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Simplified models for turbulent diffusion: a test of the decorrelation trajectory method

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Abstract

Simplified models for the turbulent diffusion are studied in order to test a new statistical approach, the decorrelation trajectory method. The results are in good agreement with the analytical solutions obtained by other methods.

1 Introduction

Test particle motion in stochastic velocity fields is a generic problem in various topics of fluid and plasma turbulence or solid state physics [1], [2]. The main difficulty in determining the resulting time dependent (running) diffusion coefficient and mean square displacement consists in calculating the Lagrangian velocity correlation function (LVC). This is a very complex quantity which, for fundamental reasons, can be evaluated only approximatively from the knowledge of the statistical properties of the stochastic trajectories determined by the random velocity field. The physical parameter which characterizes such processes of diffusion by continuous movements is the Kubo number K (defined below) which measures the particle's capacity of exploring the space structure of the stochastic velocity field before it changes. In the weak turbulence case, $K \ll 1$, the time variation of the velocity field is fast and the particles cannot "see" the space structure of the velocity field. The latter is important in the strong turbulence case ($K > 1$) and influences the LVC and the K scaling of the diffusion coefficient. This problem has been studied extensively and solutions based on Corrsin [3], [1] and direct-interaction [4], [5] approximations have been found. Both types of methods are based on the hypothesis that the stochastic trajectories are Gaussian and Markovian.

In this context, particle motion in 2-dimensional divergence-free velocity fields represents a special case. Kraichnan has shown for the first time in a study based on numerical simulations [6] that the existing analytical methods are not adequate for this type of problems. The cause of this anomaly is the non-Gaussian behavior of the trajectories determined by the trapping of the particles in the space structure of such velocity fields. A more recent analysis of the effects of trapping is presented in [7] where a non-Gaussian peaked distribution of the displacements and a long negative tail in the LVC are evidenced for numerically calculated trajectories. The latter is shown to be associated with non-Markovian behavior of the stochastic trajectories [8]. The 2-dimensional divergence-free velocity field could have been studied until now especially by means of direct numerical simulations (see [9] and the reference there in) or on the basis of simplified models [11], [10]. There is only one qualitative theoretical estimation [12], which is in agreement with the numerical calculations [9]. It is based on an analogy with the percolation process in stochastic landscapes and determines the scaling law of the diffusion coefficient in the large Kubo number. The case of collisional particle motion in such stochastic velocity fields could be analyzed by means of the renormalization group techniques [2] and the asymptotic time behavior of the mean square displacement was determined.

In a recent work [13] a rather different statistical approach (the decorrelation trajectory method) was proposed for determining the LVC for given Eulerian correlation of the velocity field. The case of collisional particles was treated in [14] and the influence of an average velocity was analyzed in [15] by means of the decorrelation trajectory method. The basic idea consists of determining an approximate form of the correlation of the Lagrangian velocity by means of a set of average Lagrangian velocities estimated in *subensembles of realizations of the stochastic field*. These subensembles are defined by given values of the velocity and of the potential in the starting point of the trajectories. It was shown that the statistical process of trapping is related to the invariance of the potential along the trajectories. The decorrelation trajectory method preserves this property and it is thus able to describe the trapping process. Its results give a clear image of the physical process and are in qualitative agreement with numerical simulations. However a detailed study of the accuracy of this method is not yet performed. A test of the method is presented in this paper. We consider some simplified diffusion-advection problems [11] which have known analytical solutions and compare these solutions with the results obtained with our method.

The paper is organized as follows. The problem of particle diffusion in stochastic potentials and the decorrelation trajectory method are presented in Section 2. Then, in Section 3, we introduce the simplifications that permitted to obtain analytical results by means of other methods and we study these models in the framework of the decorrelation trajectory method. Section 4 contains the conclusions.

2 The decorrelation trajectory method

Particle motion in a 2-dimensional stochastic velocity field is described by the nonlinear Langevin equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}\left[\mathbf{x}(t), t\right], \qquad \mathbf{x}(0) = \mathbf{0} \tag{1}$$

where $\mathbf{x}(t)$ represents the trajectory in Cartesian coordinates $\mathbf{x} \equiv (x_1, x_2)$. The stochastic velocity field $\mathbf{v}(\mathbf{x}, t)$ is divergence-free: $\boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}, t) = 0$ and thus its two components v_1 an v_2 can be determined from a stochastic scalar field $\phi(\mathbf{x}, t)$, as:

$$\mathbf{v}(\mathbf{x}, t) = \left(\frac{\partial \phi(\mathbf{x}, t)}{\partial x_2}, \quad -\frac{\partial \phi(\mathbf{x}, t)}{\partial x_1} \right) \tag{2}$$

