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BRST Extension of the Non-Linear Unfolded Formalism

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Abstract

We review the construction of gauge field theories from BRST first-quantized systems and its relation to the unfolded formalism. In particular, the BRST extension of the non-linear unfolded formalism is discussed in some details.

1 Introduction

This contribution is based on the papers [1, 2, 3].

Hamiltonian BFV-BRST quantization [4, 5, 6] as a tool for constructing gauge invariant field theories was originally used in the context of open string field theory in the mid eighties (see [7] for an early review). Soon thereafter, this approach was also used to describe higher spin gauge theories at the free level [8, 9, 10]. Despite several attempts to constructing consistent interactions in this approach [11, 12], the full interacting theory was eventually constructed in the so-called unfolded formalism [13, 14, 15].

In the latter approach, the theory is formulated at the level of equations of motion while constructing a Lagrangian is a separate problem that usually requires introducing additional structures. Recently [2], the relation between the unfolded and the BRST approach has been understood at the free level: an extended BRST parent system has been explicitly constructed which gives rise upon different reductions both to the standard Fronsdal formulation and to the unfolded form of the equations of motion. From the first-quantized point of view, the construction of the parent theory corresponds to a version of extension used in Fedosov quantization [16, 17]. More precisely, this version is adapted to the quantization of cotangent bundles [18].

In this contribution, we develop further the general considerations of [2] concerning the interacting case by describing in more details the BRST extension of the general non-linear unfolded equations. We explicitly show that this BRST extension is well suited for the problem of incorporating additional constraints and the analysis of various reductions. We also propose a geometrical interpretation of the BRST extended unfolded formalism in terms of supermanifolds and show how it generalizes the so-called AKSZ construction [19].

2 Generalities on the BRST formalism

In this section, we review the BRST construction in the non-Lagrangian/non-Hamiltonian case. We want to emphasize here that the standard construction of the BRST differential (see e.g. [20]) does

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in general not depend on the existence of an even/odd bracket structure together with a generator (BRST charge/master action) and can be constructed in terms of constraints and gauge generators alone. In the exposition we closely follow [20].

The relevance of the non-Lagrangian generalization of BRST theory was emphasized recently in [2], where a non-Lagrangian version of generalized auxiliary fields [21] was developed in order to discuss possibly non-Lagrangian theories which become Lagrangian after elimination/addition of unphysical degrees of freedom. The rationale behind this is the idea that no physical principle can force one to require unphysical dynamics to follow from a variational principle. This approach has proved useful both in the general setting and in the context of higher spin gauge theories. More recently, the BRST theory for non-Lagrangian/non-Hamiltonian systems was also studied in [22], where the non-Hamiltonian BRST formulation was extended to incorporate weak Poisson structures and to describe their quantization.

Consider a phase/configuration space $\mathcal{M}_0$ and assume the physical system to be restricted to a submanifold $\Sigma_0 \subset \mathcal{M}_0$ described by some constraints/equations of motion. For a gauge system, Σ_0 is in addition foliated by integral submanifolds (leaves = gauge orbits) of an integrable distribution and all points of a single integral submanifold describe the same physical state.

In the simplified, finite-dimensional setting we discuss now, let Σ be specified by regular equations $T_a = 0$ and the distribution determined by a set of vector fields R_α on $\mathcal{M}_0$, which restrict to Σ. The integrability condition on Σ takes the form

$$[R_\alpha, R_\beta] = U^\gamma_{\alpha\beta} R_\gamma + \dots \tag{1}$$

where $\dots$ denote terms vanishing on Σ. Note that there is some freedom for the choices of T and R.

The physical degrees of freedom are coordinates on the reduced space which is the quotient of Σ modulo the gauge orbits. The basic idea of BRST theory (either Batalin–Fradkin–Vilkovisky in the Hamiltonian or Batalin–Vilkovisky in the Lagrangian context [23, 24, 25, 26, 27]) is to describe physical quantities as the cohomology of an appropriately constructed BRST differential instead of explicitly solving constraints and taking the quotient with respect to the gauge orbits. The construction of the BRST differential involves two steps.

The first step consists in reducing to Σ. For simplicity, constraints T_a are assumed to have at most first order reducibility relations. This means that there exist functions Z^A_a such that $T_a Z^a_A = 0$ and matrix Z^a_A has maximal rank on Σ, so that that there are no further reducibility relations. We also assume that $\mathcal{M}_0$ is not a supermanifold and therefore the constraints and gauge generators are Grassmann even. The cohomological description of the reduction is given by the Koszul-Tate complex: one introduces Grassmann odd variables $\mathcal{P}_a$, $|\mathcal{P}_a| = 1$ and Grassmann even variables ρ_A, $|\rho_A| = 0$ and considers then the algebra $\mathcal{F}^0$ of smooth functions on $\mathcal{M}_0$ with values in polynomials in $\mathcal{P}$ and ρ. This algebra is graded according to the "antighost number", $\mathcal{F}^0 = \oplus_{i \geqslant 0} \mathcal{F}^0_i$, with

$$\mathrm{antigh}(\mathcal{P}) = 1\,, \qquad \mathrm{antigh}(\rho) = 2\,. \tag{2}$$

The Koszul-Tate differential $\delta : \mathcal{F}^0_i \to \mathcal{F}^0_{i-1}$ is given by

$$\delta = T_a \frac{\partial}{\partial \mathcal{P}_a} + \mathcal{P}_a Z^a_A \frac{\partial}{\partial \rho_A}\,. \tag{3}$$

Its nilpotency $\delta^2 = \delta\delta = 0$ follows from the reducibility identity $T_a Z^a_A = 0$. The cohomology of δ is concentrated in degree zero and is given by functions on Σ:

$$H_0(\delta, \mathcal{F}^0) = \mathcal{C}^\infty(\Sigma)\,, \qquad H_i(\delta, \mathcal{F}^0) = 0 \quad i > 0\,. \tag{4}$$

