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Renormalization group equations and geometric flows

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Abstract

The quantum field theory of two-dimensional sigma models with bulk and boundary couplings provides a natural framework to realize and unite different species of geometric flows that are of current interest in mathematics. In particular, the bulk renormalization group equation gives rise to the Ricci flow of target space metrics, to lowest order in perturbation theory, whereas the boundary renormalization group equation gives rise to the mean curvature flow for embedded branes. Together they form a coupled system of parabolic non-linear second order differential equations that can be further generalized to include non-trivial fluxes. Some closely related higher order curvature flows, such as the Calabi and Willmore flows associated to quadratic curvature functionals, are also briefly discussed as they arise in physics and mathematics. However, there is no known interpretation for them, as yet, in the context of quantum field theory.

Quantum field theory provides the main framework to study a large variety of problems in modern theoretical physics ranging from high energy physics to condensed matter physics. At the same time it serves as laboratory for the fruitful exchange of ideas and techniques between physics and mathematics. Geometric evolution equations are no exception to this interplay, since, as it turns out, certain curvature flows are naturally realized by the renormalization group equations of two-dimensional sigma models. The main purpose of these notes is to review the emergence of two distinct geometric flows, namely the Ricci and the mean curvature flows, as equations for the bulk and boundary couplings of two-dimensional Dirichlet sigma models that depend on the energy (or length) scale of the quantum theory. Fixed points of these flows can be reached by imposing conformal invariance of the bulk and boundary interactions, respectively, but, in general, there can be trajectories that extend from the ultra-violet to the infra-red regime of the underlying quantum field theory. A few other examples of geometric flows will also be discussed at the end.

Sigma models have all the necessary ingredients to allow for field theoretic explorations of problems in differential geometry. In their simplest form they are defined by the classical action

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2z \ \sqrt{\det\gamma}\ \gamma^{ab} G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \ , \tag{1}$$

where X^μ are local coordinates on a Riemannian manifold $\mathcal{M}$ with metric $G_{\mu\nu}(X)$, called target, and z^a are local coordinates on the base space manifold Σ, also called world-sheet, with corresponding metric γ_{ab}. Thus, the fields X^μ are maps from Σ to $\mathcal{M}$ and they are harmonic when the classical equations of motion are satisfied. The target space metric can be severely constrained by imposing additional restrictions on the two-dimensional field theory defined by S, which effectively turn non-linear sigma models into a valuable tool for studying problems of differential geometry. For example, supersymmetry of the classical action can be consistently implemented provided that $\mathcal{M}$ is Kähler manifold and generalizations thereof, [1]. Another example, which is closer related to the type of problems that will be considered next comes from the requirement of conformal invariance at the quantum level. Although the classical sigma model is always conformal, as can be readily seen by the (in)dependence of its action upon γ, conformal invariance at 1-loop is only achieved when the target space metric is Ricci flat, i.e., $R_{\mu\nu} = 0$. This simple fact provides the basis for the sigma model description of string theory in terms of two-dimensional conformal field theories. It also serves

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as starting point for the sigma model description of certain geometric flows by relaxing conformal invariance of the quantum theory.

Two-dimensional sigma models were used before, in several occasions, as toy models for non-linear interactions classically as well as quantum mechanically. It is quite remarkable that they have many properties in common with four-dimensional Yang-Mills theories including asymptotic freedom, [2], and non-trivial vacuum structure due to instantons, [3]. In this context, the physically interesting cases are described by non-linear sigma models with compact target spaces of positive curvature so that their beta function is negative and the theory becomes free at very high energies. Thus, in the ultra-violet regime, perturbation theory can be used reliably to extract information about physical processes as in non-abelian gauge theories, [4]. This particular property of strong interactions had far reaching consequences for our current understanding of the physical world (*Nobel Prize* 2004) and at the same time motivated the systematic study of renormalization group flows in simpler non-linear theories. Two-dimensional sigma models are perturbatively renormalizable quantum field theories and the scale dependence of their couplings can be computed order by order in perturbation theory. This, in effect, induces deformations of their target space metric with respect to the renormalization group time t, given by the logarithm of world-sheet length scale, which can be formulated and studied systematically in all generality. The renormalization of the metric, $G_{\mu\nu}(X;t)$, viewed as generalized coupling, takes the following form to 1-loop, [5],

$$\frac{\partial}{\partial t} G_{\mu\nu} = -\beta(G_{\mu\nu}) = -R_{\mu\nu} \qquad (2)$$

and yields the Ricci-flatness condition at the conformal fixed points, as noted before. This result is only valid at regions of weak curvature where all higher loop corrections can be dropped consistently.

