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Direct methods of construction of conservation laws

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Abstract

The basic notions and statements of conservation laws of differential equations are reviewed. The main attention is paid to constructive methods of finding conservation laws for general systems of differential equations, which are not Euler–Lagrange equations of a functional and, therefore, do not admit application of the symmetry approach based on the Noether theorem. Recently introduced notions of equivalence of conservation laws with respect to Lie symmetry groups for fixed systems of differential equations and with respect to equivalence groups or sets of admissible transformations for classes of such systems are considered. To construct conservation laws, we develop and apply a modification of the most direct method, which is effective to construct both local and potential conservation laws, especially, in the case of two independent variables. Classification of potential conservation laws of diffusion–convection equations with respect to the associated equivalence group and exhaustive list of locally inequivalent potential systems corresponding to these equations are adduced as an example on calculation of complete hierarchy of potential conservation laws. More details are presented on simpler classification of local conservation laws of variable coefficient diffusion–reaction equations.

1 Introduction

After the Emmy Noether's remarkable paper [19] had become well-known, a number of authors searched for conservation laws using the symmetry approach based on the Noether's results. In view of the generalized Noether's theorem [20], there exists one-to-one correspondence between the non-trivial generalized variational symmetries of some functional and the non-trivial conservation laws of the associated Euler–Lagrange equations, and any such symmetry is a generalized symmetry of the Euler–Lagrange equations.

The Noether's approach has a number of advantages. It reduces construction of conservation laws to finding symmetries for which there exist a number of well-developed methods, and complete description of necessary symmetry properties is known for a lot of systems of differential equations. However, this approach can be applied only to Euler–Lagrange equations that form normal systems and admit symmetry groups satisfying an additional "variational" property of leaving the variational integral invariant in some sense [20]. The latter requirements lead to restriction of class of systems that could be investigated in such way.

At the same time, the definition of conservation laws itself gives rise to a method of finding conservation laws. Technique of calculations used in the framework of this method is similar to the classical Lie method yielding symmetries of differential equations [11, Chapter 6]. As mentioned in the above reference, such algorithmic possibility was first employed by P.-S. Laplace [15] for derivation of the well-known Laplace vector of the two-body Kepler problem. Following tradition from group analysis of differential equations, we may call this method *direct* and distinguish four its versions, depending on the way of taking into account systems under investigation. (See e.g. [1, 2, 5] and Section 5 of this paper for more details as well as [30] for comparison of the versions and their realizations in computer algebra programs.) In the present paper we use the most direct version based

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on immediate solving of determining equations for conserved vectors of conservations laws on the solution manifolds of investigated systems.

Let us note that there exist other approaches for construction of conservation laws which differ from the Noether's or above direct ones, are based on exploitation of symmetry properties of differential equations and can be applied to non-Lagrangian systems. Thus, W.I. Fushchych and A.G. Nikitin [9] proposed to calculate directly bilinear combinations of solutions of motion equations, which are conserved in time by virtue of symmetries of these equations. It is possible in such way to find conservation laws corresponding to non-geometric symmetries.

To classify conservation laws, instead of the usual equivalence relation on their set (more exactly, on the set of conserved vectors) we use the natural and more general notions of equivalences of conservation laws with respect to Lie symmetry groups for fixed systems of differential equations and with respect to equivalence groups or sets of admissible point (or contact) transformations for classes of such systems. Results of classification up to these equivalences are more comprehensible, especially, if a whole class of systems is studied and blend with the framework of group analysis.

In [5] an ingenious procedure of branching iterations for finding nonlocal (potential) conservation laws of diffusion equations was proposed. Namely, on each iteration they use a conservation law from the previous iteration (one conservation law for one iteration) to introduce a potential and to construct the extended potential system. Then they study local conservation laws of the potential system, which are, generally speaking, nonlocal (potential) conservation laws for the initial equation. To the best of our knowledge, it was the first paper where the idea of hierarchy of potential systems and associated conservation laws is presented in an explicit form.

The iteration procedure was generalized in [24] by admitting dependence of conserved vectors on different finite number (from one to the maximum possible that) of new potentials on each iteration. The idea of a similar approach was adduced in [29] and was formalized in the form of notion of *universal Abelian covering* of differential equations [7, 25, 27]. Such approach naturally results in the questions on some independence of employed potentials. That is why, we also discuss the notions of linear dependence of conservation laws and of local dependence of potentials in detail.

2 Basic definitions and statements

In this and the next sections we give basic definitions and statements on conservation laws, following the spirit of the well-known textbook [20] in general outlines. Then we formulate the notion of equivalence of conservation laws with respect to transformation groups, which was first introduced in [24]. This notion is a base for modification of the direct method of construction of conservation laws, which is applied in section 7 for exhaustive classification of local conservation laws of equations from class (9).

Let $\mathcal{L}$ be a system $L(x,u_{(\rho)})=0$ of l differential equations $L^1=0,\ \dots,\ L^l=0$ for m unknown functions $u=(u^1,\dots,u^m)$ of n independent variables $x=(x_1,\dots,x_n)$. Here $u_{(\rho)}$ denotes the set of all the derivatives of the functions u with respect to x of order no greater than ρ, including u as the derivatives of the zero order. Let $\mathcal{L}_{(k)}$ denote the set of all algebraically independent differential consequences that have, as differential equations, orders no greater than k. We identify $\mathcal{L}_{(k)}$ with the manifold determined by $\mathcal{L}_{(k)}$ in the jet space $J^{(k)}$.

Definition 1. A *conserved vector* of the system $\mathcal{L}$ is an n-tuple

$$F=(F^1(x,u_{(r)}),\dots,F^n(x,u_{(r)}))$$

for which the divergence $\operatorname{Div}F:=D_iF^i$ vanishes for all solutions of $\mathcal{L}$, i.e., $\operatorname{Div}F\big|_{\mathcal{L}}=0$.

In Definition 1 and below $D_i=D_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i, i.e. $D_i=\partial_{x_i}+u^a_{\alpha,i}\partial_{u^a_\alpha}$, where u^a_α and $u^a_{\alpha,i}$ stand for the variables in jet spaces, which correspond to derivatives $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$ and $\partial u^a_\alpha/\partial x_i$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\in\mathbb{N}\cup\{0\}$, $|\alpha|:=\alpha_1+\dots+\alpha_n$. We use the summation convention for repeated indices and assume any function

as its zero-order derivative. The notation $V\big|_{\mathcal{L}}$ means that values of V are considered only on solutions of the system $\mathcal{L}$.