The next step is the factorization by the gauge orbits. Consider the space $\mathcal{F}_0$ of functions on $\mathcal{M}_0$ with values in polynomials in some Grassmann odd variables $\mathcal{C}^\alpha$. This algebra is graded according to the "pure ghost number", $\mathcal{F}_0 = \oplus_{i \geqslant 0} \mathcal{F}^i_0$, with puregh$(\mathcal{C}) = 1$. Because of (1), the vector field

$$\gamma = \mathcal{C}^\alpha R_\alpha - \frac{1}{2} \mathcal{C}^\alpha \mathcal{C}^\beta U^\gamma_{\alpha\beta} \frac{\partial}{\partial \mathcal{C}^\gamma}\,, \tag{5}$$

is nilpotent on Σ, i.e., γ^2 is proportional to constraints T_a. In pure ghost number zero, the cohomology of γ in $\mathcal{C}^\infty(\Sigma) \otimes \bigwedge(C)$ is given by functions on Σ annihilated by R_α (i.e., functions on Σ that are constant along the gauge orbits). Note that in contrast to δ, higher cohomology groups of γ do not vanish in general.

Let us consider now $\mathcal{F} = \oplus_{i,j \geqslant 0} \mathcal{F}^i_j$, the space of functions on $\mathcal{M}_0$ with values in polynomials in $\mathcal{C}, \mathcal{P}, \rho$. Algebra $\mathcal{F}$ is to be identified with the algebra of functions on a supermanifold $\mathcal{M}$ called extended phase (or configuration) space. The natural degree in $\mathcal{F}$ is given by the difference of the pure ghost number and the antighost number: for a homogeneous element f

$$\mathrm{gh}(f) = \mathrm{puregh}(f) - \mathrm{antigh}(f)\,. \tag{6}$$

The fact that γ is nilpotent up to terms vanishing on Σ can be reformulated as nilpotency of γ in the cohomology of δ. Both vector fields γ and δ can be extended to $\mathcal{F}$ in such a way that δ remains nilpotent while γ commutes with δ. The existence of additional vector fields $s_1, s_2, \ldots$ of antighost number $1, 2, \ldots$ such that

$$s = \delta + \gamma + s_1 + s_2 + \ldots\,, \qquad \mathrm{gh}(s) = 1\,, \tag{7}$$

satisfies $s^2 = 0$ is then guaranteed because the cohomology of δ is concentrated in antighost number zero.

Let us recall how the BRST construction specializes to the case where $\mathcal{M}_0$ is the field space of the Lagrangian system described by the gauge invariant action $S(\phi)$. In this case constraints T_a are equations of motion $\frac{\partial}{\partial \phi^a} S = 0$ while the gage generators are symmetries of the action i.e., Noether identities $R_\alpha S = 0$ hold. At the same time the Noether identities imply that equations of motion are not independent: $R^a_\alpha \frac{\partial}{\partial \phi^a} S = 0$.

Because the equations of motion can be considered as components of a 1-form on the field space while reducibility identities and the gauge symmetries are determined by the same generators $R_\alpha = R^a_\alpha \frac{\partial}{\partial \phi^a}$ there is a natural odd Poisson bracket (antibracket) in $\mathcal{F}$ which is determined by

$$\big(\phi^a, \mathcal{P}_b\big) = \delta^a_b\,, \qquad \big(\mathcal{C}^\alpha, \rho_\beta\big) = \delta^\alpha_\beta\,. \tag{8}$$

In this context variables $\mathcal{P}_a$ and ρ_α are called antifields and usually denoted by $\phi^*_a, \mathcal{C}^*_\alpha$. Note that the bracket is Grassmann odd and carries ghost number 1. The BRST differential of the Lagrangian gauge system can be taken to be canonically generated in the antibracket. Namely, for such an s there exists a generating function $\mathbf{S} \in \mathcal{F}$ with $|\mathbf{S}| = \mathrm{gh}(\mathbf{S}) = 0$ such that

$$s = \big(\mathbf{S}, \cdot\,\big)\,, \qquad \tfrac{1}{2}\big(\mathbf{S}, \mathbf{S}\big) = 0\,. \tag{9}$$

The function $\mathbf{S}$ is called master action while the second equation ensuring nilpotency of s is referred to as the master equation. The expansion of $\mathbf{S}$ in the antifields reads as

$$\mathbf{S} = S + \mathcal{P}_a R^a_\alpha \mathcal{C}^\alpha + \frac{1}{2}\rho_\alpha U^\alpha_{\beta\gamma}\mathcal{C}^\beta \mathcal{C}^\gamma + \ldots\,, \tag{10}$$

where $\ldots$ denote higher order terms in the expansion with respect to the the antighost number. This BRST construction and the corresponding quantization method is known as BV (Batalin–Vilkovisky) formalism.

Finally, let us recall the BRST construction in the case where $\mathcal{M}_0$ is the phase space of a first-class constrained system. In this case $\mathcal{M}_0$ is equipped with a symplectic structure and the gauge generators are Hamiltonian vector fields generated by the constraints, i.e.

$$R_\alpha = \{T_\alpha,\, \cdot\,\}\,, \tag{11}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the Poisson bracket determined by the symplectic structure on $\mathcal{M}_0$. Because the constraints and the gauge generators are not any longer independent, the Poisson bracket can be naturally extended to $\mathcal{F}$ by defining

$$\Big\{\mathcal{P}_\alpha, \mathcal{C}^\beta\Big\} = -\delta^\alpha_\beta\,, \tag{12}$$

with the bracket between ghosts and other phase space variables vanishing. In this case, the BRST differential s is canonically generated in the extended Poisson bracket: there exists $\Omega^{\rm cl} \in \mathcal{F}$ such that

$$s = \left\{\Omega^{\rm cl}, \cdot\right\}, \qquad \tfrac{1}{2}\left\{\Omega^{\rm cl}, \Omega^{\rm cl}\right\} = 0\,. \tag{13}$$

The expasion of $\Omega^{\rm cl}$ in ghost momenta $\mathcal{P}$ reads as

$$\Omega^{\rm cl} = \mathcal{C}^\alpha T_\alpha + \frac{1}{2}\mathcal{P}_\gamma U^\gamma_{\alpha\beta}\mathcal{C}^\alpha\mathcal{C}^\beta + \dots\,. \tag{14}$$

If in addition the Hamiltonian $H_0^{\rm cl} \in \mathcal{F}_0^0$ satisfies $R_\alpha H_0^{\rm cl} = T_\beta V_\alpha^\beta$ for some V_β^α (i.e. $H_0^{\rm cl}$ is gauge invariant on Σ), one can construct a BRST invariant Hamiltonian $H^{\rm cl}$ such that

$$\left\{\Omega^{\rm cl}, H^{\rm cl}\right\} = 0\,, \quad \mathrm{gh}(H^{\rm cl}) = 0\,, \quad H^{\rm cl}|_{\mathcal{P}=\mathcal{C}=0} = H_0^{\rm cl}\,. \tag{15}$$

This Hamiltonian BRST construction and the corresponding quantization method is referred to as BFV (Batalin–Fradkin–Vilkovisky) formalism.

