(\frac{\pi\alpha}{2}\right)\right)\right]. \tag{12}$$

The three labels $[\alpha, \lambda, \sigma]$ define the properties of each distribution.

- First, λ measures the **asymmetry** of the distribution. This comes from the fact that:
$$L_{\alpha,\lambda,\sigma}(y) = L_{\alpha,-\lambda,\sigma}(-y). \tag{13}$$
It can vary within $-1 \leq \lambda \leq 1$ except when $\alpha = 1, 2$, for which only $\lambda = 0$ is possible.

- Secondly, α gives the **asymptotic behavior** of the distribution at large y. All Lévy distributions exhibit heavy tails if $0 < \alpha < 2$. In fact, for $\alpha \neq 1$, it holds that:

$$L_{\alpha,\lambda,\sigma}(y) \sim \begin{cases} C_\alpha \left(\frac{1-\lambda}{2}\right) \sigma^\alpha \, |y|^{-(1+\alpha)} \, , y \to -\infty \\ C_\alpha \left(\frac{1+\lambda}{2}\right) \sigma^\alpha \, |y|^{-(1+\alpha)} \, , y \to +\infty \end{cases} \tag{14}$$

where the constant is given by:

$$C_\alpha = \frac{(\alpha-1)\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, \tag{15}$$

In the special case $\alpha = 1$, the PDF decays as $L_{1,0,\sigma}(y) \sim (\sigma/\pi)|y|^{-2}$. And when $\alpha = 2$, one recovers the standard Gaussian distribution.

- Finally σ is called a *scale parameter* because:

$$L_{\alpha,\lambda,\sigma}(ay) = P_{\alpha, \ \mathrm{sgn}(a)\lambda, |a|\sigma}(y) \tag{16}$$

Extremal Lévy distributions

Waiting-time distributions must satisfy an additional constraint: *they must be defined only for positive waiting-times*! Luckily, this kind of distribution also exists within the Lévy family. They are a subset of the **extremal** Lévy distributions, which are those with maximum skewness value: $\lambda = \pm 1$ for $\alpha \neq 1, 2$. In this case, according to the previous equations, the power-law decay is only observed in one tail, the other decaying instead exponentially. In the case of $1 < \alpha < 2$, $\lambda = +1$ implies that the exponential tail exists for $y \to -\infty$, while $\lambda = -1$ has a right exponential tail for $y \to \infty$. For $0 < \alpha < 1$ the extremal distributions are *one-sided* [35]: they are defined only for $y > 0$ if $\lambda = 1$ and for $y < 0$ if $\lambda = -1$. In that case, the exponential tail is found in the limit $y \to 0+$ for $\lambda = 1$, and for $y \to 0-$ for $\lambda = -1$.

An important property is that their Laplace transform is given by:

$$L_{\alpha,1,\sigma}(s) = \exp\left[-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}s^\alpha\right]. \tag{17}$$

As an illustration, note that the waiting-time pdf choice used by Prof. Balescu's to construct his CTRW model for stochastic magnetic subdiffusion that we discussed previously would be the extremal Lévy distribution $\psi(\Delta t) = L_{1/2,1,\tau_0}(\Delta t)$.

Moments of Lévy distributions

The reason why Lévy distributions with $\alpha < 2$ are appropriate choices for step-size pdfs if we are interested in constructing a CTRW model with non-local features is because of the following property: all moments higher than α are infinite. That is, the momenta of $L_{\alpha,\lambda,\sigma}(y)$ verify:

$$< |x|^p >= \begin{cases} \infty, & p \geq \alpha \\ [c_{\alpha,\lambda}(p)]^p \sigma^p, & p < \alpha \end{cases} \tag{18}$$

where the coefficient is not relevant for our discussion (it can be found in Ref. [34]). Thus, only the Gaussian distribution ($\alpha = 2$) has a finite variance. As a result, the characteristic transport length provided by σ in the case of a Gaussian ceases to exist for $\alpha < 2$. Transport is, in this sense, non-local and scale-free.