In the studies of turbulence in magnetized plasmas, $\phi(\mathbf{x}, t)$ is essentially the potential ($\phi = -\phi^e/B$ where $\phi^e(\mathbf{x}, t)$ is the electrostatic potential and B is the magnetic field strength assumed to be constant) and in fluid turbulence $\phi(\mathbf{x}, t)\mathbf{e}_3$ is the stream function ($\mathbf{e}_3$ is the unitary vector along the axis perpendicular on the plane (x_1, x_2)). The potential $\phi(\mathbf{x}, t)$ is considered to be a stationary and homogeneous Gaussian stochastic field, with zero average. The Eulerian two-point correlation function (EC) of $\phi(\mathbf{x}, t)$ is assumed to be of factorized form:

$$E(\mathbf{x}, t) \equiv \langle \phi(\mathbf{x}_1, t_1)\, \phi(\mathbf{x}_1 + \mathbf{x}, t_1 + t) \rangle = \beta^2 \mathcal{E}(\mathbf{x})\, h(t) \tag{3}$$

where β measures the amplitude of the potential fluctuations and $\langle ... \rangle$ denotes the statistical average over the realizations of $\phi(\mathbf{x}, t)$. As obtained in most experimental measurements, the dimensionless function $\mathcal{E}(\mathbf{x})$ has a maximum at $\mathbf{x} = \mathbf{0}$, where $\mathcal{E}(\mathbf{0}) = 1$, and tends to zero as $|\mathbf{x}| \to \infty$. The dimensionless, decreasing function of time $h(t)$ varies from $h(0) = 1$ to $h(\infty) = 0$.

The statistical properties of the velocity components are completely determined by those of the potential: they are stationary and homogeneous Gaussian stochastic fields like $\phi(\mathbf{x}, t)$. The two-point Eulerian correlations of the velocity components, $E_{ij}(\mathbf{x}, t) \equiv \langle v_i(\mathbf{x}_1, t_1)\, v_j(\mathbf{x}_1 + \mathbf{x}, t_1 + t) \rangle$, and the potential-velocity correlations, $E_{\phi i} \equiv \langle \phi(\mathbf{x}_1, t_1)\; v_i(\mathbf{x}_1 + \mathbf{x}, t_1 + t) \rangle$, are

$$\begin{aligned} E_{11} &= -\frac{\partial^2}{\partial x_2^2} E, \quad E_{22} = -\frac{\partial^2}{\partial x_1^2} E, \quad E_{12} = E_{21} = \frac{\partial^2}{\partial x_1 \partial x_2} E, \\ E_{1\phi} &= -E_{\phi 1} = -\frac{\partial}{\partial x_2} E, \quad E_{2\phi} = -E_{\phi 2} = \frac{\partial}{\partial x_1} E. \end{aligned} \tag{4}$$

The characteristic parameters of the (isotropic) stochastic velocity field are: the amplitude $V = \sqrt{E_{11}(\mathbf{0}, 0)}$, the correlation time τ_c defined by

$$\tau_c = \int_0^{\infty} E_{11}(\mathbf{x} = \mathbf{0}, t)\, dt \tag{5}$$

and the correlation length λ_c defined by

$$\lambda_c = \int_{-\infty}^{\infty} E_{11}(x_1, x_2 = 0, t = 0)\, dx_1. \tag{6}$$

These three units combine in a dimensionless Kubo number

$$K = \frac{V\tau_c}{\lambda_c} \tag{7}$$

which is the ratio of the average distance covered by the particles during τ_c to λ_c, or, equivalently, the ratio of τ_c to the average time of flight of the particles over the correlation length, $\tau_{fl} = \lambda_c/V$. Using dimensionless quantities for $\mathbf{x}$, t, $\mathbf{v}$ scaled with the units λ_c, τ_c, V, the equation of motion (1) becomes:

$$\frac{d\mathbf{x}(t)}{dt} = K\mathbf{v}\left[\mathbf{x}(t), t\right], \qquad \mathbf{x}(0) = \mathbf{0}. \tag{8}$$

The velocity unit V is related to the potential unit β by $V = c\beta/\lambda_c$ where c is a constant determined by $c = \sqrt{-\,\partial^2\mathcal{E}(\mathbf{x})/\partial x_2^2|_{\mathbf{x}=\mathbf{0}}}$. In the static case $\tau_c = \infty$, the natural unit of time is the flight time τ_{fl} and the dimensionless equation of motion has the same expression as Eq.(1).

Starting from this statistical description of the stochastic potential, we determine the Lagrangian velocity correlation (LVC), defined by:

$$L_{ij}(t) \equiv \langle v_i\left[\mathbf{x}(0), 0\right]) v_j\left[\mathbf{x}(t), t\right]\rangle\,. \tag{9}$$

and the running diffusion coefficient which is the integral of the LVC [16]:

$$D_i(t) = \int_0^t d\tau\; L_{ii}(\tau). \tag{10}$$

For small Kubo numbers (quasilinear regime), the results are well established: the diffusion coefficient is $D_{QL} = (\,\lambda_c^2/\tau_c)K^2$. For large K the time variation of the stochastic potential is slow and the trajectories approximately follow the contour lines of $\phi(\mathbf{x}, t)$. This produces a trapping effect : the trajectories are confined for long periods in small regions. A typical trajectory shows an alternation of large displacements and trapping events. The latter appear when the particles are close to the maxima or minima of the potential and consist of trajectory winding on almost closed small size paths. The large displacements are produced when the trajectories are at small absolute values of the potential. The most important effect of trajectory trapping consists of decreasing the diffusion coefficient and of changing its dependence on the Kubo number from the Bohm scaling [18], [19] ($D_B \sim (\,\lambda_c^2/\tau_c)K$) to a trapping scaling ($D_{tr} \sim (\,\lambda_c^2/\tau_c)K^{\gamma}$) with $\gamma < 1$. The first estimation of γ is based on an analogy with the percolation in stochastic landscapes [12] and yields $\gamma = 0.7$. This value appears as a critical exponent valid for any EC of the potential that decays fast enough when $\mathbf{x} \to \infty$.