3 Gauge theories associated to the first-quantized BFV-BRST systems

3.1 Operator BFV quantization

At the quantum level phase space observables become operators represented in a space of states. The BRST operator Ω and the Hamiltonian operator H satisfy

$$\frac{1}{2}[\Omega, \Omega] = 0, \qquad [\Omega, H] = 0\,, \tag{16}$$

Physical operators are described by hermitian operators A such that $[\Omega, A] = 0$ where two such operators have to be identified if they differ by a BRST exact operator $A \sim A + [\Omega, B]$. These two equations define the BRST operator cohomology. Note that if an inner product is specified on the space of states, hermitian conjugation is just standard conjugation in this inner product. In general, however, conjugation can be defined as an additional structure even if no suitable inner product is given.

Similarly, physical states are selected by the condition $\Omega\phi = 0$. Furthermore, BRST exact states should be considered as trivial, or equivalently, states that differ by BRST exact ones should be identified $\phi \sim \phi + \Omega\chi$. These two equations define the BRST state cohomology. Finally, time evolution is described by the Schrödinger equation

$$i\frac{d\phi}{dt} = H\phi. \tag{17}$$

3.2 Free gauge theory on the space of quantum states

We consider a quantum BFV-BRST system which we assume to be time-reparametrization invariant so that its Hamiltonian H vanishes. For a detailed discussion of the general case we refer to [1].

We also assume that among the degrees of freedom, there are coordinates x^μ which are interpreted as coordinates of a space-time manifold $\mathcal{X}_0$ and which are quantized in the coordinate representation. The space of states is then given by functions of x^μ taking values in an internal space $\mathcal{H}$. (In a geometrically nontrivial situation one should consider sections of a suitable vector bundle over space-time instead.) Because we are not interested in constructing proper quantum mechanics, we are not concerned with normalizability of the states. For simplicity, we thus can consider states with a smooth dependence on x^μ.

Given a graded superspace $\mathcal{H}$, one associates to each basis vector e_A a coordinate ψ^A with ghost number $\mathrm{gh}(\psi^A) = -\mathrm{gh}(e_A)$ and Grassmann parity $|\psi^A| = |e_A|$. One then considers $\mathcal{M}_\mathcal{H}$ to be the supermanifold with coordinates ψ^A. In what follows we denote by ψ^{A_k} the fields associated with the ghost number $-k$ subspace $\mathcal{H}^{(k)} \subset \mathcal{H}$, which implies in particular that $\mathrm{gh}(\psi^{A_k}) = k$. We also introduce the object $\Psi(x) = e_A\psi^A(x)$ with $|\Psi| = 0, \mathrm{gh}(\Psi) = 0$, called string field, which is in particular useful to avoid using indices (see [2] and references therein for precise definitions and the relation with the similar notion [28] used in the context of string field theory).

The configuration space of the free field theory associated with the quantum system is the space of maps from $\mathcal{X}_0$ to the submanifold $\mathcal{M}^{(0)}_{\mathcal{H}}$ of ghost number zero fields. In terms of coordinates it is described by fields $\psi^{A_0}(x)$. The equations of motion are given by

$$\Omega^{A_{-1}}_{B_0}\psi^{B_0}(x) = 0\,. \tag{18}$$

Due to $\Omega^2 = 0$, these equations are invariant under the gauge transformations $\delta_\epsilon\psi^{A_0} = \Omega^{A_0}_{B_1}\epsilon^{B_1}$, for some gauge parameters ϵ^{B_1}.

The fields associated with states in nonzero ghost number are to be interpreted as ghost fields, ghosts for ghosts, and antifields for the BRST-BV description of the theory. The BRST-BV differential s_0 on the fields $\psi^A(x)$ is then defined by

$$s_0\Psi = \Omega\Psi \quad \Longleftrightarrow \quad s_0\psi^A = \Omega^A_B\psi^B. \tag{19}$$

If in addition we are given with an inner product that makes Ω formally self-adjoint, one can build a classical action and the Batalin-Vilkovisky master action which determine the equations of motion and the BRST differential respectively (for more details, see [2] and references therein).

3.3 Algebraic structure of interactions

We now briefly discuss the general algebraic structures underlying consistent deformations of a theory described by a linear BRST differential s_0 given in (19). For simplicity, we use here De Witt's condensed notation consisting in including the space-time coordinates x^μ in the index A of the coordinates ψ^A on $\mathcal{M}_{\mathcal{H}}$.

A general nonlinear deformation s of $s_0 = \Omega^B_A\psi^A\frac{\partial}{\partial\psi^B}$ has the form

$$s = \Omega^B_A\psi^A\frac{\partial}{\partial\psi^B} + U^C_{AB}\psi^A\psi^B\frac{\partial}{\partial\psi^C} + U^D_{ABC}\psi^A\psi^B\psi^C\frac{\partial}{\partial\psi^D} + \dots\,, \tag{20}$$

with $|s| = 1$, $\mathrm{gh}(s) = 1$, and $\dots$ denoting higher order terms. The deformation is consistent if $s^2 = 0$. Such a nilpotent vector field s on a flat supermanifold associated with a superspace $\mathcal{H}$ is equivalent to an L_∞ algebra on $\mathcal{H}$ [29].