Explicit expressions of Lévy distributions

There are only three Lévy distributions for which an analytical expression exists [34]. The *Cauchy distribution*. Its real space representation is:

$$L_{1,0,\sigma}(y) = \frac{\sigma}{\pi(y^2 + \sigma^2)}, \tag{19}$$

the *Gauss distribution*,

$$L_{2,0,\sigma}(y) = \frac{1}{2\sigma\sqrt{\pi}}\, e^{-y^2/4\sigma^2}, \tag{20}$$

(note that the relation of σ with the usual width w of the Gaussian is thus $2\sigma^2 = w^2$) and the *Lévy distribution*,

$$L_{1/2,1,\sigma}(y) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{y^{3/2}}\, e^{-\sigma/2y}. \tag{21}$$

To conclude, we will give some hints on how to use the Lévy pdfs to choose the pdfs to construct a CTRW model with certain desired transport properties. As we mentioned before – and illustrated with Prof. Balescu's example–, CTRWs are useful to model transport in systems where subdiffusion or superdiffusion occurs [15, 16]. The way to do it is to remember that, if we choose the symmetric Lévy pdf $L_{\alpha,0,\sigma}(\Delta x)$ as step-size pdf, and the extremal Lévy pdf $L_{\beta,1,\tau}(\Delta t)$ as waiting-time pdf, the mean particle displacement follows the scaling:

$$\langle|\Delta x|^\alpha\rangle \propto \Gamma(1+\beta)\left(\frac{\sigma^\alpha}{\tau_0^\beta}\right)t^\beta, \qquad t>>1. \tag{22}$$

Note that this relation implies that the transport exponent is $H = \beta/\alpha$. Thus, subdiffusion ensues whenever $\beta/\alpha < 1/2$ and superdiffusion when $\beta/\alpha > 1/2$. The correct ratio between exponents is thus set by the observed transport exponent H. Next, appropriate considerations about locality and Markovianity can be used to determine their precise values. For instance, in Prof. Balescu's example about subdiffusion of guiding-centers in magnetic turbulence, $\beta/\alpha = 1/4$. He includes memory effects but there is no non-locality in the model. Thus, the correct choices are $\alpha = 2$ and $\beta = 1/2$. The scale factors are then chosen to match the observed pre-factor of the mean displacement scaling [2].

4 Fluid limit of CTRWs: fractional differential equations

To conclude our review of the fundamentals of CTRWs we will prove in what follows that their "fluid limit" is rewritten in terms of fractional differential operators [14]. But before doing that, we will briefly introduce what these operators are.

4.1 Crash course on fractional differential operators

The *Riemann-Liouville fractional derivative operators* can be defined explicitly by means of the integral operators [36]:

$$\begin{aligned} {}_aD_x^\alpha\ f(x) &\equiv \frac{1}{\Gamma(p-\alpha)}\frac{d^p}{dx^p}\left[\int_a^x \frac{f(x')dx'}{(x-x')^{\alpha-p+1}}\right], \\ {}^bD_x^\alpha\ f(x) &\equiv \frac{-1}{\Gamma(p-\alpha)}\frac{d^p}{d(-x)^p}\left[\int_x^b \frac{f(x')dx'}{(x'-x)^{\alpha-p+1}}\right]. \end{aligned} \tag{23}$$

In this expressions, $\Gamma(x)$ is the usual Euler Gamma function, and p represents one plus the integer part of α. a [or b] is called the start [end] point of the operator. In the cases in which the start point a or the end point b extend all the way to infinity, we will use the notation:

$$\frac{d^\alpha f}{dx^\alpha} \equiv {}_{-\infty}D_x^\alpha\ f(x) \qquad \frac{d^\alpha f}{d(-x)^\alpha} \equiv {}^{+\infty}D_x^\alpha\ f(x) \tag{24}$$

These operators have very interesting properties. For integer α they reduce to the standard derivatives. Like them, they are linear. But it is not true that the fractional derivative of a constant is zero. Also, they must be combined appropriately with integer and non-integer derivatives and they do not satisfy the simple chain rule [36]. Their non-local character comes from the fact that, to compute the value of the fractional derivative of some quantity at a given point, one has to integrate that quantity over the whole domain! So why bother with them at all if they are so complicated? The reason is that, under Fourier transformations, they satisfy that:

$$F\left[\frac{d^\alpha f}{dx^\alpha}\right] = (-ik)^\alpha f(k), \qquad F\left[\frac{d^\alpha f}{d(-x)^\alpha}\right] = (ik)^\alpha f(k). \tag{25}$$

This property is the key to their prominence in CTRW theory, as we will show shortly.