The main idea in our method is to study the Langevin equation (8) in subensembles S of realizations of the stochastic field, which are determined by given values of the potential and of the velocity in the starting point of the trajectories:

$$\phi(\mathbf{0}, 0) = \phi^0, \quad \mathbf{v}(\mathbf{0}, 0) = \mathbf{v}^0. \tag{11}$$

The LVC (9) for the whole set of realizations is obtained by summing up the contributions of each subensemble. The latter can be written as $\langle v_i\left[\mathbf{x}(0), 0\right]\; v_j\left[\mathbf{x}(t), t\right]\rangle_S = v_i^0\; \langle v_j\left[\mathbf{x}(t), t\right]\rangle_S$ where $\langle ...\rangle_S$ denotes the average in S. The LVC (9) is represented by:

$$L_{ij}(t) = \int\!\!\int d\phi^0\, d\mathbf{v}^0\, P_1(\phi^0, \mathbf{v}^0, \mathbf{0}, 0)\; v_i^0\; \langle v_j\left[\mathbf{x}(t), t\right]\rangle_S \tag{12}$$

where $P_1(\phi^0, \mathbf{v}^0, \mathbf{0}, 0)$ is the Gaussian probability density for the values of the potential and velocity in the point $(\mathbf{0}, 0)$.

Thus, the problem of evaluating the LVC reduces to the determination of the average Lagrangian velocity in each subensemble S. This is one of the advantages brought by the subensemble analysis: the two-point LVC (9) can be expressed as a function of one-point averages ($\langle v_j\left[\mathbf{x}(t),t\right]\rangle_S$) in subensembles which correspond to given initial velocity. Another important advantage comes from the definition (2) of the subensembles. The general idea consists of introducing in this definition not only the initial velocity but also *the "initial" values of the invariants of the motion*. Then, the Lagrangian averages of these invariant quantities are known in each subensemble and the relation between them and the trajectory can be used in order to find some deterministic trajectory such that the subensemble average of the Eulerian invariant calculated along this trajectory equal the (known) Lagrangian average of the invariant. In the problem studied here the motion invariant is the potential. This is the reason why the subensembles were defined by Eq.(2) that contains the initial potential ϕ^0. In the static case ($\tau_c \to \infty$, hence $K \to \infty$), the potential is an exact invariant along the trajectory in each realization:

$$\frac{d\phi\left[\mathbf{x}(t)\right]}{dt} = \frac{\partial\phi\left[\mathbf{x}(t)\right]}{\partial x_i}\frac{dx_i}{dt} = 0 \tag{13}$$

due to the definition of the velocity (2) which shows that the latter is at any moment tangent to the contour line of the potential, thus perpendicular to $\boldsymbol{\nabla}\phi$. In the time-dependent case there is a variation of the potential along the trajectory that is determined only by the explicit time dependence:

$$\frac{d\phi\left[\mathbf{x}(t),t\right]}{dt} = \frac{\partial\phi\left[\mathbf{x}(t),t\right]}{\partial t}, \tag{14}$$

since the velocity is still tangent to the local contour line of the potential.

As suggested by the above discussions, the following steps are pursued in the development of the method. First, the statistical properties of the stochastic potential and velocity, reduced in the subensemble S defined by condition (2) are derived. Then, the subensemble average Lagrangian velocity is evaluated. Finally, analytical expressions for the LVC and the running diffusion coefficient are obtained on the basis of Eq.(12).

2.1 Eulerian statistics in the subensemble S

The probability density for the potential in the subensemble S is the conditional probability corresponding to (2) and is deduced from the 2-point probability of having the potential ϕ in $\mathbf{x}$, t and ϕ^0 and $\mathbf{v}^0$ in $\mathbf{x} = \mathbf{0}$, $t = 0$ (see e.g. [17]). One can show that the potential in the subensemble is a non-stationary and non-homogeneous Gaussian field having a space-time dependent average

$$\Phi^S(\mathbf{x},t) \equiv \langle\phi(\mathbf{x},t)\rangle_S = \phi^0\frac{E(\mathbf{x},t)}{E(\mathbf{0},0)} + v_1^0\frac{E_{1\phi}(\mathbf{x},t)}{E_{11}(\mathbf{0},0)} + v_2^0\frac{E_{2\phi}(\mathbf{x},t)}{E_{22}(\mathbf{0},0)} \tag{15}$$

The subensemble average potential (15) equals ϕ^0 in $\mathbf{x} = \mathbf{0}$, $t = 0$ and it decays to zero as $\mathbf{x} \to \infty$ and/or $t \to \infty$. As in the whole ensemble, the statistical properties of the velocity field in the subensemble S are deduced from those of the potential in S. The probability density for the velocity in the subensemble (2) is a non-stationary and non-homogeneous Gaussian distribution with an average given by

$$V_i^S(\mathbf{x},t) \equiv \langle v_i(\mathbf{x},t)\rangle_S = \phi^0\frac{E_{\phi i}(\mathbf{x},t)}{E(\mathbf{0},0)} + v_1^0\frac{E_{1i}(\mathbf{x},t)}{E_{11}(\mathbf{0},0)} + v_2^0\frac{E_{2i}(\mathbf{x},t)}{E_{22}(\mathbf{0},0)}. \tag{16}$$

which depends on the subensemble parameters ϕ^0, $\mathbf{v}^0$; it equals $\mathbf{v}^0$ in $\mathbf{x} = \mathbf{0}$, $t = 0$ and decays to zero as $\mathbf{x} \to \infty$ and/or $t \to \infty$.