The simplest example is provided by an s that is at most quadratic. In this case, $\mathcal{H}$ is a differential graded Lie algebra with $[e_A, e_B] = e_C U^C_{AB}$. In the more general case, the U^A_{BC} determine a Lie algebra structure only in cohomology of Ω and there are higher order brackets related to the higher orders terms in s.

In the case of field theories, some complications arise because $\mathcal{H}$ is an infinite dimensional space of field configurations and the algebraic structures are really represented by differential operators. Moreover, space-time locality of the deformation is to be taken into account. However, in some cases one can explicitly separate the space-time dependence in such a way that interactions do not involve an explicit x-dependence or x-derivatives. We now turn to the discussion of systems of this type.

4 Geometry of the Vasiliev unfolded formalism

4.1 First-quantized BRST picture

Consider the special class of free theories associated with a BRST first quantized model for which fields are defined on a supermanifold $\mathcal{X}$ with Grassmann-even coordinates x^μ and Grassmann-odd coordinates θ^μ. The latter coordinates are space-time ghosts with $\mathrm{gh}(\theta^\mu)=1$ and can be identified with the basic differentials dx^μ. The fields take values in a supermanifold $\mathcal{M}_{\mathcal{H}}$ associated with the graded superspace $\mathcal{H}$ and the coordinates on $\mathcal{M}_{\mathcal{H}}$ are denoted by Ψ^A. The components in the expansion of $\Psi^A(x,\theta)$ in θ^μ can be considered as differential forms on $\mathcal{X}_0$:

$$\Psi^A(x,\theta)=(\psi_0)^A(x)+\theta^\mu(\psi_1)^A_\mu(x)+\,\theta^\mu\theta^\nu\,(\psi_2)^A_{\mu\nu}(x)+\dots\,, \tag{21}$$

with $\mathrm{gh}((\psi_p)^A_{\mu_1\dots\mu_p})=\mathrm{gh}(\Psi^A)-p$ and $|(\psi_p)^A_{\mu_1\dots\mu_p}|=|\Psi^A|-p\;mod\;2$. We also assume that the BRST differential s_0 can be represented in the form

$$s_0\Psi(x,\theta)=\boldsymbol{d}\Psi(x,\theta)+\bar{\Omega}\Psi(x,\theta) \tag{22}$$

with $\boldsymbol{d}=\theta^\mu\frac{\partial}{\partial x^\mu}$ and $\bar{\Omega}$ a linear operator in $\mathcal{H}$, i.e. $\bar{\Omega}\Psi=e_A\bar{\Omega}^A_B\Psi^B$. Note that the system described by s_0 is explicitly space-time covariant.

Under sufficiently general conditions, one can show that nonlinear deformations of the theory preserving the general covariance can be assumed to contain neither x^μ and θ^μ derivatives nor an explicit dependence on these variables [30, 31]. The deformed differential s is then determined by an odd vector field Q on $\mathcal{M}_{\mathcal{H}}$

$$Q=\bar{\Omega}^A_B\Psi^B\frac{\partial}{\partial\Psi^A}+\Psi^B\Psi^C U^A_{BC}\frac{\partial}{\partial\Psi^A}+\Psi^B\Psi^C\Psi^D U^A_{BCD}\frac{\partial}{\partial\Psi^A}+\dots\,, \tag{23}$$

with $\mathrm{gh}(Q)=|Q|=1$ and satisfying the compatibility condition $Q^2=0$. In other words, $\mathcal{H}$ is equipped with an L_∞ algebra structure.

Given such a Q, the BRST differential itself is then determined by

$$s\Psi^A=\boldsymbol{d}\Psi^A+Q^A(\Psi)\,. \tag{24}$$

The dynamical equations of the system determined by s are

$$\big(\boldsymbol{d}\Psi^A+Q^A(\Psi)\big)\Big|_{\psi^{(l)}=0,\,l\neq0}=0 \tag{25}$$

where we have put to zero all the component fields $(\psi_p)^A_{\mu_1\dots\mu_p}(x)$ entering $\Psi(x,\theta)$ except those of ghost number zero. In the case where $\mathrm{gh}(\Psi^A)\geqslant0$, this is exactly the form of the general unfolded equations proposed in [32, 33, 34]. Equations of this form are also known as defining the structure of a free differential algebra [35].

Some comments are in order. Note that for each coordinate function Ψ^A of ghost number $\mathrm{gh}(\Psi^A)=p_A$, there is at most one component field $(\psi_p)^A_{\mu_1\dots\mu_p}$ with a given ghost number. Note also that if $p_B<0$ for some B, then equations (25) reduce to the constraint equations, $Q^B(\Psi)|_{\psi^{(l)}=0,\,l\neq0}=0$ because in Ψ^B there is no ghost number zero component field so that the first term in (25) vanishes.

The BRST differential also determines gauge transformations for physical fields. Let e_a be a basis in the subspace of $\mathcal{H}$ with zero or negative ghost number, i.e., $\mathrm{gh}(e_a)\leqslant0$ so that the associated coordinates Ψ^a carry nonnegative ghost numbers. It then follows that among component fields in the expansion of Ψ^a with respect to θ^μ there is a field $\psi^a_{\mu_1\dots\mu_p}$ with $p=\mathrm{gh}(\Psi^a)$ of zero ghost number. The gauge transformation of ψ^a is given by

$$\delta_\epsilon\psi^a=s\psi^a\big|_{\psi^{(l)}=0,\,l\neq0,1} \tag{26}$$

with the ghost number 1 fields $\psi^{(1)}$ replaced by gauge parameters ϵ. Observing that the right hand side is linear in $\psi^{(1)}$ and of the same form degree as ψ^a, one arrives at a more explicit form for the gauge transformations:

$$\delta_\epsilon\psi^a=\boldsymbol{d}\epsilon^a+\epsilon^A\frac{\partial Q^a}{\partial\Psi^A}\Big|_{\psi^{(l)}=0,\,l\neq0}\,. \tag{27}$$

The BRST differential (24) can naturally be considered as an extension of the unfolded equations (25). It allows for cohomological tools to be used at the level of the field theory, e.g., for the introduction or elimination of generalized (non-Lagrangian) auxiliary fields [21, 2].