Another useful fractional operator is the so-called *Riesz fractional derivative operator* [36], which defined as the symmetrization:

$$\frac{d^\alpha}{d|x|^\alpha} \equiv -\frac{1}{2\cos\left(\pi\alpha/2\right)}\left[\frac{d^\alpha}{dx^\alpha} + \frac{d^\alpha}{d(-x)^\alpha}\right]. \tag{26}$$

Its usefulness comes from the fact that the Riesz operator verifies, under Fourier transform, that:

$$F\left[\frac{d^\alpha f}{d|x|^\alpha}\right] = -|k|^\alpha f(k). \tag{27}$$

The last fractional operator we will introduce is the *Caputo fractional derivative operator*, which is defined as [36]:

$$\frac{d_c^\beta f}{d_c t^\beta}(x) \equiv \frac{1}{\Gamma(\beta-p)}\int_0^t \frac{d^p f}{dt^p}(\tau)\frac{d\tau}{(t-\tau)^{\beta+1-p}}, \tag{28}$$

where p is one plus the integer part of β. The Caputo fractional derivative is usually associated to derivatives in time. Its non-Markovian character is also clear: to calculate the Caputo time derivative of any quantity, one has to integrate that quantity over all its past history! Its importance comes from the fact that the Laplace transform of the Caputo derivative verifies [36]:

$$\mathsf{L}\left[\frac{d_c^\beta f}{d_c t^\beta}(t)\right] = s^\beta f(s) - \sum_{k=0}^{p-1} s^{\beta-k-1}\cdot\frac{d^k f}{dt^k}(0), \tag{29}$$

which depends only on the initial values of $f(t)$ and its integer derivatives.

4.2 Finding the fluid limit of CTRWs

We are now in good shape to calculate the "fluid limit" of any CTRW [14]. But for simplicity, we will restrict the calculation to the case in which we choose the symmetric Lévy pdf $L_{\alpha,0,\sigma}(\Delta x)$ as step-size pdf, and the extremal Lévy pdf $L_{\beta,1,\tau}(\Delta t)$ as waiting-time pdf. The calculation is very simple. By "fluid limit" one should understand an equation that captures the characteristic features of the CTRW transport in the limit of very long distances and very long times. Formally, we do this calculation in the limit of an infinite system. Then, the limit of long distances is equivalent to making $k \to 0$ in Fourier space. Similarly, the limit of long times can be carried out in Laplace space by making $s \to 0$. Thus, we take the Fourier-Laplace transform of the GME (Eq. 3):

$$sn(k,s) - n(k,0) = \phi(s)(p(k)-1)n(k,s), \tag{30}$$

where we have applied the convolution theorem and the definition of the Laplace transform of a derivative. This equation can in fact be solved to give the Fourier-Laplace transform of the density of walkers:

$$n(k,s) = \frac{n(k,0)}{s - \phi(s)(p(k)-1)} = \frac{n_0(k)(1-\psi(s))}{s(1-\psi(s)p(k))}, \tag{31}$$

where we have rewritten the memory function in terms of the Laplace transform of the waiting-time pdf. $n_0(k)$ is the prescribed (Fourier transform of the) initial density of walkers. Eq. 31 is known as the *Montroll-Weiss equation* [4].

We can now take the fluid limit by taking $k \to 0$ and $s \to 0$ in either the Montroll-Weiss equation or in Eq. 30. To do it, we simply assume that both p and ψ are chosen from within the Lévy family, as the central limit theorem advices. Then, it is trivial to realize using the properties we discussed before that for small k:

$$p(k) = L_{\alpha,0,\sigma}(k) \simeq 1 - \sigma^\alpha |k|^\alpha. \tag{32}$$

Similarly, the Laplace transform of positive extremal Lévy pdfs, given by Eq. 17, behaves at small s as:

$$\psi(s) = L_{\beta,1,\tau_0} \simeq 1 - A_\beta^{-1}\tau_0^\beta\ s^\beta. \tag{33}$$

where we have also included the exponential law if $\beta = 1$ and defined the constant:

$$A_\beta = \begin{cases} \cos\left(\frac{\pi\beta}{2}\right), & \beta < 1 \\ 1, & \beta = 1 \end{cases} \tag{34}$$

Inserting now Eqs. 33 and 32 in Eq. 30, the fluid limit of the Montroll-Weiss equation becomes:

$$n(s,k) \simeq n_0(k)\left[s + D_{[\alpha,\beta]} s^{1-\beta}|k|^\alpha\right]^{-1}. \tag{35}$$

where the coefficient $D_{[\alpha,\beta]} = A_\beta \sigma^\alpha / \tau^\beta$ has been defined. Eq. 35 can be rewritten as:

$$sn(s,k) - n_0(k) = -D_{[\alpha,\beta]} s^{1-\beta}|k|^\alpha n(s,k). \tag{36}$$