Definition 2. A conserved vector F is called *trivial* if $F^i = \hat{F}^i + \check{F}^i$, $i = \overline{1,n}$, where $\hat{F}^i$ and $\check{F}^i$ are, likewise F^i, functions of x and derivatives of u (i.e. differential functions), $\hat{F}^i$ vanish on the solutions of $\mathcal{L}$ and the n-tuple $\check{F} = (\check{F}^1, \dots, \check{F}^n)$ is a null divergence (i.e. its divergence vanishes identically).

The triviality concerning the vanishing conserved vectors on solutions of the system can be easily eliminated by confining on the manifold of the system, taking into account all its necessary differential consequences. A characterization of all null divergences is given by the following lemma (see e.g. [20]).

Lemma 1. *The n-tuple $F = (F^1, \dots, F^n)$, $n \geq 2$, is a null divergence ($\operatorname{Div} F \equiv 0$) iff there exist smooth functions v^{ij} $(i,j = \overline{1,n})$ of x and derivatives of u, such that $v^{ij} = -v^{ji}$ and $F^i = D_j v^{ij}$.*

The functions v^{ij} are called *potentials* corresponding to the null divergence F. If $n = 1$ any null divergence is constant.

Definition 3. Two conserved vectors F and F' are called *equivalent* if the vector-function $F' - F$ is a trivial conserved vector.

The above definitions of triviality and equivalence of conserved vectors are natural in view of the usual "empiric" definition of conservation laws of a system of differential equations as divergences of its conserved vectors, i.e. divergence expressions which vanish for all solutions of this system. For example, equivalent conserved vectors correspond to the same conservation law. It allows us to formulate the definition of conservation law in a rigorous style (see e.g. [31]). Namely, for any system $\mathcal{L}$ of differential equations the set $\mathrm{CV}(\mathcal{L})$ of conserved vectors of its conservation laws is a linear space, and the subset $\mathrm{CV}_0(\mathcal{L})$ of trivial conserved vectors is a linear subspace in $\mathrm{CV}(\mathcal{L})$. The factor space $\mathrm{CL}(\mathcal{L}) = \mathrm{CV}(\mathcal{L})/\mathrm{CV}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{CV}(\mathcal{L})$ with respect to the equivalence relation adduced in Definition 3.

Definition 4. The elements of $\mathrm{CL}(\mathcal{L})$ are called *conservation laws* of the system $\mathcal{L}$, and the whole factor space $\mathrm{CL}(\mathcal{L})$ is called as *the space of conservation laws* of $\mathcal{L}$.

That is why we assume description of the set of conservation laws as finding $\mathrm{CL}(\mathcal{L})$ that is equivalent to construction of either a basis if $\dim \mathrm{CL}(\mathcal{L}) < \infty$ or a system of generatrices in the infinite dimensional case. The elements of $\mathrm{CV}(\mathcal{L})$ which belong to the same equivalence class giving a conservation law $\mathcal{F}$ are considered all as conserved vectors of this conservation law, and we will additionally identify elements from $\mathrm{CL}(\mathcal{L})$ with their representatives in $\mathrm{CV}(\mathcal{L})$. For $F \in \mathrm{CV}(\mathcal{L})$ and $\mathcal{F} \in \mathrm{CL}(\mathcal{L})$ the notation $F \in \mathcal{F}$ will denote that F is a conserved vector corresponding to the conservation law $\mathcal{F}$. In contrast to the order r_F of a conserved vector F as the maximal order of derivatives explicitly appearing in F, the *order of the conservation law* $\mathcal{F}$ is called $\min\{r_F \,|\, F \in \mathcal{F}\}$. Under linear dependence of conservation laws we understand linear dependence of them as elements of $\mathrm{CL}(\mathcal{L})$. Therefore, in the framework of "representative" approach conservation laws of a system $\mathcal{L}$ are considered *linearly dependent* if there exists linear combination of their representatives, which is a trivial conserved vector.

3 Characteristics of conservation laws

Let the system $\mathcal{L}$ be totally nondegenerate [20]. Then application of the Hadamard lemma to the definition of conserved vector and integrating by parts imply that divergence of any conserved vector of $\mathcal{L}$ can be always presented, up to the equivalence relation of conserved vectors, as a linear combination of left side of independent equations from $\mathcal{L}$ with coefficients λ^μ being functions on a suitable jet space $J^{(k)}$:

$$\operatorname{Div} F = \lambda^\mu L^\mu. \tag{1}$$

Here the order k is determined by $\mathcal{L}$ and the allowable order of conservation laws, $\mu = \overline{1,l}$.

Definition 5. Formula (1) and the l-tuple $\lambda = (\lambda^1, \dots, \lambda^l)$ are called the *characteristic form* and the *characteristic* of the conservation law $\operatorname{Div} F = 0$ correspondingly.

The characteristic λ is *trivial* if it vanishes for all solutions of $\mathcal{L}$. Since $\mathcal{L}$ is nondegenerate, the characteristics λ and $\tilde{\lambda}$ satisfy (1) for the same F and, therefore, are called *equivalent* iff $\lambda - \tilde{\lambda}$ is a trivial characteristic. Similarly to conserved vectors, the set $\operatorname{Ch}(\mathcal{L})$ of characteristics corresponding to conservation laws of the system $\mathcal{L}$ is a linear space, and the subset $\operatorname{Ch}_0(\mathcal{L})$ of trivial characteristics is a linear subspace in $\operatorname{Ch}(\mathcal{L})$. The factor space $\operatorname{Ch}_{\mathrm{f}}(\mathcal{L}) = \operatorname{Ch}(\mathcal{L})/\operatorname{Ch}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\operatorname{Ch}(\mathcal{L})$ with respect to the above characteristic equivalence relation.

The following result [20] forms the cornerstone for the methods of studying conservation laws, which are based on formula (1), including the Noether theorem and the direct method in the version by Anco and Bluman [1, 2].