Arbitrary unfolded equations can be embedded in such a BRST system. Indeed, suppose that the equations of motion of a system are given by

$$d\psi^A + Q^A(\psi) = 0\,, \tag{28}$$

where ψ^A are differential forms on $\mathcal{X}_0$ with form degree denoted by p_A. Suppose furthermore that Q^A are polynomial (in the sense of the wedge product of differential forms) functions in ψ^A satisfying the compatibility condition

$$Q^B \frac{\partial Q^A}{\partial \psi^B} = 0. \tag{29}$$

For simplicity we assume that these compatibility conditions are satisfied without making use of the relations of the Grassmann algebra of basis 1-forms dx^μ besides the supercommutativity of ψ^A-s with respect to the wedge product.[1] This can be equivalently formulated in terms of an auxiliary linear supermanifold $\mathcal{M}$ with independent coordinates Ψ^A with $|\Psi^A| = |(\psi^{p_A})^A_{\mu_1\dots\mu_{p_A}}| + p_A$ and $\mathrm{gh}(\Psi^A) = p_A$ as the nilpotency condition $Q^2 = 0$ for an odd vector field

$$Q = Q^A(\Psi)\frac{\partial}{\partial \Psi^A}\,, \tag{30}$$

on $\mathcal{M}$. One can then introduce additional fields on $\mathcal{X}_0$ to define superfields $\psi^A(x,\theta)$ on $\mathcal{X}$ taking values in $\mathcal{M}$ with

$$\mathrm{gh}((\psi^{p_A})^A_{\mu_1\dots\mu_{p_A}}) = \mathrm{gh}(\Psi^A) - p_A\,, \qquad |(\psi^{p_A})^A_{\mu_1\dots\mu_{p_A}}| = |\Psi^A| - p_A\,, \tag{31}$$

so that the original fields $\psi^A_{\mu_1\dots\mu_{p_A}}$ appear as ghost number zero component fields from $\Psi^A(x,\theta)$. If one now considers the BRST differential (24) determined by Q, it is straightforward to verify that the dynamical equations (25) coincides with the original unfolded equations (28).

4.2 Geometric picture – non-Lagrangian AKSZ procedure

When reformulated in BRST terms, the unfolded equations allow for a nice geometrical interpretation. Consider two supermanifolds: a supermanifold $\mathcal{X}$ equipped with a degree, an odd nilpotent vector field $\boldsymbol{d}$, $\mathrm{gh}_{\mathcal{X}}(\boldsymbol{d}) = 1$, and a volume form $d\mu$ preserved by $\boldsymbol{d}$ and a supermanifold $\mathcal{M}$ equipped with another degree, an odd nilpotent vector field Q, $\mathrm{gh}_{\mathcal{M}}(Q) = 1$. As implied by the notation, the basic example for $\mathcal{X}$ is the odd tangent bundle $\Pi T\mathcal{X}_0$ which has a natural volume form and is equipped with the De Rham differential. Note that supermanifolds equipped with an odd nilpotent vector field are often called Q-manifolds [37].

Consider then the manifold of maps from $\mathcal{X}$ to $\mathcal{M}$ (more generally, one could of course consider the space of sections of a bundle over $\mathcal{X}$ with fibers isomorphic to $\mathcal{M}$). This space is naturally equipped with the total degree denoted by $\mathrm{gh}(\cdot)$ and an odd nilpotent vector field s, $\mathrm{gh}(s) = 1$. If z are local coordinates on $\mathcal{X}$ (in the case where $\mathcal{X} = \Pi T\mathcal{X}_0$ coordinates z split into x^μ and θ^μ) and Ψ^A are coordinates on $\mathcal{M}$, the expression for s reads

$$s = \int_{\mathcal{X}} d\mu (-1)^{|d\mu|} \left[\boldsymbol{d}\Psi^A(z) + Q^A(\Psi(z))\right] \frac{\delta}{\delta \Psi^A(z)}\,. \tag{32}$$

Vector field s can be considered as a BRST differential of a field theory on $\mathcal{X}$. Indeed, the basic properties $s^2 = 0$ and $\mathrm{gh}(s) = 1$ hold. In what follows we refer to this system as a quadruple $(\mathcal{X}, \boldsymbol{d}, \mathcal{M}, Q)$, where manifolds $\mathcal{X}$ and $\mathcal{M}$ are equipped with the odd nilpotent vector fields $\boldsymbol{d}$ and Q respectively. In addition, $\mathcal{X}$ is equipped with a $\boldsymbol{d}$-invariant volume form and the ghost grading on $\mathcal{X}$ and $\mathcal{M}$ is such that $\mathrm{gh}_{\mathcal{X}}(\boldsymbol{d}) = 1$ and $\mathrm{gh}_{\mathcal{M}}(Q) = 1$.

For the system $(\mathcal{X}, \boldsymbol{d}, \mathcal{M}, Q)$ it is easy to check using the explicit form (32) that $s\Psi^A = \boldsymbol{d}\Psi^A + Q^A$. This shows that, locally, (32) describes the same theory as s defined in (24) if $\mathcal{M} = \mathcal{M}_{\mathcal{H}}$, $\mathcal{X} = \Pi T\mathcal{X}_0$, and $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$.

[1] Such free differential algebras are called *universal*, for details see e.g. [36].

In the case where the "target" manifold $\mathcal{M}$ is in addition equipped with a compatible (odd) Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ and $Q = \{S,\,\cdot\,\}$ is generated by a "master action" S satisfying the classical master equation $\frac{1}{2}\{S,S\} = 0$, one can construct a field theory master action $\mathbf{S}$ on the space of maps. This procedure was proposed in [19] as an approach for constructing BV-BRST formulations of topological sigma models. Further developments can be found in [38, 39, 40, 41, 42, 43, 44] and references therein. A generalization that also includes the Hamiltonian BRST formulation has been proposed in [45] and covers the case where $\mathbf{S}$ is Grassmann odd and is to be interpreted as a BRST charge of the BFV-BRST formulation of the theory.