We can now use the properties of the fractional operators with respect to the Fourier or Laplace transforms. For instance, using Eq. 27 we can Fourier-invert Eq. 36. The result is thus an FDE in space:

$$sn(s,x) - n_0(x) = D_{[\alpha,\beta]} s^{1-\beta} \frac{\partial^\alpha n}{\partial |x|^\alpha}. \tag{37}$$

Analogously, we carry out the Laplace inversion of Eq. 37 next. We do it by multiplying first both sides by $s^{\beta-1}$ and then using the properties of the Caputo fractional differential operator (Eq. 28) with respect to the Laplace transform (Eq. 29) to write the following FDE in space and time:

$$\frac{\partial_c^\beta n}{\partial t_c^\beta} = D_{[\alpha,\beta]} \frac{\partial^\alpha n}{\partial |x|^\alpha}. \tag{38}$$

Of course, if $\alpha = 2$ and $\beta = 1$, one recovers the standard diffusive equation.

One thing must be made clear at this point. Using a FDE equation like Eq. 38 with exponents β and α and appropriate effective fractional diffusivity $D_{[\alpha,\beta]}$ is almost equivalent (at least if we only care about long distances and times) to using a CTRW with $p = L_{\alpha,0,\sigma}(\Delta x)$ and $\psi = L_{\beta,1,\tau_0}(\Delta t)$. Then, what are the advantages of FDEs? Mainly, that many properties of FDEs are known analytically. For instance, their propagators [15]. From a numerical point of view, both CTRW and FDEs can be implemented numerically quite easily [36]. Each has certain advantages and disadvantages. Probably, CTRWs are easier to implement when one needs to use absorbing boundary conditions. On the other hand, FDEs are probably easier when we want to apply Neumann or Dirichlet boundary conditions [37], but one has to regularize first the fractional operators close to the limits of the integrals to avoid divergences [38].

5 Towards a "fractional" renormalization

We hope that the attentive reader will be already convinced of the usefulness of CTRWs and FDEs to model transport in systems with dynamics that exhibit non-local and/or memory effects. But what is the physical basis of these models? Can they be derived from some reasonable form of the microscopic dynamics? This was probably the problem that excited Prof. Balescu's interest the most during the last two years of his life. How can one construct a formal procedure to derive equations like Eq. 38 from a simplified form of the microscopic dynamics of the particles being transported? As the quest for answers to these questions advanced, we exchanged numerous e-mails with him in which we discussed our respective (slow) progress. We will use this last section to explain the initial point of our investigations and sketch the two possible solutions: Prof. Balescu's and ours. But we will not discuss them in detail (that will require a pretty long article in itself), pointing instead the interested reader instead to some references that contain our own solution to it.

The idea is to deal with particles which are advected by a background turbulent flow and to assume that the flow characteristics are not affected by the tracer motion. That is, the particles behave like tracers. The equation of motion of each tracer is then given by:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}(\mathbf{r}, t), \tag{39}$$

where $\mathbf{V}(\mathbf{r}, t)$ is the incompressible n_D-dimensional turbulent flow with some prescribed statistical properties. The associated 'kinetic' equation is the continuity equation for the the density of tracers [5],

$$\frac{\partial n}{\partial t} + \mathbf{V} \cdot \nabla n = 0. \tag{40}$$

What we look for now is any transformation procedure that converts Eq. 40 into a linear transport equation of the class defined by Eq. 38. This procedure will be referred to as a **renormalization**. In the case in which the final result is the classical diffusion equation, the simplest (and oldest) renormalization known is that based on quasi-linear theory (QLT) [18]. It starts by separating the ensemble-averaged $\langle\ \rangle$ and the fluctuating $(\tilde{\ })$ parts of Eq. 40 (note that we use the notation $\langle A \rangle \equiv A_0$ all throughout this section),

$$\frac{\partial n_0}{\partial t} + \mathbf{V}_0 \cdot \nabla n_0 = -\left\langle \tilde{\mathbf{V}} \cdot \nabla \tilde{n} \right\rangle \frac{\partial \tilde{n}}{\partial t} + \mathbf{V}_0 \cdot \nabla \tilde{n} + \tilde{\mathbf{V}} \cdot \nabla \tilde{n} = -\tilde{\mathbf{V}} \cdot \nabla n_0 + \left\langle \tilde{\mathbf{V}} \cdot \nabla \tilde{n} \right\rangle \tag{41}$$

where the ensemble is carried out over multiple realizations of the flow $\mathbf{V}(\mathbf{r}, t)$.