Theorem 1 ([20])**.** *Let $\mathcal{L}$ be a normal, totally nondegenerate system of differential equations. Then representation of conservation laws of $\mathcal{L}$ in the characteristic form* (1) *generates a one-to-one linear mapping between* $\operatorname{CL}(\mathcal{L})$ *and* $\operatorname{Ch}_{\mathrm{f}}(\mathcal{L})$*.*

Using properties of total divergences, we can exclude the conserved vector F from (1) and obtain a condition for the characteristic λ only. Namely, a differential function f is a total divergence, i.e. $f = \operatorname{Div} F$ for some n-tuple F of differential functions iff $\mathsf{E}(f) = 0$. Hereafter the Euler operator $\mathsf{E} = (\mathsf{E}^1, \dots, \mathsf{E}^m)$ is the m-tuple of differential operators

$$\mathsf{E}^a = (-D)^\alpha \partial_{u^a_\alpha}, \quad a = \overline{1,m},$$

where $(-D)^\alpha = (-D_1)^{\alpha_1} \dots (-D_m)^{\alpha_m}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ runs the multi-indices set (i.e., $\alpha_i \in \mathbb{N} \cup \{0\}$). Therefore, action of the Euler operator on (1) results to the equation

$$\mathsf{E}(\lambda^\mu L^\mu) = \mathsf{D}^*_\lambda(L) + \mathsf{D}^*_L(\lambda) = 0, \tag{2}$$

which is a necessary and sufficient condition on characteristics of conservation laws for the system $\mathcal{L}$. The matrix differential operators D^*_λ and D^*_L are the adjoints of the Fréchet derivatives D_λ and D_L, i.e.

$$\mathsf{D}^*_\lambda(L) = \left((-D)^\alpha \left(\frac{\partial \lambda^\mu}{\partial u^a_\alpha} L^\mu\right)\right), \qquad \mathsf{D}^*_L(\lambda) = \left((-D)^\alpha \left(\frac{\partial L^\mu}{\partial u^a_\alpha} \lambda^\mu\right)\right).$$

Since $\mathsf{D}^*_\lambda(L) = 0$ automatically on solutions of $\mathcal{L}$ then equation (2) implies a necessary condition for λ to belong to $\operatorname{Ch}(\mathcal{L})$:

$$\mathsf{D}^*_L(\lambda)\big|_{\mathcal{L}} = 0. \tag{3}$$

Condition (3) can be considered as adjoint to the criteria $\mathsf{D}_L(\eta)\big|_{\mathcal{L}} = 0$ for infinitesimal invariance of $\mathcal{L}$ with respect to evolutionary vector field having the characteristic $\eta = (\eta^1, \dots, \eta^m)$. That is why solutions of (3) are called sometimes as *cosymmetries* [25, 4] or *adjoint symmetries* [2].

4 Equivalence of conservation laws with respect to transformation groups

We can essentially simplify and order classification of conservation laws, taking into account additionally symmetry transformations of a system or equivalence transformations of a whole class of systems. Such problem is similar to one of group classification of differential equations.

Proposition 1. *Any point transformation g maps a class of equations in the conserved form into itself. More exactly, the transformation g: $\tilde{x} = x_g(x,u)$, $\tilde{u} = u_g(x,u)$ prolonged to the jet space $J^{(r)}$*

transforms the equation $D_iF^i=0$ to the equation $D_iF^i_g=0$. The transformed conserved vector F_g is determined by the formula

$$F^i_g(\tilde{x},\tilde{u}_{(r)})=\frac{D_{x_j}\tilde{x}_i}{|D_x\tilde{x}|}\,F^j(x,u_{(r)}),\quad i.e.\quad F_g(\tilde{x},\tilde{u}_{(r)})=\frac{1}{|D_x\tilde{x}|}(D_x\tilde{x})F(x,u_{(r)}) \tag{4}$$

in the matrix notions. Here $|D_x\tilde{x}|$ is the determinant of the matrix $D_x\tilde{x}=(D_{x_j}\tilde{x}_i)$.

Note 1. In the case of one dependent variable ($m=1$) g can be a contact transformation: $\tilde{x}=x_g(x,u_{(1)})$, $\tilde{u}_{(1)}=u_{g(1)}(x,u_{(1)})$. Similar notes are also true for the statements below.

Definition 6. Let G be a symmetry group of the system $\mathcal{L}$. Two conservation laws with the conserved vectors F and F' are called *G-equivalent* if there exists a transformation $g\in G$ such that the conserved vectors F_g and F' are equivalent in the sense of Definition 3.

Any transformation $g\in G$ induces a linear one-to-one mapping g_* in $\mathrm{CV}(\mathcal{L})$, transforms trivial conserved vectors only to trivial ones (i.e. $\mathrm{CV}_0(\mathcal{L})$ is invariant with respect to g_*) and therefore induces a linear one-to-one mapping g_{f} in $\mathrm{CL}(\mathcal{L})$. It is obvious that g_{f} preserves linear (in)dependence of elements in $\mathrm{CL}(\mathcal{L})$ and maps a basis (a set of generatrices) of $\mathrm{CL}(\mathcal{L})$ in a basis (a set of generatrices) of the same space. In such way we can consider the G-equivalence relation of conservation laws as well-determined on $\mathrm{CL}(\mathcal{L})$ and use it to classify conservation laws.

Proposition 2. *If the system $\mathcal{L}$ admits a one-parameter group of transformations then the infinitesimal generator $X=\xi^i\partial_i+\eta^a\partial_{u^a}$ of this group can be used for construction of new conservation laws from known ones. Namely, differentiating equation (4) with respect to the parameter ε and taking the value $\varepsilon=0$, we obtain the new conserved vector*

$$\widetilde{F}^i=-X_{(r)}F^i+(D_j\xi^i)F^j-(D_j\xi^j)F^i. \tag{5}$$

Here $X_{(r)}$ denotes the r-th prolongation [20, 21] of the operator X.

Note 2. Formula (5) can be directly extended to generalized symmetry operators (see, for example, [3]). A similar statement for generalized symmetry operators in evolutionary form ($\xi^i=0$) was known earlier [10, 20]. It was used in [16] to introduce a notion of basis of conservation laws as a set which generates a whole set of conservation laws with action of generalized symmetry operators and the operation of linear combination.