4.3 Generalized auxiliary fields

The restriction that the compatibility condition (29) holds without making use of the Grassmann algebra relations for the basic differential forms is not really necessary. Moreover, in practice it often happens that there are some other constraints on $\mathcal{M}$. Nevertheless, it is still possible to bring the system to the form (24) by explicitly solving these constraints or by appropriately extending $\mathcal{M}$.

To show how constraints on $\mathcal{M}$ can be incorporated in the BRST differential, suppose that we are in the setting of the previous subsection and let also $\Sigma \subset \mathcal{M}$ be a submanifold in $\mathcal{M}$ such that Q restricts to Σ. In terms of some constraints T_a determining Σ, this means that $QT_a|_\Sigma = 0$. The system described in this way is just a system without constraints but with $\mathcal{M}$ replaced with Σ and Q replaced by its restriction $Q|_\Sigma$ to Σ. For this system to be well defined, it is actually enough to require that Q^2 be zero in $\mathcal{M}$ only up to terms vanishing on Σ.

For simplicity, let T_a be independent, regular constraints. One then introduces variables $\mathcal{P}_a$ with $\mathrm{gh}(\mathcal{P}_a) = -1$, $|\mathcal{P}_a| = |T_a| + 1$ and extends $\mathcal{M}$ to $\mathcal{M}_\mathcal{P} = \mathcal{M} \times \Lambda$ where Λ is a linear supermanifold with coordinates $\mathcal{P}_a$. Exactly the same arguments as in Section **2** then show that one can construct

$$Q_\mathcal{P} = T_a \frac{\partial}{\partial \mathcal{P}_a} + Q + Q_0 + Q_1 + Q_2 + \ldots \tag{33}$$

satisfying $Q_\mathcal{P}^2 = 0$. Here, Q_i denote terms of degree i in $\mathcal{P}_a$.

We claim that the system $(\mathcal{X}, \boldsymbol{d}, \mathcal{M}_\mathcal{P}, Q_\mathcal{P})$ is equivalent to the system $(\mathcal{X}, \boldsymbol{d}, \Sigma, Q|_\Sigma)$ through elimination of generalized auxiliary fields (in the non-Lagrangian sense, see [2]). Indeed, let $s_\mathcal{P}$ be a BRST differential of $(\mathcal{X}, \boldsymbol{d}, \mathcal{M}_\mathcal{P}, Q_\mathcal{P})$ then $\mathcal{P}_a$ and $s_\mathcal{P}\mathcal{P}_a$ are independent constraints because $s_\mathcal{P}\mathcal{P}_a = T_a + \ldots$. Moreover, equations $s_\mathcal{P}\mathcal{P}_a = 0$ at $\mathcal{P} = 0$ are equivalent to $T_a = 0$, while T_a can be taken (locally) as independent fields so that one concludes that $\mathcal{P}_a, s_\mathcal{P}\mathcal{P}_a$ are generalized auxiliary fields and can be eliminated. The reduced system is obviously $(\mathcal{X}, \boldsymbol{d}, Q|_\Sigma, \Sigma)$.

Conversely, let w^a be some constraints on $\mathcal{M}$ such that w^a, Qw^a are independent constraints determining a surface $\Sigma \subset \mathcal{M}$. The same arguments then show that the system $(\mathcal{X}, \boldsymbol{d}, \mathcal{M}, Q)$ can be reduced to the system $(\mathcal{X}, \boldsymbol{d}, \Sigma, Q|_\Sigma)$ through the elimination of generalized auxiliary fields

Consequently, if one works in BRST terms, one can assume without loss of generality that all constraints on the fields are already incorporated in Q, which can be useful from various points of view. In particular, this also shows that it is enough to consider the case where $\mathcal{M}$ is a linear supermanifold with all the nontrivial geometry encoded in Q.

4.4 Elimination of pure gauge variables in space-time

As explained in general in Subsection **3.2**, the field theory differentials $\boldsymbol{d}$ and the operator $\bar{\Omega}$ determining the linear part of Q can be understood in first quantized terms as a BRST operator acting in a space of quantum states. The equivalence under elimination of generalized auxiliary fields for Q, or more precisely, its linear part, can then be understood as a natural equivalence of first quantized systems under elimination of pure gauge degrees of freedom.

Among the first quantized degrees of freedom, variables x^μ, θ^μ and their conjugate momenta are represented in the coordinate representations on functions in x^μ and θ^μ and are identified with space-time coordinates, while the other degrees of freedom are represented in the target space $\mathcal{H}$. Of course one could as well represent θ^μ in $\mathcal{H}$. This makes no difference because the respective representation space is finite dimensional. That introduction/elimination of pure gauge variables represented in $\mathcal{H}$ leads to theories related by introduction/elimination of generalized auxiliary fields was shown in details in [2]. One can expect the same for pure gauge variables represented on functions on $\mathcal{X}$. As we are going to see, the respective theories are also related by elimination of generalized auxiliary fields if one allows for nonlocal transformations in the sector of the pure gauge variables.

Consider then a not necessarily linear system determined by $\mathcal{X}, \boldsymbol{d}, \mathcal{M}, Q$ and replace $\mathcal{X}$ with $\mathcal{X}\times\mathcal{M}_{t,\theta}$ and $\boldsymbol{d}$ with $\boldsymbol{d}' = \boldsymbol{d} + \theta\frac{\partial}{\partial t}$. Here, $\mathcal{M}_{t,\theta}$ denotes a linear supermanifold with coordinates t, θ with $|t| = 0$, $|\theta| = 1$ and $\mathrm{gh}(t) = 0$, $\mathrm{gh}(\theta) = 1$. Note that from a first quantized point of view, for the free part of the system, this corresponds to adding a pair of pure gauge variabels t, π_t together with their associated ghost variables θ, π_θ with commutation relations $[\pi_t, t] = -1$, $[\pi_\theta, \theta] = -1$ and adding the respective term $\theta\pi_t$ to the BRST charge. These pure gauge variables are represented on functions of t, θ so that the additional term in the BRST charge acts as $\theta\frac{\partial}{\partial t}$.