Standard QLT proceeds now by neglecting all second-order terms **only** in Eq. 41. Then $\tilde{n}$ is solved in terms of $\mathbf{V}_0$ and ∇n_0 by means of a Green function or propagator. The result is then inserted in Eq. 41, that becomes an advection-diffusion equation with a renormalized *eddy diffusivity* [18].

In our case, it is however better to retain $\tilde{\mathbf{V}} \cdot \nabla\tilde{n}$ in Eq. 41 and neglect only the contribution to $\tilde{n}$ from $<\tilde{\mathbf{V}} \cdot \nabla\tilde{n}>$, which can be shown by iteration that yields a vanishing contribution (to lowest order) when inserted in Eq. 41 [5]. The reason for this non-standard approach is that one deals then with the exact propagator, not that associated to the mean flow $\mathbf{V}_0$, which opens up the possibility of examining the case $\mathbf{V}_0 = 0$ and avoid any asymmetries introduced by a mean flow. The propagator for Eq. 41 (assuming $G(\vec{r}, 0) = 0$) then satisfies:

$$\frac{\partial G}{\partial t} + \tilde{\mathbf{V}} \cdot \nabla G = \delta(\mathbf{r} - \mathbf{r}', t - t'), \tag{42}$$

whose formal solution is (for $t > t'$),

$$G(\mathbf{r} - \mathbf{r}', t - t') = \delta\left(\mathbf{r}' - \mathbf{R}(t'|\mathbf{r}, t)\right). \tag{43}$$

The characteristic $\mathbf{R}(t'|\mathbf{r}, t)$ results from solving Eq. 39 backwards in time (from t to t'):

$$\frac{d\mathbf{R}}{d\tau} = \tilde{\mathbf{V}}(\mathbf{R}, \tau), \qquad \mathbf{R}(t) = \mathbf{r}. \tag{44}$$

We can then write the solution for $\tilde{n}(\mathbf{r}', t')$ in terms of the propagator (Eq. 43) and insert the result in Eq. 41 to obtain:

$$\frac{\partial n_0}{\partial t} = \nabla \cdot \left[\int_0^t dt' \left\langle \tilde{\mathbf{V}}(\mathbf{r}, t)\tilde{\mathbf{V}}(\mathbf{R}(t'|\mathbf{r}, t), t') \cdot \nabla n_0(\mathbf{R}(t'|\mathbf{r}, t), t')\right\rangle\right]. \tag{45}$$

Note that, in contrast to the more standard QLT result [18], ∇n_0 depends on the turbulent part of the flow through $\mathbf{R}(t'|\mathbf{r}, t)$ and cannot be taken out of the ensemble average. This implies that Eq. 45 is clearly non-Markovian (due to the time integral over the past history of the mean density gradient) and probably non-local (due to the characteristic appearing in the first argument of the mean density gradient. The traditional QLT shortcut is to assume *the localization hypothesis* [5]: $\nabla n_0(\mathbf{R}(t'|\mathbf{r}, t), t') \simeq \nabla n_0(\mathbf{r}, t')$. Then, Eq. 45 turns into:

$$\frac{\partial n_0}{\partial t} \simeq \nabla \cdot \left[\int_0^t dt' \mathbf{C}^{\mathbf{L}}(t', t)\nabla n_0(\mathbf{r}, t')\right]. \tag{46}$$

where we have defined the standard **Lagrangian correlation matrix** as:

$$\mathbf{C}^{\mathbf{L}}(t', t) = \int d\mathbf{r}\left\langle \tilde{\mathbf{V}}(\mathbf{r}, t)\tilde{\mathbf{V}}(\mathbf{R}(t'|\mathbf{r}, t), t')\right\rangle. \tag{47}$$

In this way, the new equation (Eq. 46) is still non-Markovian, but becomes local!

An additional assumption of the standard QLT procedure eliminates also the non-Markovian character of Eq. 46. After assuming homogeneity and isotropy (i.e., $\mathbf{C}^{\mathbf{L}}(t', t) = \mathbf{C}^{\mathbf{L}}(t - t')$), one chooses an exponentially decaying Lagrangian correlation with some decorrelation time τ_c,

$$\mathbf{C}^{\mathrm{L}}{}_{ij}(\tau) = V_c^2 \exp(-\tau/\tau_c)\delta_{ij}. \tag{48}$$

This assumption is tantamount to saying that the flow "forgets" its past local history after a lapse of time τ_c has passed. Memory effects thus disappear for time lags larger than τ_c and Eq. 46 then becomes [18]

$$\frac{\partial n_0}{\partial t} \simeq D\nabla^2 n_0(\mathbf{r},t), \qquad D = (V_c^2 \tau_c), \tag{49}$$

with an eddy diffusivity D that depends only on the flow properties.