The renormalization group equation of the target space metric is no other but the *Ricci flow* which arose independently in mathematics, [6]; see also the recent textbook [7] for detailed account of the technical aspects and an extensive list of references to the original mathematical papers. The Ricci flow was introduced as tool to address a variety of non-linear problems in differential geometry and, in particular, the uniformization of compact Riemannian manifolds. Under some general conditions, appropriate rescaling, and surgery, when it is necessary, solutions exist and tend to a constant curvature metric on $\mathcal{M}$. Fixed points with constant curvature metrics occur naturally in a variant of the Ricci flow that follows by suitable rescaling and time reparametrization so that the overall volume of space remains fixed through out the evolution. The resulting normalized flow is equivalent to the Ricci flow but expressed in different variables. This approach has been successfully implemented in three dimensions leading to a complete proof of Poincaré's conjecture, [8] (*Fields Medal* 2006). Entropy functionals, some of which have their origin in known monotonicity formulae of two-dimensional quantum field theory, such as decreasing c-function, [9], play important role in the whole subject. The Ricci flow, which is the simplest example of *intrinsic* curvature flows, in more general mathematical context, provides a non-linear generalization of the heat equation. It is driven by the intrinsic curvature tensor and exhibits dissipative properties that wash away any deviations of the metric from its canonical (constant curvature) form in a continuous fashion.

The Ricci flow defines a dynamical system in superspace, which is an infinite dimensional space consisting of all possible metrics on $\mathcal{M}$. It can be described as gradient flow of the Einstein-Hilbert action with respect to the DeWitt metric in superspace. Since metrics on $\mathcal{M}$ serve as generalized couplings of the non-linear sigma model, the identification of the Ricci flow with the renormalization group equation on the space of couplings seems natural. It also provides physical interpretation to the deformation variable t that otherwise appears adhoc in mathematics. Note, however, a difference of philosophy between physics and mathematics regarding the applicability of the Ricci flow. In high energy physics one is typically interested in local quantum field theories that exist at very short distances and so it is more important to obtain valid solutions by integrating the renormalization group flow backward in t. They correspond to ancient solutions of the Ricci flow, in the mathematical terminology, that exist at $t = -\infty$ and evolve forward in time until the formation of singularities. Said differently, ancient solutions yield unlimited trajectories of the backward time evolution without running to singularities, and, hence, two-dimensional quantum field theories that are asymptotically free in the ultra-violet regime. In mathematics, on the other hand, one is only interested in the forward evolution, starting from a given initial configuration $G_{\mu\nu}(X; t = 0)$, regardless its past existence, and study its behavior after sufficiently long time. Although the short time existence of forward solutions is always guaranteed by the parabolic nature of the flow, there are several intricate things that may happen further ahead, such as the formation of singularities. It is precisely here that all mathematical studies concentrated in recent years and made the Ricci flow into an effective tool for studying the geometrization problem of manifolds in low dimensions.

Close to the singularities higher order curvature terms become important and introduce modifications of the Ricci flow as predicted by the higher loop perturbative corrections to the metric beta function. We will not discuss them at all nor try to incorporate the effect of non-perturbative corrections to the beta function due to instanton. Both can have dramatic effect on the resolution of singularities that may otherwise arise and they certainly deserve proper mathematical attention. Here, we only refer to another way to (partially) avoid the formation of singularities via the well known mechanism of flux stabilization of collapsing cycles. This possibility arises perturbatively by introducing an anti-symmetric tensor field $B_{\mu\nu}(X)$, together with the metric in target space, so that the sigma model action generalizes to

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 z \, \sqrt{\det\gamma} \left(\gamma^{ab} G_{\mu\nu}(X) - i\epsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu \, . \tag{3}$$