The resulting system is again a system of the same type, but living on the extended space–time manifold $\mathcal{X}\times\mathcal{M}_{t,\theta}$, and the question arises as to how it relates to the original system. To be able to compare these two field theories, we first need to consider them as field theories determined on the same space–time manifold. To this end we identify $(\mathcal{X}\times\mathcal{M}_{t,\theta}, \boldsymbol{d}+\theta\frac{\partial}{\partial t}, \mathcal{M}, Q)$ with $(\mathcal{X}, \boldsymbol{d}, \mathcal{M}', Q')$ where $\mathcal{M}'$ and Q' are the configuration space and the BRST differential of the system $(\mathcal{M}_{t,\theta}, \theta\frac{\partial}{\partial t}, \mathcal{M}, Q)$. In other words, the space-time coordinate t becomes a continuous index for fields on $\mathcal{X}$, while $\theta\frac{\partial}{\partial t}$ becomes a part of the target space BRST differential Q'.

The supermanifold $\mathcal{M}'$ can then be identified with the manifold of smooth maps from $\mathcal{M}_{t,\theta}$ to $\mathcal{M}$ while the BRST differential Q' is determined in coordinates by

$$Q'\Psi^A(t,\theta) = \theta\frac{\partial}{\partial t}\Psi^A(t,\theta) + Q^A(\Psi(t,\theta))\,. \tag{34}$$

On $\mathcal{M}'$, it is useful to take the coordinates $\widetilde{\Psi}^A, \Psi^A_{0t}, \Psi^A_{1t}$ with $\Psi^A_{0t}\big|_{t=0} = 0$ so that a general map has the form

$$\Psi^A(t,\theta) = \widetilde{\Psi}^A + \Psi^A_{0t} + \theta\Psi^A_{1t}\,. \tag{35}$$

It then follows that fields $\Psi^A_{0t}(z)$ and $\Psi^A_{1t}(z)$ on $\mathcal{X}$, with z denoting coordinates on $\mathcal{X}$, are generalized auxiliary fields. Indeed, in terms of the coordinates $\widetilde{\Psi}^A, \Psi^A_{0t}, \Psi^A_{1t}$, the term in Q' corresponding to $\theta\frac{\partial}{\partial t}$ acts as $\int dt(\frac{\partial}{\partial t}\Psi^A_{0t})\frac{\delta}{\delta\Psi^A_{1t}}$. This shows that at $\Psi^A_{1t} = 0$, equations $Q'\Psi^A_{1t} = 0$ takes the form $\frac{\partial}{\partial t}\Psi^A_{0t} = 0$ which has as unique solution $\Psi^A_{0t} = 0$ taking into account $\Psi^A_{0t}\big|_{t=0} = 0$. The arguments from the end of the subsection **4.3** then show that the fields $\Psi^A_{0t}(z)$ and $\Psi_{1t}(z)$ are indeed generalized auxiliary fields and can be eliminated, showing the equivalence to the original system on $\mathcal{X}$. For systems in unfolded form, the possibility to add/eliminate space time coordinates together with their differentials was first observed in the context of higher spin gauge theories in [46] (see also [47, 48, 49] for a more recent discussion).

More generally, if on $\mathcal{X}$ one can find constraints t^α such that t^α and $\boldsymbol{d}t^\alpha$ are independent, similar arguments show that, locally in space-time, one can consistently reduce the theory to the "constraint surface" $\widetilde{\mathcal{X}}\subset\mathcal{X}$ determined by the constraints $t^\alpha, \boldsymbol{d}t^\alpha$. In the case where $\boldsymbol{d} = \theta^\mu\frac{\partial}{\partial x^\mu}$, this means that locally in $\mathcal{X}$ one can consistently reduce the theory to a point. From the BRST theory point of view, this can also be understood as a version of the statement that, for theories of this type, representatives of various cohomology groups can be taken not to depend on space-time derivatives [30, 31]. Hence, cohomology groups are described by functions, tensor fields, etc. on the target space $\mathcal{M}$. In particular, this implies that possible consistent deformations and conserved currents of the system are determined by appropriate Q-cohomology classes in $\mathcal{M}$.

As a final remark, we comment on the BRST extension of the unfolded formalism as a generally-covariant first-order formalism. Indeed, manifold $\mathcal{M}$ can be considered as the space of initial data for the equations of motion, while the equations determine a multi-parametric flow, the number of parameters being the space-time dimension. If one mods out by the constraints and the gauge freedom, this multi-parametric evolution is uniquely determined by the initial data.

As an illustration, it is useful to consider the one-dimensional case which corresponds to a time reparametrization-invariant Hamiltonian system. Such a system is described by a BRST charge Ω and a symplectic structure on the phase space $\mathcal{M}$ with coordinates Ψ^A. The BV-BRST extension of the dynamics is governed by the master action $\mathbf{S}$ [50, 51, 52, 53]

$$\mathbf{S} = \int dtd\theta(V_A(\Psi)\theta\frac{\partial}{\partial t}\Psi^A - \Omega(\Psi))\,, \tag{36}$$

which we wrote in the superfield form proposed in [45]. Here V_A is the symplectic potential and θ is the superpartner of the "time" variable t with $|\theta| = 1, \mathrm{gh}(\theta) = 1$ (for details and precise definitions see [45, 1]). The BRST differential determined by $\mathbf{S}$ can be written as

$$s\Psi^A = \boldsymbol{d}\Psi^A + Q^A(\Psi)\,, \qquad \boldsymbol{d} = \theta\frac{\partial}{\partial t}\,, \quad Q^A = \left\{\Omega, \Psi^A\right\}\,, \tag{37}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the Poisson bracket corresponding to the symplectic structure on $\mathcal{M}$. On the one hand, s is just the standard BRST differential of the BV formulation for the reparametrization-invariant Hamiltonian system on $\mathcal{M}$ written in terms of superfields. On the other hand, it can be considered as the BRST differential describing the one-dimensional system in unfolded form. This illustrates, in particular, the role of space-time coordinates in the unfolded formalism. They play exactly the same role as an evolution parameter in the Hamiltonian formulation of time-reparametrization invariant systems.

Note added: While this contribution was being completed, there appeared reference [54] where, among other things, related aspects of the unfolded formalism are also discussed.