The average velocity (16) and the average potential (15) in S are coupled by a relation similar to (2) which can easily be deduced using Eqs.(4):

$$\mathbf{V}^S(\mathbf{x},t) = \left(\frac{\partial\Phi^S(\mathbf{x},t)}{\partial x_2}, \quad -\frac{\partial\Phi^S(\mathbf{x},t)}{\partial x_1}\right). \tag{17}$$

It shows that the average velocity in the subensemble S is divergence-free: $\boldsymbol{\nabla}\cdot\mathbf{V}^S(\mathbf{x},t) = 0$.

It is interesting to note that the potential and the velocity in the subensemble S are deterministic quantities in $\mathbf{x} = \mathbf{0}$ and $t = 0$ ($\phi(\mathbf{0},0) = \phi^0$, $\mathbf{v}(\mathbf{0},0) = \mathbf{v}^0$ for all realizations in S). As $|\mathbf{x}|$ and/or t grow, the average values decay to zero and the fluctuations build up progressively and eventually become the same as in the global statistical ensemble.

Thus, in the zero-average stochastic velocity field, *a set of average velocities* (labeled by ϕ^0, $\mathbf{v}^0$) was identified. It contains the statistical characteristics of the velocity field (the correlation and the constraint imposed in the problem, i.e. the zero-divergence condition). We have to determine the average Lagrangian velocity, corresponding the Eulerian average velocity (16), in each subensemble S. According to Eq.(12) this is sufficient to determine the global LVC and the running diffusion coefficient.

2.2 Average Lagrangian velocity in the subensemble S

We first consider the static case $\phi(\mathbf{x})$, corresponding to $\tau_c \to \infty$, $K \to \infty$ and $h(t) = 1$ in the EC of the potential (3). The natural unit of time is $\tau_{fl} = \lambda_c/V$, defined in Section II and the dimensionless equation of motion is (1). The potential is an invariant of the motion: the Lagrangian potential in each realization in S is $\phi\left[\mathbf{x}(t)\right] = \phi(\mathbf{0}) = \phi^0$ and consequently its average in S is:

$$\langle \phi\left[\mathbf{x}(t)\right]\rangle_S = \phi^0 \tag{18}$$

at any time. Actually, in the Lagrangian frame, there is only one non-zero cumulant of the potential in the subensemble (the average (18)) since the fluctuation of the Lagrangian potential is zero at any time. Thus, when passing from Eulerian to Lagrangian coordinates, the distribution function of the potential in the subensemble S simplifies considerably: it transforms from a non-homogeneous Gaussian distribution with space-dependent average and dispersion into a deterministic distribution $\delta\left[\phi\left[\mathbf{x}(t)\right] - \phi^0\right]$.

We define in each subensemble S a *deterministic trajectory* $\mathbf{X}(t; S)$ such that the average of the Eulerian potential in S calculated along this trajectory equals the average Lagrangian potential:

$$\langle \phi\left[\mathbf{X}(t;S)\right]\rangle_S = \langle \phi\left[\mathbf{x}(t)\right]\rangle_S = \phi^0. \tag{19}$$

This means that in each realization in S, the trajectory $\mathbf{X}(t; S)$ is not on the contour line of the potential like the real trajectory $\mathbf{x}(t)$, but it is on the contour line of the average potential in S. Thus, the invariant $\Phi^S\left[\mathbf{X}(t; S)\right]$ plays the role of the Hamiltonian: the trajectory $\mathbf{X}(t; S)$ can be determined from the following Hamiltonian system of equations

$$\frac{d\mathbf{X}(t;S)}{dt} = \left(\frac{\partial \Phi^S\left[\mathbf{X}(t;S)\right]}{\partial X_2}, \quad -\frac{\partial \Phi^S\left[\mathbf{X}(t;S)\right]}{\partial X_1}\right) \tag{20}$$

with the initial condition is $\mathbf{X}(0; S) = \mathbf{0}$. The solution of this equation ensures the invariance of its time-independent Hamiltonian and, since the Eulerian average potential (15) has the value ϕ^0 in $\mathbf{x} = \mathbf{0}$, the required condition (18) is fullfiled. The average Lagrangian velocity in the subensemble S is approximated by the corresponding Eulerian quantity calculated along the deterministic trajectory $\mathbf{X}(t; S)$:

$$\langle \mathbf{v}\left[\mathbf{x}(t)\right]\rangle_S \cong \mathbf{V}^S\left[\mathbf{X}(t;S)\right] \tag{21}$$

This is the approximation on which the decorrelation trajectory method is based.