The 1-loop beta functions of this theory form the following coupled system of evolution equations, [10],

$$\frac{\partial}{\partial t} G_{\mu\nu} = -\beta(G_{\mu\nu}) = -R_{\mu\nu} + \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda} \, , \tag{4}$$

$$\frac{\partial}{\partial t} B_{\mu\nu} = -\beta(B_{\mu\nu}) = \frac{1}{2} \nabla_\lambda H^\lambda{}_{\mu\nu} \, , \tag{5}$$

generalizing the Ricci flow. The field strength of the anti-symmetric tensor field, $H = dB$, enters quadratically in the beta function of the metric and can balance the shrinking effect of positively curved spaces. Then, fixed points arise by imposing the conditions

$$R_{\mu\nu} = \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda} \, , \qquad \nabla_\lambda H^\lambda{}_{\mu\nu} = 0 \tag{6}$$

and correspond to non-trivial conformal field theories (see, also, [11]). A prime example is provided by the $SU(2)$ Wess-Zumino-Witten model, [12], that represents a round 3-sphere stabilized by fluxes. Thus, singularities that otherwise seem inevitable on pure metric backgrounds can be avoided this way. Such generalizations have not been investigated at all in mathematics and they certainly deserve further study. The dilaton field $\Phi(X)$ can also be included in the above equations by adding reparametrizations generated by a gradient vector field $\xi_\mu = -\partial_\mu \Phi$. For review of the subject see, for instance, [13].

Two-dimensional sigma models can also be used to explore problems associated with the geometry of submanifolds $\mathcal{N}$ embedded in their target space $\mathcal{M}$. The key point here is to consider world-sheets with boundary, which for all practical purposes are taken to have the topology of a disc so that $\partial\Sigma = S^1$. Then, by imposing Dirichlet boundary conditions on (some of) the sigma models fields,

$$X^\mu \mid_{\partial\Sigma} = f^\mu(y^A) \, , \tag{7}$$

amounts to introducing branes in $\mathcal{M}$ as embedded submanifolds with local coordinates y^A. The embedding functions can be arbitrary, but they are fixed once and for all in the classical theory. Equivalently, one can think of Dirichlet sigma models as two-dimensional field theories with generalized bulk and boundary couplings,

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 z \, \sqrt{\det\gamma} \gamma^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau G_{\mu\nu}(X) V^\mu \partial_n X^\nu \, , \tag{8}$$

where τ is the parameter along the world-sheet boundary and ∂_n is the derivative operator normal to it. Variation of this classical action leads to the following set of compatible boundary conditions,

$$f^{\mu}_{,A} G_{\mu\nu} \partial_n X^{\nu} = 0 \ , \qquad V^{\mu} = 0 \ . \tag{9}$$

Usual D-branes are *minimal submanifolds* whose extrinsic curvature vanishes and correspond to conformal invariant boundary conditions in the quantum theory.

Note, however, that the embedding functions may depend on the energy scale of the corresponding quantum field theory, as $f^{\mu}(y^A; t)$. This, in turn, induces a non-vanishing beta function for the boundary coupling V^{μ}, which, in principle, can be calculated to all generality order by order in perturbation theory. The boundary renormalization group equation is driven by the extrinsic curvature of the brane embedded in $\mathcal{M}$ and the 1-loop result yields the deformation, [14],

$$\frac{\partial}{\partial t} f^{\mu} = \sum_{\sigma=1}^{\text{codim}} H^{\sigma} \hat{n}^{\mu}_{\sigma} \tag{10}$$

perpendicular to the brane. The right-hand side is the mean curvature vector inward to the brane, which is defined in terms of the extrinsic curvature tensor

$$K^{\sigma}_{AB} = G_{\mu\nu} \hat{n}^{\mu}_{\sigma} \left(D_A D_B f^{\nu} + \Gamma^{\nu}_{\rho\lambda} f^{\rho}_{,A} f^{\lambda}_{,B} \right) , \tag{11}$$

using the trace $H^{\sigma} = g^{AB} K^{\sigma}_{AB}$. The induced metric on the brane is defined, as usual, by $g_{AB}(y) = G_{\mu\nu} f^{\mu}_{,A} f^{\nu}_{,B}$ and $\hat{n}^{\mu}_{\sigma}$ form a system of mutually orthogonal unit normal vectors labeled by the codimension of the brane. The fixed points correspond to D-branes enjoying conformal boundary conditions irrespective of the bulk conformal invariance.