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References

[1] G. Barnich and M. Grigoriev, *Commun. Math. Phys.* **254** (2005) 581, hep-th/0310083.

[2] G. Barnich, M. Grigoriev, A. Semikhatov, and I. Tipunin, *to appear in Commun. Math. Phys.* (2005), hep-th/0406192.

[3] G. Barnich, G. Bonelli, and M. Grigoriev, hep-th/0502232.

[4] E. S. Fradkin and G. A. Vilkovisky, *Phys. Lett.* **B55** (1975) 224.

[5] I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **B69** (1977) 309.

[6] E. S. Fradkin and T. E. Fradkina, *Phys. Lett.* **B72** (1978) 343.

[7] C. B. Thorn, *Phys. Rept.* **175** (1989) 1.

[8] S. Ouvry and J. Stern, *Phys. Lett.* **B177** (1986) 335.

[9] A. K. H. Bengtsson, *Phys. Lett.* **B182** (1986) 321.

[10] M. Henneaux and C. Teitelboim, *First and second quantized point particles of any spin*, ch. 9, pp. 113–152. Quantum mechanics of fundamental systems 2, Centro de Estudios Científicos de Santiago. Plenum Press, 1987.

[11] L. Cappiello, M. Knecht, S. Ouvry, and J. Stern, *Ann. Phys.* **193** (1989) 10.

[12] A. K. H. Bengtsson, *Class. Quant. Grav.* **5** (1988) 437.

[13] M. A. Vasiliev, *Phys. Lett.* **B243** (1990) 378.

[14] M. A. Vasiliev, *Phys. Lett.* **B285** (1992) 225.

[15] M. A. Vasiliev, *Phys. Lett.* **B567** (2003) 139, hep-th/0304049.

[16] B. Fedosov, *J. Diff. Geom.* **40** (1994) 213.

[17] B. Fedosov, "Deformation quantization and index theory,". Berlin, Germany: Akademie-Verl. (1996) 325 p. (Mathematical topics: 9).

[18] M. Bordemann, N. Neumaier, and S. Waldmann, q-alg/9711016.

[19] M. Alexandrov, M. Kontsevich, A. Schwartz, and O. Zaboronsky, *Int. J. Mod. Phys.* **A12** (1997) 1405, hep-th/9502010.

[20] M. Henneaux and C. Teitelboim, "Quantization of Gauge Systems,". Princeton, USA: Univ. Pr. (1992) 520 p.

[21] M. Henneaux, *Phys. Lett.* **B238** (1990) 299.

[22] S. L. Lyakhovich and A. A. Sharapov, *JHEP* **03** (2005) 011, hep-th/0411247.

[23] I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **B102** (1981) 27.

[24] I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **B120** (1983) 166.

[25] I. A. Batalin and G. A. Vilkovisky, *Phys. Rev.* **D28** (1983) 2567.

[26] I. A. Batalin and G. A. Vilkovisky, *Nucl. Phys.* **B234** (1984) 106.

[27] I. A. Batalin and G. A. Vilkovisky, *J. Math. Phys.* **26** (1985) 172.

[28] M. R. Gaberdiel and B. Zwiebach, *Nucl. Phys.* **B505** (1997) 569, hep-th/9705038.

[29] T. Lada and J. Stasheff, *Int. J. Theor. Phys.* **32** (1993) 1087, hep-th/9209099.

[30] F. Brandt, *Commun. Math. Phys.* **190** (1997) 459, hep-th/9604025.

[31] G. Barnich and M. Henneaux, *J. Math. Phys.* **37** (1996) 5273, hep-th/9601124.

[32] M. A. Vasiliev, *Phys. Lett.* **B209** (1988) 491.

[33] M. A. Vasiliev, *Annals Phys.* **190** (1989) 59.

[34] M. A. Vasiliev, *Class. Quant. Grav.* **11** (1994) 649.

[35] D. Sullivan, *Inst. des Haut Etud. Sci. Pub. Math.* **47** (1977) 269.

[36] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. A. Vasiliev, hep-th/0503128.

[37] A. Schwarz, *Commun. Math. Phys.* **158** (1993) 373, hep-th/9210115.

[38] A. S. Cattaneo and G. Felder, *Commun. Math. Phys.* **212** (2000) 591, math.qa/9902090.

[39] I. Batalin and R. Marnelius, *Phys. Lett.* **B512** (2001) 225, hep-th/0105190.

[40] I. Batalin and R. Marnelius, "Superfield algorithms for topological field theories," in *"Multiple facets of quantization and supersymmetry"*, M. Olshanetsky and A. Vainshtein, eds., pp. 233–251. World Scientific, 2002. hep-th/0110140.

[41] A. S. Cattaneo and G. Felder, *Lett. Math. Phys.* **56** (2001) 163, math.qa/0102108.

[42] J.-S. Park, hep-th/0012141.

[43] D. Roytenberg, math.sg/0203110.

[44] L. Edgren and N. Sandstrom, *JHEP* **09** (2002) 036, hep-th/0205273.

[45] M. A. Grigoriev and P. H. Damgaard, *Phys. Lett.* **B474** (2000) 323, hep-th/9911092.

[46] M. A. Vasiliev, *Phys. Rev.* **D66** (2002) 066006, hep-th/0106149.

[47] M. A. Vasiliev, hep-th/0301235.

[48] O. A. Gelfond and M. A. Vasiliev, hep-th/0304020.

[49] J. Engquist, E. Sezgin, and P. Sundell, *Nucl. Phys.* **B664** (2003) 439, hep-th/0211113.

[50] I. Batalin and E. Fradkin, *Ann. Inst. Henri Poincare (Phys. Theor.)* **49** (1988) 145.

[51] W. Siegel, '*Int. J. Mod. Phys.* **A4** (1989) 3951.

[52] C. Batlle, J. Gomis, J. París, and J. Roca, *Phys. Lett.* **B224** (1989) 288.

[53] J. M. L. Fisch and M. Henneaux, *Phys. Lett.* **B226** (1989) 80.

[54] M. A. Vasiliev, hep-th/0504090.