Proposition 3. *Any point transformation g between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping g_* from $\mathrm{CV}(\mathcal{L})$ into $\mathrm{CV}(\tilde{\mathcal{L}})$, which maps $\mathrm{CV}_0(\mathcal{L})$ into $\mathrm{CV}_0(\tilde{\mathcal{L}})$ and generates a linear one-to-one mapping g_{f} from $\mathrm{CL}(\mathcal{L})$ into $\mathrm{CL}(\tilde{\mathcal{L}})$.*

Corollary 1. *Any point transformation g between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping $\hat{g}_{\mathrm{f}}$ from $\mathrm{Ch}_{\mathrm{f}}(\mathcal{L})$ into $\mathrm{Ch}_{\mathrm{f}}(\tilde{\mathcal{L}})$.*

It is possible to obtain an explicit formula for correspondence between characteristics of $\mathcal{L}$ and $\tilde{\mathcal{L}}$. Let $\tilde{\mathcal{L}}^\mu=\Lambda^{\mu\nu}\mathcal{L}^\nu$, where $\Lambda^{\mu\nu}=\Lambda^{\mu\nu\alpha}D^\alpha$, $\Lambda^{\mu\nu\alpha}$ are differential functions, $\alpha=(\alpha_1,\dots,\alpha_n)$ runs the multi-indices set ($\alpha_i\in\mathbb{N}\cup\{0\}$), $\mu,\nu=\overline{1,l}$. Then

$$\lambda^\mu=\Lambda^{\nu\mu*}(|D_x\tilde{x}|\tilde{\lambda}^\nu).$$

Here $\Lambda^{\nu\mu*}=(-D)^\alpha\cdot\Lambda^{\mu\nu\alpha}$ is the adjoint to the operator $\Lambda^{\nu\mu}$. For a number of cases, e.g. if $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are single partial differential equations ($l=1$), the operators $\Lambda^{\mu\nu}$ are simply differential functions (i.e. $\Lambda^{\mu\nu\alpha}=0$ for $|\alpha|>0$) and, therefore, $\Lambda^{\nu\mu*}=\Lambda^{\mu\nu}$.

Consider the class $\mathcal{L}|_{\mathcal{S}}$ of systems $\mathcal{L}_\theta$: $L(x,u_{(\rho)},\theta(x,u_{(\rho)}))=0$ parameterized with the parameter-functions $\theta=\theta(x,u_{(\rho)})$. Here L is a tuple of fixed functions of x, $u_{(\rho)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(x,u_{(\rho)})=(\theta^1(x,u_{(\rho)}),\dots,\theta^k(x,u_{(\rho)}))$ running the set $\mathcal{S}$ of solutions of the system $S(x,u_{(\rho)},\theta_{(q)}(x,u_{(\rho)}))=0$. This system consists of differential equations on θ, where x

and $u_{(\rho)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. In what follows we call the functions θ arbitrary elements. Denote the point transformations group preserving the form of the systems from $\mathcal{L}|_{\mathcal{S}}$ as $G^{\sim} = G^{\sim}(L, S)$.

Consider the set $P = P(L, S)$ of all pairs each of which consists of a system $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ and a conservation law $\mathcal{F}$ of this system. In view of Proposition 3, action of transformations from $G^{\sim}$ on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathrm{CV}(\mathcal{L}_\theta) \,|\, \theta \in \mathcal{S}\}$ together with the pure equivalence relation of conserved vectors naturally generates an equivalence relation on P.

Definition 7. Let $\theta, \theta' \in \mathcal{S}$, $\mathcal{F} \in \mathrm{CL}(\mathcal{L}_\theta)$, $\mathcal{F}' \in \mathrm{CL}(\mathcal{L}_{\theta'})$, $F \in \mathcal{F}$, $F' \in \mathcal{F}'$. The pairs $(\mathcal{L}_\theta, \mathcal{F})$ and $(\mathcal{L}_{\theta'}, \mathcal{F}')$ are called $G^{\sim}$-*equivalent* if there exists a transformation $g \in G^{\sim}$ which transform the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\theta'}$ and such that the conserved vectors F_g and F' are equivalent in the sense of Definition 3.

Classification of conservation laws with respect to $G^{\sim}$ will be understood as classification in P with respect to the above equivalence relation. This problem can be investigated in the way that is similar to group classification in classes of systems of differential equations, especially it is formulated in terms of characteristics. Namely, we construct firstly the conservation laws that are defined for all values of the arbitrary elements. (The corresponding conserved vectors may depend on the arbitrary elements.) Then we classify, with respect to the equivalence group, arbitrary elements for each of that the system admits additional conservation laws.

In an analogues way we also can introduce equivalence relations on P, which are generated by either generalizations of usual equivalence groups or all admissible point or contact transformations [22] (called also form-preserving ones [17, 18]) in pairs of equations from $\mathcal{L}|_{\mathcal{S}}$.

Note 3. It can be easy shown that all the above equivalences are indeed equivalence relations, i. e. they have the usual reflexive, symmetric and transitive properties.

Note 4. Inclusion of the equivalence with respect to transformations to the framework of conservation laws allows us to investigate different classification problems on conservation laws (construction of generating sets of conservation laws for a system of differential equations with respect to the corresponding point symmetry group, classification of conservation laws for a class of systems of differential equations with respect to its equivalence group or the associated set of admissible transformations, investigation of generating sets of conservation laws for classes of systems of differential equations etc). What kind of the problem is necessary to solve depends on the way of further usage of conservation laws.

5 Direct iteration method of finding conservation laws

To construct conservation laws of a system $\mathcal{L}$ of differential equations, we iterate a modification of the most *direct method* based on Definition 1. More precisely, the algorithm is as follows.

Zeroth iteration. At first we construct local conservation laws of $\mathcal{L}$. We fix an (arbitrary) order r of conserved vectors under consideration. Then we introduce local coordinates ("unconstrained variables") on the manifold $\mathcal{L}_{(r+1)}$ determined by the system $\mathcal{L}$ and its differential consequences in $J^{(r+1)}$. The other ("constrained") variables of $J^{(r+1)}$ are expressed via unconstrained ones by means of using the equations of $\mathcal{L}_{(r+1)}$. We substitute the expressions for constrained variables into a conservation law and split the obtained condition with respect to the unconstrained variables. This procedure results in a first-order linear system of determining equations for conserved vectors. Solving the determining equations up to the usual equivalence relation on $\mathrm{CV}(\mathcal{L})$, we obtain complete description of local conservation laws of $\mathcal{L}$. To classify conservation laws in easier and more systematic way (especially for classes of systems of differential equations), instead of usual equivalence we use the introduced above equivalence with respect to symmetry or equivalence transformations.

First iteration. After applying Lemma 1 to constructed conservation laws on the set of solutions of $\mathcal{L}=\mathcal{L}^0$, we introduce potentials as additional dependent variables and attach the equations connecting the potentials with components of corresponding conserved vectors to $\mathcal{L}^0$. (If $n>2$ the attached equations of such kind form an underdetermined system with respect to the potentials. Therefore, we can also attach gauge conditions on the potentials to $\mathcal{L}^0$.)

We have to use linear independent conservation laws since otherwise the introduced potentials will be *dependent* in the following sense: there exists a linear combination of the potential tuples, which is, for some $r'\in\mathbb{N}$, a tuple of functions of x and $u_{(r')}$ only.