As discussed in the paper [15], the boundary renormalization group equation for $f^{\mu}(y; t)$ coincides with the so called *mean curvature flow*, which was introduced independently in mathematics; for extensive reviews see, for instance, the textbooks [16], and references therein. It is the prime example of another class of geometric evolution equations, the *extrinsic* curvature flows, which attracted a lot of attention in recent years. The arena for dynamics is now provided by the infinite dimensional space of all possible embedding functions of a submanifold in Riemannian spaces. The simplest example is described by the linear theory of two free bosons, represented by the plane, in which curves can deform by their extrinsic curvature. Then, the mean curvature flow is also known as curve shortening flow. Closed curves have the tendency to shrink, whereas open ones tend to stretch until they become geodesic lines, [17]. Similar considerations apply to higher dimensions and to hypersurfaces of arbitrary codimension satisfying appropriate technical conditions. Branes can also be immersed allowing for self-intersections. In all cases, the mean curvature flow favors the dissipation of extrinsic curvature until an equilibrium configuration is reach or a singularity. There are also entropy functionals that help us understand the structure of singularities, [18], but their physical interpretation is investigated less than the Ricci flow.

When the ambient space is flat or a more general fixed point of the Ricci flow, the mean curvature flow is defined with respect to a fixed background metric. However, Dirichlet sigma models realize the possibility to have the combined system of Ricci and mean curvature flows and examine competing effects of shrinking curves on deforming backgrounds. Of course, the deformations of the target space metric are insensitive to the presence of embedded branes, while the deformations of branes depend on the metric through their extrinsic curvature. Such equations have not been really studied in mathematics and they are left open to future work.

Further generalizations include the effect of fluxes. An anti-symmetric tensor field coupling in target space can affect the boundary renormalization group equation when it has non-trivial components induced on the brane. There can also be additional $U(1)$ gauge fields coupled to the world-sheet boundary that are quite familiar from the theory of open strings and have their own beta function. Thus, taking into consideration all possible bulk and boundary couplings, one arrives at a generalized system of evolution equations. The bulk flows will be the standard renormalization group equations for the metric $G_{\mu\nu}$, the anti-symmetric tensor field $B_{\mu\nu}$ and the dilaton Φ, if all are present, and they do not depend on boundary conditions. The boundary flows follow by variation of the so called Dirac-Born-Infeld action on the submanifold $\mathcal{N}$, as gradient flows,

$$S_{\rm DBI} = \int_{\mathcal{N}} d^n y \; e^{-\Phi} \sqrt{\det(g + b + F)} \; , \tag{12}$$

where g_{AB} and b_{AB} are the induced components of the metric and anti-symmetric tensor field on the brane and F_{AB} is the field strength of the $U(1)$ gauge field living on it. Specializing to pure metric backgrounds, one recovers the well known description of the mean curvature flow as gradient flow of the area functional of the hypersurface $\mathcal{N}$ so that minimal submanifolds sit at its extremal (critical) points. More generally, the equations that result from the action $S_{\rm DBI}$ lead to interesting extensions of the basic system of flows in sigma models, but they will not be discussed any further in these notes. We refer the interested reader to the original work [14] (but see also [19] for some technical problems) and the publication [15].

There are several simple solutions of the Ricci and mean curvature flows that help us understand how geometric objects can deform. They are designed to depend on a few moduli that satisfy a truncated system of simpler evolution equations, and, as such, they reduce the general problem to mini-superspace models. Here, we will only present solutions that depend on one moduli, namely the radius of uniformly contracting spheres, and refer the interested reader to the literature for more examples and further details (see, for instance, [15], and references therein). Note that spherically symmetric solutions exist for both Ricci and mean curvature flows because the time evolution will respect all isometries, if they are initially present. We also introduce the important concept of solitonic solutions for geometric flows and mention a few examples in two dimensions.

First, let us consider a round n-sphere of radius $a(t)$ undergoing deformations by the Ricci flow. The evolution truncates to a single differential equation for $a(t)$ which is solved by

$$a(t) = \sqrt{a^2(0) - (n-1)t} \; . \tag{13}$$

The solution exists for all time from $-\infty$ to $a^2(0)/(n-1)$, until the space fully collapses to a point for all $n \geq 2$. In the context of $O(n+1)$ non-linear sigma models, it corresponds to the familiar running of the coupling constant $g^2 = a^{-2}$,

$$\frac{1}{\tilde{g}^2} = \frac{1}{g^2} + (n-1)\log\frac{\tilde{\Lambda}}{\Lambda} \tag{14}$$

with respect to the world-sheet renormalization scale Λ^{-1}, setting $t = \log\Lambda^{-1}$. The field theory of a free compact boson (Gaussian model) corresponds to $n = 1$ and does not run. Sigma models with more general target spaces of positive curvature asymptote this special solution close to the big crunch singularity, since they tend to become rounder and rounder as they shrink to a point.