In the case of time-dependent potentials $\phi(\mathbf{x}, t)$ (finite τ_c and K), if the space and time dependences are statistically independent such that the EC of the potential is factorized like in Eq.(3), it is also possible to determine the average Lagrangian potential in S and properties similar to those obtained in the static case are found. The Lagrangian potential is not constant along the trajectory but the velocity is still perpendicular to $\boldsymbol{\nabla}\phi\left[\mathbf{x}(t), t\right]$ at any moment and only the explicit time-dependence contributes to the variation of ϕ along the trajectory which is given by Eq.(14). Due to the factorized EC (3) considered here, the average Eulerian potential and velocity (15), (16) have a similar factorized structure. It can be shown that the average Lagrangian potential in S is:

$$\langle \phi\left[\mathbf{x}(t), t\right]\rangle_S = \phi^0\, h(t). \tag{22}$$

We define in S a deterministic trajectory $\overline{\mathbf{X}}(t; S)$ as the solution of the time-dependent Hamiltonian system with $\Phi^S(\overline{\mathbf{X}}, t)$ as Hamiltonian function:

$$\frac{d\overline{\mathbf{X}}(t;S)}{dt} = K\left(\frac{\partial \Phi^S\left[\overline{\mathbf{X}}(t;S)\right]}{\partial \overline{X}_2}, \quad -\frac{\partial \Phi^S\left[\overline{\mathbf{X}}(t;S)\right]}{\partial \overline{X}_1}\right) h(t) \tag{23}$$

Performing the change of variable $t \to \theta(t)$ defined by:

$$\theta(t) = K \int_0^t d\tau \, h(\tau), \tag{24}$$

the time dependent Hamiltonian system reduces to Eq.(20) and thus the trajectory $\overline{\mathbf{X}}(t;S)$ can be written as:

$$\overline{\mathbf{X}}(t;S) = \mathbf{X}\left[\theta(t);S\right] \tag{25}$$

where $\mathbf{X}(\theta;S)$ is the deterministic trajectory obtained in the static case. Using Eq.(22) and the definition of $\overline{\mathbf{X}}(t;S)$, one finds that the average Eulerian potential calculated along the deterministic trajectory $\overline{\mathbf{X}}(t;S)$ equals, as in the static case, the average Lagrangian potential:

$$\Phi^S \left[\overline{\mathbf{X}}(t;S)\right] h(t) = \left\langle \phi\left[\mathbf{x}(t), t\right]\right\rangle_S = \phi^0 \, h(t). \tag{26}$$

The average Lagrangian velocity in S is approximated by:

$$\left\langle \mathbf{v}\left[\mathbf{x}(t), t\right]\right\rangle_S \cong \mathbf{V}^S \left[\overline{\mathbf{X}}(t;S)\right] \, h(t). \tag{27}$$

The LVC (9) for the global statistical ensemble of realizations is determined using Eqs. (12) and (27). The average Lagrangian velocity (27) is obtained by calculating the decorrelation trajectories in the subensembles. The running diffusion coefficient is obtained from Eq.(12) by time integration

$$D_{ij}(t) = \int\int d\phi^0 \, d\mathbf{v}^0 \, P_1(\phi^0, \mathbf{v}^0, \mathbf{0}, 0) \; v_i^0 \, \overline{X_j}(t;S). \tag{28}$$

3 Results for the simplified models

The aim of this paper is to study the accuracy of the approximation (21) or (27) for some simplified problems which have been studied by completely different methods. They are based on particular choices of the above general stochastic potential. We consider two problems. In the first case, 1-dimensional stochastic potentials are considered depending only on one space variable, x_1 [20], [12]. In the second case the stochastic potential is a sum of two independent functions each depending on one component of $\mathbf{x}$ [21], [22], [23].

The above models contain, besides the velocity determined by the potential, an additional component which is a time dependent Gaussian noise. It determines a Brownian component of the motion that represents particle collisions. We have studied in [14] such doubly stochastic process for a general potential of the type described in the above section. The stochastic velocity $\boldsymbol{\eta}(t)$ determines a collisional displacement $\boldsymbol{\xi}(t) = \int_0^t d\tau \; \boldsymbol{\eta}(\tau)$ which is a Gaussian non-stationary Markov stochastic variable known as the Wiener process. Since the collisions and the stochastic potential are statistically independent variables, one can perform the change of variable

$$\mathbf{x}'(t) = \mathbf{x}(t) - \boldsymbol{\xi}(t) \tag{29}$$

which leads to equations similar to (1) but with the collisional displacements contained in the argument of the potential which becomes a doubly stochastic variable (stochastic function of a stochastic variable). We have show that the doubly stochastic field $\phi(\mathbf{x} + \boldsymbol{\xi}(t), t)$ preserves the statistical properties of the potential $\phi(\mathbf{x}, t)$, namely it is stationary, homogeneous, isotropic and has a Gaussian one-point probability density like $\phi(\mathbf{x}, t)$. The average effect of the collisional noise $\boldsymbol{\eta}(t)$ consists of the modification of the EC of the potential. More specifically, the space dependence of the correlation $\mathcal{E}(\mathbf{x})$ is transformed into

$$\mathcal{E}^{coll}(\mathbf{x}, t) \equiv \int\int d^2\xi \; \mathcal{E}(\mathbf{x} + \boldsymbol{\xi}) \; P_1^c\left(\boldsymbol{\xi}, t\right) \tag{30}$$