Then we exclude the unnecessary equations (i.e. the equations that are dependent on equations of $\mathcal{L}^0$ and attached equations simultaneously) from the extended (potential) system $\mathcal{L}^1$ which will be called a *potential system of the first level*. Any conservation law of $\mathcal{L}^0$ is a one of $\mathcal{L}^1$. We iterate the above procedure of the direct method for $\mathcal{L}^1$ to find its conservation laws which are linear independent with ones from the previous iteration and will be called *potential conservation laws of the first level*.

Further iterations. We make iterations while it is possible (i.e. the iteration procedure has to be stopped if all the conservation laws of a *potential system* $\mathcal{L}^{k+1}$ *of the* $(k+1)$*-th level* are linear dependent with the ones of $\mathcal{L}^k$) or construct infinite chains of conservation laws by means of induction. This process may yield *purely potential* conservation laws of the initial system $\mathcal{L}$, which are linear independent with local conservation laws and, therefore, depend explicitly on potential variables.

Any conservation law from the previous step of iteration procedure will be a conservation law for the next step and vice versa, conservation laws which are obtained on the next step and depend only on variables of the previous step are linear dependent with conservation laws from the previous step. It is also obvious that the conservation laws used for construction of a potential system of the next level are trivial on the manifold of this system.

Since gauge conditions on potentials can be chosen in many different ways, exhaustive realization of iterations is improbable in the case $n>2$.

The procedure of exclusion of constrained variables (which are described above in detail only for the zeroth iteration) is called in classical group analysis as "confining to the manifold of $\mathcal{L}$". Taking into account $\mathcal{L}$ in the above way, we automatically eliminate the ambiguity connected with vanishing conserved vectors on the solutions of $\mathcal{L}$. However, the second kind ambiguity arising via existence of null divergences is preserved, and it is the main reason of difficulties in realization of this algorithm with symbolic computation systems [30].

The modification of the most direct method with usage of equivalence with respect to symmetry or equivalence transformations is especially effective in the two-dimensional case. See, e.g., [12, 14, 24, 26] for examples on calculation of conservation laws for different classes of evolution equations. A new example is given in Section 7. At the same time, this method can be effectively applied also to multi-dimensional equations [13].

To find conservation laws on each step of iteration procedure, one can apply other methods which are based on the characteristic form (1) or its consequences (2) and (3). These methods are also called as direct [1, 2]. Following [30], for convenience we will numerate them as the second, third and fourth versions of the direct method in contrast to the above first one. They are close to the symmetry group method by Noether since in the case of Euler–Lagrange equations the coefficients λ^a are nothing else than Noether's characteristics. Taking into account the equivalence relation on $\mathrm{Ch}(\mathcal{L})$, one can assume during calculations that characteristics depend only on unconstrained variables.

In the second version of the direct method the representation (1) is regarded as an equation defined on an open subset of $J^{(k)}$ with respect to conserved vectors and characteristics simultaneously.

In the framework of the third version, sought quantities are characteristics only. Determining equation (2) is defined on an open subset of $J^{(k)}$. Conserved vectors are reconstructed from known characteristics via explicit integral formulas. An algorithm of this (third) version of the direct method was developed for Cauchy–Kovalevskaya systems by S. Anco and G. Bluman [1, 2] (see also [20, 31] for a theoretical background).

The fourth version is based on equation (3) which is defined on the manifold $\mathcal{L}$ and is only a necessary condition on characteristic of conservation laws. Therefore, one has to choose characteristics from

the set of adjoint symmetries using additional conditions. See, e.g., [7] or [28] and other papers of the same journal issue for the corresponding theoretical background, numerous examples of applications and references on this version of the direct method.

Each from four above versions of the direct method has its advantages and disadvantages. A detailed comparative analysis of all the versions and their realizations in computer algebra programs are given by T. Wolf [30].

6 Two-dimensional case

The case of two independent variables is singular, in particular, with respect to possible (constant) indeterminacy after introduction of potentials and high effectiveness of application of potential symmetries. That is why we consider some notions connected with conservation laws in this case separately. We denote independent variables as t (the time variable) and x (the space one). Any local conservation law has the form

$$D_tF(t,x,u_{(r)})+D_xG(t,x,u_{(r)})=0. \tag{6}$$

Here D_t and D_x are the operators of the total differentiation with respect to t and x. F and G are called the *conserved density* and the *flux* of the conservation law correspondingly.

Two conserved vectors (F,G) and (F',G') are *equivalent* if there exist such functions $\hat F$, $\hat G$ and H of t, x and derivatives of u that $\hat F$ and $\hat G$ vanish on $\mathcal{L}_{(k)}$ for some k and

$$F'=F+\hat F+D_xH, \qquad G'=G+\hat G-D_tH. \tag{7}$$

Any conservation law (6) of $\mathcal{L}$ allows us to deduce the new dependent (potential) variable v by means of the equations

$$v_x=F, \qquad v_t=-G. \tag{8}$$

To construct a number of potentials in one step, we have to use a set of linear independent conservation laws (see the previous section) since otherwise the potentials will be dependent in the following sense: there exists a linear combination of the potentials, which is, for some $r'\in\mathbb{N}$, a function of t, x and $u_{(r')}$ only.

In the case of two independent variables we can also introduce the more general notion of potential dependence.

Definition 8. The potentials v^1, …, v^p are called *locally dependent on the set of solution of the system* $\mathcal{L}$ (or, briefly speaking, *dependent*) if there exist $r'\in\mathbb{N}$ and a function H of the variables t, x, $u_{(r')}$, v^1, …, v^p such that $H(t,x,u_{(r')},v^1,\dots,v^p)=0$ for any solution $(u,v^1,\dots,v^p)$ of the united system determining the set of potentials v^1, …, v^p.

Proof of local dependence or independence of potentials for general classes of differential equations is difficult since it is closely connected with precise description of possible structure of conservation laws. An example of such proof for diffusion–convection equations is presented below.

In the case of single equation $\mathcal{L}$, equations of form (8) combine into the complete potential system since $\mathcal{L}$ is a differential consequence of (8). As a rule, systems of such kind admit a number of nontrivial symmetries and so they are of a great interest.

Equations (4) and (8) imply the following statement.