Likewise, for the mean curvature flow, we consider the case of concentric round n-spheres in R^{n+1} with radius $a(t)$. Their evolution reduces to a single differential equation for $a(t)$ that yields

$$a(t) = \sqrt{a^2(0) - 2nt} \; . \tag{15}$$

The solution exists for all time from $-\infty$ to $a^2(0)/2n$, until it fully collapses to a point. The evolution of more general convex hypersurfaces in R^{n+1} asymptotes this special solution close to the big crunch singularity. Note, however, that the contraction rate of deforming n-spheres by mean curvature is always larger than that of the Ricci flow. It is also non-zero for $n = 1$, unlike the obvious invariance of S^1 under the Ricci flow. Thus, it is natural to expect that closed curves on positive curvature spaces will also shrink faster than the ambient space, as they do in most cases, when the combined system of Ricci and mean curvature flows is under consideration. However, there are examples of branes on curved spaces where the configurations become singular simultaneously.

Next, let us define *gradient solitons* of the Ricci flow by the following relation

$$R_{\mu\nu} = -2\nabla_\mu \nabla_\nu \Phi \tag{16}$$

for appropriately chosen field $\Phi(X)$. Solutions of this kind arise when the metric deforms by pure diffeomorphisms with generating vector field $\xi_\mu = -\partial_\mu \Phi$. Gradient solitons can be regarded as fixed points of the metric-dilaton flow and correspond to non-trivial conformal field theories. The Euclidean black-hole in two dimensions with metric and dilaton fields

$$ds^2 = dr^2 + \tanh^2 r d\theta^2 \ , \qquad \Phi(r) = -\log(\cosh r) \tag{17}$$

provides example of a rotationally symmetric gradient Ricci soliton on R^2 with the geometry of a semi-infinite long cigar, [20], [21].

Likewise, gradient solitons of the mean curvature flow are defined by equation

$$\sum_{\sigma=1}^{\text{codim}} H^\sigma \hat{n}^\mu_\sigma = -\partial^\mu \Phi \ , \tag{18}$$

which describes deformations of branes by pure diffeomorphisms generated by a vector field $\xi^\mu = -\partial^\mu \Phi$. The two-dimensional plane supports translating solitons of the form

$$y(x) = -\log(\cos x) \tag{19}$$

with linear dilaton, $\Phi(y) = -y$, that does not affect the conformal field theory of two free bosons on the bulk. This soliton is a curve that asymptotes the lines $x = \pm\pi/2$ as $y \to \infty$ and has a tip located at the orgin of coordinates. It is has become known as *grim-reaper* in the mathematics literature, [17], or *hair-pin* in the physics literature, [22]. Finally, there are simple solutions that arise by placing mean curvature solitons on Ricci solitons, like a hair-pin on a cigar.

As mentioned in the beginning, supersymmetry puts severe constraints on the geometry of the target space manifold and restricts it to be Kähler. Thus, the natural question arises how the bulk renormalization group equation combines with supersymmetry. Since supersymmetry remains unbroken to all orders in perturbation theory, the deformations of the metric should preserve the Kähler class of the metric. Indeed, by introducing a system of complex coordinates $(Z^a, \bar{Z}^a)$ on $\mathcal{M}$, the renormalization of the metric follows the so called Ricci-Kähler flow,

$$\frac{\partial}{\partial t} G_{a\bar{b}} = -R_{a\bar{b}} \tag{20}$$

with the required property. This is a great simplification since all components of the metric depend on a single scalar function, the Kähler potential, which evolves accordingly, and the non-vanishing components of the Ricci curvature tensor are simply expressed as

$$R_{a\bar{b}} = -\frac{\partial^2}{\partial Z^a \partial \bar{Z}^b} \log(\det G) \ . \tag{21}$$

On S^2 with Kähler metric

$$ds^2 = 2e^{\Phi(Z, \bar{Z}; t)} dZ d\bar{Z} \tag{22}$$

the Ricci flow takes the form

$$\frac{\partial}{\partial t} e^\Phi = \frac{\partial^2 \Phi}{\partial Z \partial \bar{Z}} \tag{23}$$

and can be (formally) integrated by group theoretical methods based on Bäcklund transformations, [23]. Here, Φ should not be confused with the dilaton field.