getting an additional time dependence. Here

$$P_1^c\left(\boldsymbol{\xi}, t\right) = \frac{1}{4\pi\chi t} \exp\left(-\frac{\xi^2}{4\chi t}\right) \tag{31}$$

is the probability distribution of the collisional displacements and χ is the collisional diffusion coefficient. As $\mathcal{E}^{coll}(\mathbf{x},t)$ is the solution of the diffusion equation

$$\frac{\partial}{\partial t}P_1^c(\boldsymbol{\xi},t)=\chi\left(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}\right)P_1^c(\boldsymbol{\xi},t), \tag{32}$$

the average effect of collisions consists of smoothing out progressively the EC of the potential and of eliminating asymptotically the space dependence of the correlation. Thus, collisional particle diffusion in stochastic potentials can be studied with the decorrelation trajectory method as presented in the previous section.

3.1 Slab turbulence model

In this specific model the stochastic potential is invariant along x_1 axis and depends only on x_2. Introducing particle collisions, the equations of motion (1) becomes

$$\frac{dx_1}{dt}=\frac{d\phi(x_2)}{dx_2}+\eta_1(t),\quad \frac{dx_2}{dt}=\eta_2(t). \tag{33}$$

The nonlinearity is so eliminated and the problem simplifies considerably. The Eulerian correlation of this potential is represented by Eq.(3) with $h=1$ and $\mathcal{E}=\mathcal{E}(x_2)$.

The collisional displacement is introduced in the argument of the potential by the change of variable (29) and the Eulerian correlation of the potential $\phi(x_2+\xi_2(t))$ is determined according to Eq.(30). The decorrelation trajectories can be written explicitly as

$$\frac{dX_1}{dt}=\frac{d}{dX_2}\left(v_1^0\frac{d}{dX_2}+\phi^0\right)\mathcal{E}^{coll}(X_2,t),\quad \frac{dX_2}{dt}=0$$

where Eqs.(15) and (4) were used. The solution is $X_2(t;S)=0$ and

$$X_1(t;S)=v_1^0\int_{-\infty}^{\infty}d\xi\frac{d^2\mathcal{E}}{d\xi^2}\int_0^t d\tau P_1^c(\xi,\tau).$$

Integrating two times by parts and using the diffusion equation (32) for the Gaussian probability $P_1^c(\xi,\tau)$, one obtains

$$X_1(t;S)=\frac{v_1^0}{\chi}\int_{-\infty}^{\infty}d\xi\,\mathcal{E}(\xi)\,\left[P_1^c(\xi,\tau)-P_1^c(\xi,0)\right].$$

The running diffusion coefficient along x_1 axis is obtained from Eq.(28) as

$$D_{11}(t)=\chi+\frac{1}{\chi}\int_{-\infty}^{\infty}dv_1^0P(v_1^0)v_1^0X_1(t;S)$$

where χ, the direct contribution of the collisional noise $\boldsymbol{\eta}(t)$, was introduced and the integral over ϕ^0 was performed. One obtains

$$D_{11}(t)=\chi-\frac{1}{\chi}\int_{-\infty}^{\infty}d\xi\;\mathcal{E}(\xi)\;\left[P_1^c(\xi,\tau)-P_1^c(\xi,0)\right]$$

and since $P_1^c(\xi,0)=\delta(\xi)$

$$D_{11}(t)=\chi+\frac{\mathcal{E}(0)}{\chi}-\frac{1}{\chi}\int_{-\infty}^{\infty}d\xi\;\mathcal{E}(\xi)\,P_1^c(\xi,\tau). \tag{34}$$

Thus the time dependent diffusion coefficient is determined for given Eulerian correlation of the potential. We show that this result determines the known asymptotic diffusion coefficient for both short and long range correlations. The short range correlations are characterized by integrable functions $\mathcal{E}(\xi)$. In this case the integral in Eq.(34) can be approximated at large time by $(4\pi\chi t)^{-1/2}\int_{-\infty}^{\infty}d\xi\;\mathcal{E}(\xi)=const.(4\pi\chi t)^{-1/2}$ which shows that the last term decays to zero and the asymptotic diffusion coefficient is

$$D_{11}(t)=\chi+\frac{\mathcal{E}(0)}{\chi}$$

This is the exact result obtained in [20], [12]. The long range correlated potentials have divergent Eulerian correlations with $\mathcal{E}(\mathbf{x}) \sim |\mathbf{x}|^{2H}$ at large distances. The exponent H is a positive subunitary number, $0 < H < 1$. Thus the EC of the potential goes to infinity at large distances but the EC of the velocity (4) decays to zero, as imposed by the physical relevance of the problem, $E_{11}(\mathbf{x}) \sim |\mathbf{x}|^{2H-2} \to 0$. The integral in Eq.(34) can be approximated at large time by

$$\int_{-\infty}^{\infty} d\xi \; \mathcal{E}(\xi) \, P_1^c(\xi,\tau) \cong \frac{1}{\sqrt{2\chi t}} \int_{-\sqrt{2\chi t}}^{\sqrt{2\chi t}} d\xi \; \xi^{2H} \approx (\chi t)^H .$$

Thus the diffusion coefficient is a growing function of time, which means that particle transport in such long range correlated potentials is superdiffusive. The asymptotic time dependence of the mean square displacement can be easily obtained as

$$\left\langle x_1^2(t) \right\rangle^{1/2} \approx V \left(\frac{\lambda_c^2}{\chi} \right)^{(1-H)/2} t^{(1+H)/2}$$

which is the same result as in [20].