Proposition 4. *Any point transformation connecting two systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ of PDEs with two independent variables generates a one-to-one mapping between the sets of potential systems, which correspond to $\mathcal{L}$ and $\tilde{\mathcal{L}}$. Generation is made via trivial prolongation on the space of introduced potential variables, i.e. we can assume that the potentials are not transformed.*

Corollary 2. *The Lie symmetry group of a system $\mathcal{L}$ of differential equations generates an equivalence group on the set of potential systems corresponding to $\mathcal{L}$.*

Corollary 3. *Let $\widehat{\mathcal{L}}|_S$ be the set of all potential systems constructed for systems from the class $\mathcal{L}|_S$ with their conservation laws. Action of transformations from $G^{\sim}(L,S)$ together with the equivalence relation of potentials naturally generates an equivalence relation on $\widehat{\mathcal{L}}|_S$.*

Note 5. Proposition 4 and its Corollaries imply that the equivalence group for a class of systems or the symmetry group for single system can be prolonged to potential variables for any step of the direct iteration procedure. It is natural the prolonged equivalence groups are used to classify possible conservation laws and potential systems in each iteration. Additional equivalences which exist in some subclasses of the class or arise after introducing potential variables can be used for deeper analysis of connections between conservation laws.

7 Conservation laws of variable coefficient diffusion–reaction equations

To demonstrate effectiveness of the modified direct method and to illustrate some calculation details, we study local conservation laws of variable coefficient diffusion–reaction equations of the general form

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u), \tag{9}$$

where $f = f(x)$, $g = g(x)$, $h = h(x)$, $A = A(u)$ and $B = B(u)$ are arbitrary smooth functions of their variables, $f(x)g(x)A(u) \neq 0$.

Conservation laws were investigated for some subclasses of class (9). In particular, V.A. Dorodnitsyn and S.R. Svirshchevskii [8] (see also [11, Chapter 10]) constructed the local conservation laws for the class of reaction–diffusion equations of the form $u_t = (A(u)u_x)_x + B(u)$, which is a subclass of the class under consideration.

The equivalence group $G^{\sim}$ of class (9) is formed by the nondegenerate point transformations in the space of (t,x,u,f,g,h,A,B), which are projectible on the space of (t,x,u), i.e., they have the form

$$(\tilde t,\tilde x,\tilde u) = (\mathcal{T}^t,\mathcal{T}^x,\mathcal{T}^u)(t,x,u),$$
$$(\tilde f,\tilde g,\tilde h,\tilde A,\tilde B) = (\mathcal{T}^f,\mathcal{T}^g,\mathcal{T}^h,\mathcal{T}^A,\mathcal{T}^B)(t,x,u,f,g,h,A,B),$$

and transform any equation from class (9) for the function $u = u(t,x)$ with the arbitrary elements (f,g,h,A,B) to an equation from the same class for the function $\tilde u = \tilde u(\tilde t,\tilde x)$ with the new arbitrary elements $(\tilde f,\tilde g,\tilde h,\tilde A,\tilde B)$. Complicated calculations according to the direct method of finding equivalence transformations implies that $G^{\sim}$ consists of the transformations

$$\tilde t = \delta_1 t + \delta_2, \quad \tilde x = \varphi(x), \quad \tilde u = \delta_3 u + \delta_4,$$
$$\tilde f = \frac{\delta_0\delta_1}{\varphi_x} f, \quad \tilde g = \frac{\delta_0\varphi_x}{\delta_5}\, g, \quad \tilde h = \frac{\delta_0}{\delta_5\varphi_x} h, \quad \tilde A = \delta_5 A, \quad \tilde B = \delta_3\delta_5 B,$$

where δ_j $(j = \overline{0,5})$ are arbitrary constants, $\delta_0\delta_1\delta_3\delta_5 \neq 0$, φ is an arbitrary smooth function of x, $\varphi_x \neq 0$. The equivalence transformation

$$\tilde t = t, \quad \tilde x = \int \frac{dx}{g(x)}, \quad \tilde u = u \tag{10}$$

maps equation (9) to the equation $\tilde f(\tilde x)\tilde u_{\tilde t} = (A(\tilde u)\tilde u_{\tilde x})_{\tilde x} + \tilde h(\tilde x)B(\tilde u)$, where $\tilde f(\tilde x) = g(x)f(x)$, $\tilde g(\tilde x) = 1$ and $\tilde h(\tilde x) = g(x)h(x)$. That is why, without loss of generality we can restrict ourselves to investigation of the equations

$$f(x)u_t = (A(u)u_x)_x + h(x)B(u). \tag{11}$$

In view of results of section 4 it is sufficient for exhaustive investigation if we classify conservation laws of equations only from class (11) and then extend the obtained results to class (9) with the transformations (10).

Since there are two independent variables t and x in equations under consideration, which play a part of the time and space variables correspondingly, the conservation laws will have the general form (6).

At first we prove the lemma on order of local conservation laws for more general class of second-order evolution equations, which covers class (9).

Lemma 2. *Any local conservation law of any second-order $(1+1)$-dimensional quasi-linear evolution equation has the first order and, moreover, there exists its conserved vector with the density depending at most on t, x, and u and the flux depending at most on t, x, u and u_x.*

Proof. Consider a conservation law (6) of a second-order $(1+1)$-dimensional quasi-linear evolution equation

$$u_t = S(t,x,u,u_x)u_{xx} + R(t,x,u,u_x), \tag{12}$$

where $S \neq 0$. In view of equation (12) and its differential consequences, we can assume that F and G depend only on t, x and $u_k = \partial^k u/\partial x^k$, $k = \overline{0,r'}$, where $r' \leq 2r$. Suppose that $r' > 1$. We expand the total derivatives in (6) and take into account differential consequences of the form $u_{tj} = D_x^j(Su_{xx}+R)$, where $u_{tj} = \partial^{j+1}u/\partial t\partial x^k$, $j = \overline{0,r'}$. As a result, we obtain the following condition

$$F_t + F_{u_j} D_x^j(Su_{xx} + R) + G_x + G_{u_j}u_{j+1} = 0. \tag{13}$$

Let us decompose (13) with respect to the highest derivatives u_j. Thus, the coefficients of $u_{r'+2}$ and $u_{r'+1}$ give the equations $F_{u_{r'}} = 0$, $G_{u_{r'}} + SF_{u_{r'-1}} = 0$ that implies

$$F = \hat{F}, \quad G = -S\hat{F}_{u_{r'-1}}u_{r'} + \hat{G},$$

where $\hat{F}$ and $\hat{G}$ are functions of t, x, u, u_1, $\ldots$, $u_{r'-1}$. Then, after selecting the terms containing $u_{r'}^2$, we obtain that $-S\hat{F}_{u_{r'-1}u_{r'-1}} = 0$. It yields that $\hat{F} = \check{F}^1 u_{r'-1} + \check{F}^0$, where $\check{F}^1$ and $\check{F}^0$ depend only on t, x, u, u_1, $\ldots$, $u_{r'-2}$.