Extended supersymmetry puts more stringent constraints on the target space geometry of sigma models, turning the Kähler condition into hyper-Kähler, [1]. In this case there are three independent Kähler structures and the metric is necessarily Ricci flat. Thus, the models are conformal, since they correspond to fixed points of the Ricci flow, and do not renormalize. If, however, one is willing to drive them away from the fixed points, by mathematical curiosity, the Ricci-Kähler flow will only preserve one Kähler structure and break the others. Conversely, a whole sphere of Kähler structures will arise at the fixed points of the flow on Kähler manifolds of dimension $4k$ and supersymmetry will be enhanced.

It should be noted at this point that Ricci flow is not the only deformation of the metric compatible with supersymmetry. Other geometric flows have been introduced on Kähler manifolds, which also preserve the Kähler class of the metric, but they lead to higher order non-linear differential equations. The prime example is provided by the so called *Calabi flow* that is only defined on Kähler manifolds and assumes the following form, [24],

$$\frac{\partial}{\partial t} G_{a\bar{b}} = \frac{\partial^2 R}{\partial Z^a \partial \bar{Z}^b} \tag{24}$$

in terms of the Ricci scalar curvature R. Clearly, it is a fourth order equation, as opposed to the much simpler second order Ricci-Kähler flow, and turns out to be much harder to investigate by mini-superspace truncations. It preserves the total volume of the space $\mathcal{M}$, unlike the Ricci flow, and the fixed points, called *extremal metrics*, extremize the quadratic curvature functional

$$\mathcal{C} = \int_{\mathcal{M}} dV(G) \ R^2 \ . \tag{25}$$

$\mathcal{C}$ decreases monotonically along the flow, and, as such, it acts as Lyapunov functional for the system. This is another example of intrinsic curvature flows which is closely related to the Ricci-Kähler flow by superevolution, [25].

The Calabi flow has been used in mathematics as tool to study a variety of non-linear problems in Kähler geometry, such as uniformization, since extremal metrics are typically constant curvature metrics. The round 2-sphere is the most elementary example of this kind. On the other hand, the type of deformations that are induced on the metrics are quite different from the renormalization of the bulk coupling of sigma models, and, as a result, Calabi flow has not yet found its proper place in quantum field theory. Of course, this does not exclude the possibility to arrive at this flow in quantum systems other than the sigma model. It is conceivable that the renormalization group analysis of some (yet unknown) quantum field theory may yield the combined system of second order equations

$$\frac{\partial}{\partial t} G_{a\bar{b}} = \frac{\partial^2 \Psi}{\partial Z^a \partial \bar{Z}^b} \ , \qquad \Psi = -G^{c\bar{d}} \frac{\partial^2}{\partial Z^c \partial \bar{Z}^d} \log(\det G) \tag{26}$$

when restricted on a certain line in the space of couplings (G, Ψ). Then, the Calabi flow for $G_{a\bar{b}}$ will simply follow after elimination of Ψ. Yet another possibility is that the Calabi flow can be accommodated in the study of transitions among different vacua of superstring theory, as for the Ricci flow that accounts for the off-shell description of tachyon condensation. Although we will leave these questions open to future work, it is worth mentioning here a concrete physical manifestation of the two-dimensional Calabi flow in the classical theory of general relativity.