3.2 The Manhattan system

A generalization of the above problem to a 2-dimensional system is obtained by considering stochastic potentials of the type $\phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$ where ϕ_1 and ϕ_2 are independent stochastic functions of one argument. The equations of motion (1) for this potential are

$$\frac{dx_1}{dt} = \frac{d\phi_2(x_2)}{dx_2} + \eta_1(t), \quad \frac{dx_2}{dt} = -\frac{d\phi_1(x_1)}{dx_1} + \eta_2(t)$$

We consider for simplicity that the two stochastic functions have the same Eulerian correlation, The name of this problem was suggested by the solutions of these equations which are similar with random trajectories on a grid that could model the streets of Manhattan. The Eulerian correlation of this potential is $E(x_1, x_2) = e(x_1) + e(x_2)$ and the correlations for the velocity components obtained using (4) are $E_{11}(x_1, x_2) = -e''(x_2)$, $E_{22}(x_1, x_2) = -e''(x_1)$ and $E_{12}(x_1, x_2) = 0$, where $e''(x)$ is the second derivative of $e(x)$.

In the absence of collisions ($\boldsymbol{\eta}(t) = 0$) the average potential in the subensemble (15) which is the Hamiltonian of the decorrelation trajectories (20) is

$$\Phi^S[\mathbf{X}] = v_1^0 e'(X_2) - v_2^0 e'(X_1) + \phi^0 \left(e(X_1) + e(X_2) \right) \tag{35}$$

Some examples of decorrelation trajectories obtained by solving Eq.(20) for this potential are shown in Figures 1 and 2. A short range potential with $e(x) = 1/(1 + x^2/2)$ is considered in Figure 1 and a long range potential with $e(x) = 1/(1 + x^2/2) - \exp(1/|x|) \, |x|^{2H}$, $H = 5/8$, in Figure 2. One can note an important difference in the topology of the two sets of trajectories. In the first case (short range potential), there is a dense set of opened trajectories and it can be easily shown that the motion on these trajectories is asymptotically free, $\mathbf{X}(t) \sim t$. The diffusion coefficient (28) is thus linear in t at large t showing that the transport is superdiffusive of ballistic type. In a long range potential with $e(x) \approx 1/|x|^{5/4}$ at large distances, the decorrelation trajectories are similar with those obtained in [13] for the general stochastic potential presented in Section 2. The trajectories lie on closed paths (except for $\phi^0 = 0$ which correspond to a straight line along $\mathbf{v}^0$). They are periodic functions of time with period which increases with the size of the path. Thus, for a given time t, the trajectories corresponding to small paths rotate many times, while those along large enough paths are still open. Consequently, when calculating the integrals in Eq.(28), the contribution of the small paths is progressively eliminated as t increases, due to incoherent mixing in the integral. This is the trapping process described in [13] which leads to a subdiffusive transport with $D(t)$ decaying algebraically to zero.

In the presence of collisional diffusivity ($\boldsymbol{\eta}(t) \neq 0$) the transport in the long range correlated potential is completely changed: it becomes superdiffusive. In order to determine the asymptotic diffusion coefficient,

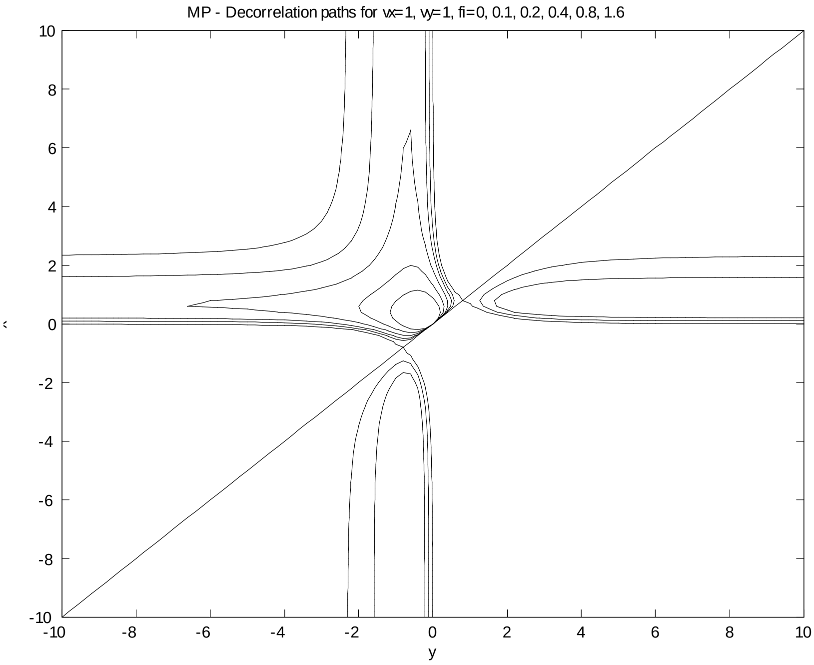

Figure 1: Decorrelation trajectory differing by the value of the potential ϕ^0 in a short range potential.