Consider the conserved vector with the density $\tilde{F} = F - D_x H$ and the flux $\tilde{G} = G + D_t H$, where $H = \int \check{F}^1 du_{r'-2}$. This conserved vector is equivalent to the initial one, and

$$\tilde{F} = \tilde{F}(t,x,u,u_1,\ldots,u_{r'-2}), \quad \tilde{G} = \tilde{G}(t,x,u,u_1,\ldots,u_{r'-1}).$$

Iterating the above procedure a necessary number of times, we result in an equivalent conserved vector depending only on t, x, u and u_x, i.e. we can assume at once that $r' \leq 1$. Then the coefficients of u_{xxx} and u_{xx} in (13) lead to the equations $F_{u_x} = 0$, $G_{u_x} + SF_u = 0$ that implies $F = F(t,x,u)$ and, moreover, $G = -F_u \int S\, du_x + \widehat{G}$, where $\widehat{G} = \widehat{G}(t,x,u)$. □

Note 6. A similar statement is true for an arbitrary (1+1)-dimensional evolution equation $\mathcal{L}$ of the even order $r = 2\bar{r}$, $\bar{r} \in \mathbb{N}$. For example [10], for any conservation law of $\mathcal{L}$ we can assume up to equivalence of conserved vectors that F and G depend only on t, x and derivatives of u with respect to x, and the maximal order of derivatives in F is not greater than $\bar{r}$.

Lemma 2 gives a stronger result for a more restricted class of equations. In the above proof we specially use the most direct method based on the definition of conservation laws to demonstrate its effectiveness in quite general cases. This proof can be easily extended to other classes of (1+1)-dimensional evolution equations of even orders and some systems connected with evolution equations [24].

Theorem 2. *A complete list of equations* (11) *having nontrivial conservation laws is exhausted by the following ones*

1. $A_u \neq 0,\ B = \beta_1 \int A$: $\quad (\varphi^i f u,\ -\varphi^i A u_x + \varphi^i_x \int A),\ \varphi^i,\ i = 1,2.$
2. $A_u \neq 0,\ B = \beta_1 \int A + \beta_2 u,\ \beta_2 \neq 0,\ h = f$:
   $(e^{-\beta_2 t}\varphi^i f u,\ e^{-\beta_2 t}(-\varphi^i A u_x + \varphi^i_x \int A)),\ e^{-\beta_2 t}\varphi^i,\ i = 1,2.$

3. $A=1,\quad B=\beta_1u+\beta_0\colon\quad (\,\alpha f u-\beta_0 h\int\alpha\,dt,\ -\alpha u_x+\alpha_x u\,),\ \alpha.$

Here β_1 and β_2 are arbitrary constants, $\alpha=\alpha(t,x)$ runs the set of solutions of the linear equation $f\alpha_t+\alpha_{xx}+\beta_1h\alpha=0$. The functions $\varphi^i=\varphi^i(x)$, $i=1,2$, form a fundamental system of solutions of the second-order linear ordinary differential equation $\varphi_{xx}+\beta_1h\varphi=0$. Hereafter $\int A=\int A\,du$.

(Together with constraints on the parameter-functions A, B, f and h we also adduce conserved vectors and characteristics of the basis elements of the corresponding space of conservation laws.)

Proof. In view of lemma 2, we can assume at once that $F=F(t,x,u)$ and $G=G(t,x,u,u_x)$. Let us substitute the expression for u_t deduced from (11) into (6) and decompose the obtained equation with respect to u_{xx}. The coefficient of u_{xx} gives the equation $AF_u+fG_{u_x}=0$, therefore $G=-Af^{-1}F_uu_x+\widehat{G}(t,x,u)$. Taking into account the latter expression for G and splitting the rest of equation (6) with respect to the powers of u_x, we obtain the system of PDEs on the functions F and $\widehat{G}$ of the form

$$F_{uu}=0,\quad -A\Big(\frac{F_u}{f}\Big)_x+\widehat{G}_u=0,\quad F_t+\frac{h}{f}BF_u+\widehat{G}_x=0. \tag{14}$$

Solving first two equations of (14) yields

$$F=\Phi(t,x)fu+F^0(t,x),\quad \widehat{G}=\Phi_x\int A+G^0(t,x),\quad \text{i.e.}\quad G=-\Phi Au_x+\Phi_x\int A+G^0(t,x).$$

(It is convenient to separate f as a multiplier in the coefficient of u in the expression of F.) In further consideration the major role is played by a differential consequence of system (14) that can be written as

$$f\Phi_t+A\Phi_{xx}+B_uh\Phi=0. \tag{15}$$

Indeed, it is the unique classifying condition for this problem. In all the classification cases we obtain the equation $F^0_t+G^0_x=0$. Therefore, we can assume $F^0=G^0=0$ up to conserved vectors equivalence and have to suppose for existence of non-trivial conserved vectors that $\Phi\neq0$.

It follows from (15) that equation (11) possesses non-trivial conserved vectors only for the special values of the parameter-functions if $B_u\in\langle1,A\rangle$. Hence it is sufficient to study the following cases. There exist three different possibilities for values of A and B_u.

1. $A\notin\langle1\rangle$, $B_u\in\langle1,A\rangle$. Therefore, $B_u=\beta_2+\beta_1A$, or $B=\beta_2u+\beta_1\int A$. (Without loss of generality we can take the same values of the integral of A as in the expression of G.) Equation (15) is split to two equations $\Phi_t+\beta_2\psi\Phi=0$ and $\Phi_{xx}+\beta_1h\Phi=0$ which imply, as a compatibility condition, the equation $\beta_2(2\psi_x\Phi_x+\psi_{xx}\Phi)=0$. Here the function ψ denotes the ratio h/f. Further consideration depends on values of β_2 and ψ.

If $\beta_2\neq0$ and $\psi\neq\text{const}$ then $\Phi=0$, hence equation (11) has only trivial conserved vectors.

The supposition $\beta_2=0$ implies $\Phi_t=0$, i.e. Φ depends only on x and runs the set of solutions of the second-order linear ordinary differential equation $\Phi_{xx}+\beta_1h\Phi=0$ (case 1).

If $\beta_2\neq0$ and $\psi=\text{const}$, we can put $\psi=1$ in view of gauge equivalence transformations in class (11). Therefore, $\Phi=e^{-\beta_2t}\varphi(x)$, where φ is an arbitrary solution of the same ordinary differential equation $\varphi_{xx}+\beta_1h\varphi=0$ (case 2).