Our problem can then be formulated as the following question: How can one modify the QLT procedure to allow the derivation of FDEs, which are in general non-local and non-Markovian? The answer is clear. One has to avoid making the two ansatzs just discussed: the locality hypothesis and the exponentially-vanishing memory. The latter is the easiest one to avoid. One needs simply to make a different choice for the Lagrangian correlation in Eq. 46. For instance, one that decays as a power-law instead [39]. Indeed, this permits the derivation of FDEs with fractional time derivatives (even when there are important subtleties in the process that need to be taken into account [19]). But how can one also include the non-local effects? That is, how can one derive equations in which $\alpha < 2$ in Eq. 38? This is the question that both Prof. Balescu and us started to attack in 2004.

Clearly, the key to finding an answer is to avoid assuming the locality hypothesis. That is, to deal with the full Eq. 45 and compute the triple-product ensemble average. This is a challenging problem. To find a solution, it is worthwhile to we rewrite Eq. 45 in its form prior to integrating the propagator G:

$$\frac{\partial n_0}{\partial t} = \nabla \cdot \left[\int_0^t dt' \int d\mathbf{r}' \; \nabla n_0(\mathbf{r}',t') \cdot \left\langle \tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{r}',t') G(\mathbf{r},t|\mathbf{r}',t') \right\rangle \right]. \tag{50}$$

The idea is then to find ways to estimate the ensemble average $\left\langle \tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{r}',t') G(\mathbf{r},t|\mathbf{r}',t') \right\rangle$ and manipulate it until it becomes the kernel that appears in the definition of both temporal and spatial fractional differential operators (Eq. 23).

Our approach to this calculation, which is described in detail in Ref. [19], consisted in applying functional integration techniques to the fluctuating particle trajectories (assumed self-similar). In the one-dimensional case, it is shown that the renormalization scheme we constructed, allows to reduce Eq. 40 to the usual fractional differential equations (i.e., like Eq. 38) under quite general conditions. The "fractional order" of the resulting transport FDE depends on two exponents, H and α, which are respectively related to the degree of correlation of the Lagrangian velocity series of the flow (i.e., H is its Hurst exponent) and to the exponent of the asymptotic tail of the pdf of the Lagrangian cumulative velocities rescaled appropriately [19]. These two exponents can be combined to define the third exponent, $\beta \equiv \alpha H$, that provides the temporal FDO appearing in Eq. 38 and that determines whether the resulting FDE is Markovian ($\beta = 1$)or not ($\beta \neq 1$).

Prof. Balescu's idea, on the other hand, was rooted in the search for generalizations of the so-called Corrsin hypothesis, which assumes that [18]:

$$\left\langle \tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{r}',t') G(\mathbf{r},t|\mathbf{r}',t') \right\rangle \simeq \left\langle \tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{r}',t') \right\rangle \langle G(\mathbf{r},t|\mathbf{r}',t') \rangle . \tag{51}$$

Then, one needs to insert reasonable ansatzs for both the Eulerian correlation function, $C_E \equiv \left\langle \tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{r}',t') \right\rangle$ and the averaged flow propagator. In his last communication with us, Prof. Balescu told us that he had found a solution along these lines but, regretfully, the details of his derivation will probably remain unknown [17].

6 Conclusions

Prof. Balescu was a man with profound physical insight and a never-ending curiosity for everything scientific. We enjoyed greatly our short interaction with him and benefited greatly from it both personally and professionally. He will certainly be missed. We also hope that the short stroll we just took through his latest interests will contribute to make his memory even larger for the younger physicists that did not have the privilege of meeting or interacting with him.

As we have shown, the CTRW/FDE may provide with a powerful stochastic/probabilistic framework to model transport in systems where non-diffusive behaviour is observed. The essential information required to formulate these models are the probability distributions (for the CTRWs) or the relevant exponents (for the FDEs), but both are intrinsically connected. The physics of anomalous transport in magnetically confined plasmas appears to be one of these systems where non-diffusive effects may be dominant, as suggested by numerical simulations and some experimental evidence. We feel confident that some of the ideas reviewed in this paper will find wide application in this field in the near future.

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