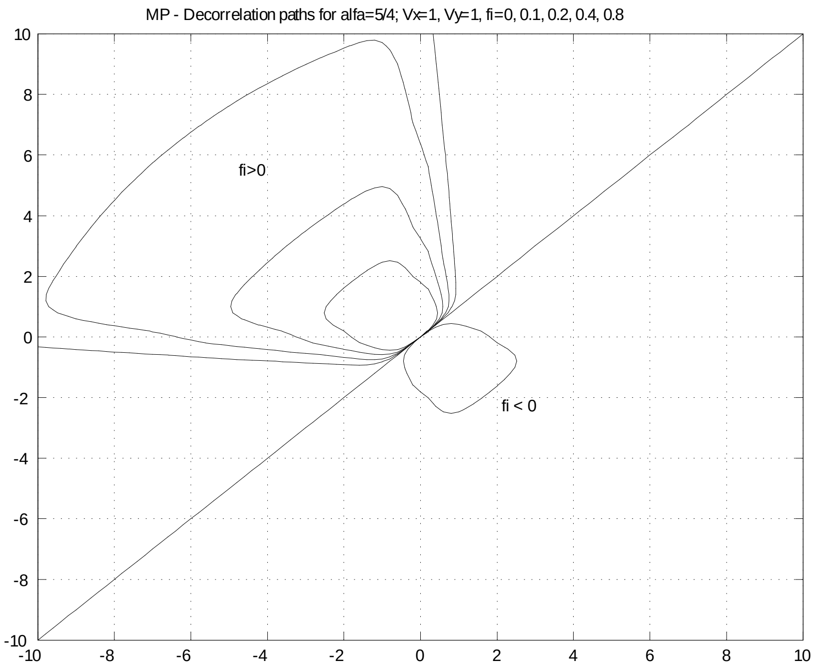

Figure 2: Decorrelation trajectories differing by the value of the potential ϕ^0 in a long range potential.

we first determine the Eulerian correlation of the potential $\phi_1(x+\xi(t))$ at large time such that $2\chi t \gg \lambda_c^2$. Using Eq.(30), the latter can be written as

$$
\begin{aligned}
e^{coll}(x,t) &\cong \frac{1}{\sqrt{2\pi}} \int d\xi \left| x+\xi\sqrt{2\chi t}\right|^{2H} \exp\left(-\frac{\xi^2}{2}\right) \\
&\cong \frac{1}{\sqrt{2\pi}} (2\chi t)^H \int d\xi \, |\xi|^{2H} \exp\left(-\frac{\xi^2}{2}\right)
\end{aligned}
$$

Similar results are obtained for the derivatives of this function. The equations for the decorrelation trajectories (23) at large time (when the collisions have already smoothed out the space dependence of the correlations) can be approximated by

$$
\frac{dX_1}{dt} \cong v_1^0 \left(\frac{2\chi t}{\lambda_c^2}\right)^{H-1} c_1 + \phi^0 \left(\frac{2\chi t}{\lambda_c^2}\right)^{H-1/2} c_2
$$

$$
\frac{dX_2}{dt} \cong v_2^0 \left(\frac{2\chi t}{\lambda_c^2}\right)^{H-1} c_1 - \phi^0 \left(\frac{2\chi t}{\lambda_c^2}\right)^{H-1/2} c_2
$$

where c_1 and c_2 are constants which can be calculated. After integrating these equations and using Eq.(28) one obtains an isotropic diffusion coefficient with the following asymptotic behavior

$$
D(t) \to c_1 V^2 \tau_d \left(\frac{t}{\tau_d}\right)^H \tag{36}
$$

where $\tau_d = \lambda_c^2/2\chi$ is the characteristic time for the collisional decorrelation. Thus the transport of collisional particles in long range potential is superdiffusive. The asymptotic mean square displacement obtained from Eq.(36) is

$$
\left\langle x_1^2(t)\right\rangle^{1/2} \approx V\tau_d \left(\frac{t}{\tau_d}\right)^{(H+1)/2}.
$$

Using completely different methods, this quantity was evaluated in [21] as

$$
\left\langle x_1^2(t)\right\rangle^{1/2} \approx \lambda_c \left(\frac{Vt}{\lambda_c}\right)^{1/(2-H)}
$$

Thus, both methods obtain a superdiffusive transport. The time exponents are however different. The relative difference between them remains small: it has a maximum of about 11% and goes to zero at the two limits of the range of $H \in [0,1]$.

4 Conclusions

We have developed in the last years a new approach for studying the particle transport in stochastic potentials, the decorrelation trajectory method. It determines analytical expressions for the Lagrangian velocity correlation and for the time dependent diffusion coefficient which are valid for arbitrary values of the Kubo number and describe the complicated process of dynamic trajectory trapping in the structure of the stochastic field. A study of the accuracy of this method is presented. Two problems of advection-diffusion are studied using the decorrelation trajectory method and the solutions are compared with known analytical results obtained in the literature with different methods. A perfect agreement is obtained for the slab turbulence model. In the case of the Manhattan system a superdiffusive transport is obtained from both approaches but with different time exponents. The relative difference between the two exponents varies with the shape of the Eulerian correlation of the potential and remains however small.

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