2. $A,B_u\in\langle1\rangle$, i.e. A can be assumed equal to 1 without loss of generality, $B=\beta_1u+\beta_0$ and $f\Phi_t+\Phi_{xx}+\beta_1h\Phi=0$ (the linear case 3). □

Note 7. Taking into account transformations from the equivalence group $G_1^{\sim}$ of class (11) and additional equivalence transformations, we can reduce some parameters in cases of theorem 2. In particular, $\beta_2=1 \bmod G_1^{\sim}$ in case 2.

8 Hierarhy of conservation laws of diffusion–convection equations

As an example of exhaustive investigation of both local and potential conservation laws, we adduce results of [24] on the hierarchy of potential conservation laws and potential systems of the class of "constant coefficient" nonlinear diffusion–convection equations

$$u_t = (A(u)u_x)_x + B(u)u_x, \tag{16}$$

where $A = A(u)$ and $B = B(u)$ are arbitrary smooth functions of u, $A(u) \neq 0$.

The complete equivalence group $G^{\sim}$ (including the both continuous and discrete transformations) of class (16) is formed by the transformations

$$\tilde{t} = \varepsilon_4 t + \varepsilon_1, \quad \tilde{x} = \varepsilon_5 x + \varepsilon_7 t + \varepsilon_2, \quad \tilde{u} = \varepsilon_6 u + \varepsilon_3, \quad \tilde{A} = \varepsilon_4^{-1}\varepsilon_5^2 A, \quad \tilde{B} = \varepsilon_4^{-1}\varepsilon_5 B - \varepsilon_7,$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6 \neq 0$.

Theorem 3. *Any equation from class* (16) *has the conservation law* (6) *where*

1. $F = u, \qquad G = -Au_x - \int B.$

A complete list of $G^{\sim}$-inequivalent equations (16) *having additional conservation laws is exhausted by the following ones*

2. $\forall A, \quad B = 0: \qquad F = xu, \quad G = \int A - xAu_x,$
3. $\forall A, \quad B = A: \qquad F = (e^x + \varepsilon)u, \quad G = -(e^x + \varepsilon)Au_x - \varepsilon \int A,$
4. $A = 1, \quad B = 0: \qquad F = \alpha u, \quad G = \alpha_x u - \alpha u_x,$

where $\varepsilon \in \{0, \pm 1\} \bmod G^{\sim}$, $\int A = \int A(u)du$, $\int B = \int B(u)du$, $\alpha = \alpha(t, x)$ is an arbitrary solution of the backward linear heat equation $\alpha_t + \alpha_{xx} = 0$.

Using the conservation laws adduced in Theorem 3, we can introduce potentials for different values of the parameter-functions A and B and construct the corresponding potential systems. The important question for our consideration is whether the introduced potentials are locally independent. If we know the precise structure of conservation laws the answer is almost obvious.

Theorem 4. *For any equation* (16) *potentials are locally dependent on the equation manifold iff the corresponding conservation laws are linear dependent.*

We introduce potentials using the obtained conservation laws and investigate local conservation laws of the corresponding potential systems. These laws can be considered as nonlocal (*potential*) conservation laws of equations from class (16). The potential conservation laws obtained with attaching only one potential to the local variables are called *simplest* potential conservation laws. We classify conservation laws up to the equivalence relation with respect to the transformation group $G^{\sim}_{\rm pr}$ which is a result of the trivial prolongation of the group $G^{\sim}$ to the space of the potential v. Then we iterate the above procedure.

Up to $G^{\sim}$-equivalence the hierarchy of conservation laws for diffusion–convection equations (16) has the form:

- the "common" local conservation law (for any values of A and B);
- two independent local conservation laws if $B = 0$ or $B = A$;
- one "common" local conservation law and one simplest potential that with $F = e^v$, $G = -e^v \int A$ if $B = \int A + uA$;
- the infinite series of local conservation laws for the linear heat equation;

- one "common" local conservation law and the infinite series of simplest potential conservation laws with $F=\alpha e^v$, $G=\alpha_x e^v-\alpha u e^v$ for the Burgers equation $u_t=u_{xx}+2uu_x$;
- two independent local conservation laws for the u^{-2}-diffusion equation $u_t=(u^{-2}u_x)_x$ as a subcase of $B=0$ and additionally the infinite series of simplest potential conservation laws with $(F,G)=(\sigma,\sigma_v u^{-1})$;
- two independent local conservation laws for the equation $u_t=(u^{-2}u_x)_x+u^{-2}u_x$ as a subcase of $B=A$ and additionally the infinite series of simplest potential conservation laws with $(F,G)=(\sigma e^x,\sigma_v u^{-1}e^x)$.

Here $\alpha=\alpha(t,x)$ and $\sigma=\sigma(t,v)$ are arbitrary solutions of the backward linear heat equation, i.e. $\alpha_t+\alpha_{xx}=0$, $\sigma_t+\sigma_{vv}=0$. The potential v in the above simplest potential conservation laws is introduced with the "common" local conservation law and, therefore, determined by the "common" potential system $v_x=u,\ v_t=Au_x+\int B$. Similarity of the conservation law hierarchies for the equations in two latter cases is explained by the fact that the equation $u_t=(u^{-2}u_x)_x+u^{-2}u_x$ is reduced to the u^{-2}-diffusion equation by the additional equivalence transformation $\tilde t=t$, $\tilde x=e^x$, $\tilde u=e^{-x}u$.

The hierarchy of conservation laws generates the complete set of locally inequivalent potential systems for the class under consideration:

- the "common" potential system $v_x=u,\ v_t=Au_x+\int B$;
- the additional simplest potential systems
  $v_x=xu,\ v_t=xAu_x-\int A$ if $B=0$ and
  $v_x=(e^x+\varepsilon)u,\ v_t=(e^x+\varepsilon)Au_x+\varepsilon\int A$ if $B=A$;
- the second level potential systems
  $v_x=u,\ w_x=e^xv,\ w_t=e^x\int A$ if $B=0$ and
  $v_x=u,\ w_x=\sigma e^x,\ w_t=-\sigma_v u^{-1}e^x$ if $B=A$
  (which are equivalent to the united potential systems of the first level);
- the systems of the general form
  $v^s_x=\alpha^s u,\ v^s_t=\alpha^s u_x-\alpha^s_x u$, where $\alpha^s_t+\alpha^s_{xx}$, $s=\overline{1,p}$,
  with an arbitrary number p of locally independent potentials for the linear heat equation.

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