Recall the theory of spherical gravitational waves in vacuum as described by the class of four-dimensional Robinson-Trautman radiative metrics, [26],

$$ds^2 = 2r^2 e^{\Phi(Z, \bar{Z}; t)} dZ d\bar{Z} - 2dtdr - H(Z, \bar{Z}, r, t) dt^2 \ . \tag{27}$$

Closed surfaces of constant r and t represent topological 2-spheres with complex coordinates $(Z, \bar{Z})$ and metric coefficient given by $\Phi(Z, \bar{Z}; t)$. It can be verified that some components of Einstein's equations restrict the form of the function H to

$$H = r \frac{\partial \Phi}{\partial t} - \Delta \Phi - \frac{2m}{r} \ , \tag{28}$$

where m is an integration constant with the interpretation of mass parameter and

$$\Delta = e^{-\Phi} \partial \bar{\partial} \tag{29}$$

is the Laplace-Beltrami operator on S^2. When $m > 0$, one may set without loss of generality $3m = 1$ in appropriate units. Then, the remaining Einstein equations yield

$$\Delta \Delta \Phi + \frac{\partial \Phi}{\partial t} = 0 \ , \tag{30}$$

which is known as Robinson-Trautman equation. It is identical to the Calabi flow on S^2 with deformation parameter t given by the retarded time of the four-dimensional metric, [27]. In this case the extremal metric on S^2 is that of a round sphere, with constant curvature, and the corresponding Robinson-Trautman metric is no other but the static Schwarzschild solution of mass m in Eddington-Finkelstein frame. Time dependent solutions represent purely outgoing gravitational radiation in space-time.

Finally, note that the Calabi flow on S^2 can be (formally) integrated by group theoretical methods, [25], as for the Ricci flow. The integration of both Ricci and Calabi flows on two-dimensional surfaces relies on their formulation as zero curvature conditions $[\partial + A, \bar{\partial} + B] = 0$ with respect to Z and $\bar{Z}$. The deformation variable t is fully encoded in the structure of the algebra where the gauge connections $A(Z, \bar{Z}; t)$ and $B(Z, \bar{Z}; t)$ take their values. The relevant framework is provided by the class of infinite dimensional *continual Lie algebras*, [28], with appropriately chosen Cartan operator. Thus, curvature dissipation becomes internal affair of the symmetry group and does not contradict integrability in two dimensions. Further details can be found in the original papers [23], [25], where both evolution equations are treated as continual Toda systems.

The last item that will be discussed here is the so called *Willmore flow*. It describes deformations of embedded branes in Riemannian geometry driven by their extrinsic curvature, but it is fourth order equation as compared to the second order mean curvature flow. As such, it can be regarded as an extrinsic curvature analogue of the Calabi flow. More explicitly, let us consider the case of immersed submanifolds $\mathcal{N}$ in $\mathcal{M}$, which is taken to be R^n for simplicity, and define the Willmore flow as gradient flow of the quadratic curvature functional

$$\mathcal{W} = \frac{1}{2} \int_{\mathcal{N}} dV(g) \; H^2 \; , \tag{31}$$

following, for example, [29], [30], and references therein. H^2 is short-hand notation for the inner product of the mean curvature vector with itself. $\mathcal{W}$ is called Willmore functional although its roots date back to the 18th century. One can add to $\mathcal{W}$ a multiple of the volume functional of the embedded submanifold $\mathcal{N}$,

$$S_{\rm ext} = s_0 \int_{\mathcal{N}} dV(g) \; H^2 \; - t_0 \int_{\mathcal{N}} dV(g) \; , \tag{32}$$

with arbitrary coefficients s_0 and t_0, and obtain interesting modifications of the evolution equation as gradient flow; normalized evolutions can also be obtained in this fashion.

$\mathcal{W}$ provides the elastic energy of curves, when the submanifold is one-dimensional, which by the classic Bernoulli-Euler theory exhibits critical points (under appropriate conditions) called *elasticae*; see, for instance, [31] for a historic overview. For two-dimensional submanifolds, the critical points of $\mathcal{W}$ are called *Willmore surfaces*; see, for instance, [32] for a comprehensive account of Willmore surfaces and related mathematical problems. Since $\mathcal{W}$ is analogous to Calabi's quadratic curvature functional $\mathcal{C}$ one may think of its critical points as being extrinsic curvature analogues of extremal metrics. Thus, one expects that the Willmore flow will evolve a given submanifold towards one of the critical points, modulo singularities. Note, however, a difference with the Calabi flow which does not derive from $\mathcal{C}$ as gradient flow; variants of the Willmore flow can be introduced to match the difference, but they are not be important here. Physical applications of fourth order extrinsic curvature flows include interface diffusion.

In string theory there is already a well studied modification of the classical Nambu-Goto action by adding quadratic extrinsic curvature terms, [33], as in the action $S_{\rm ext}$ above, thinking of $\mathcal{N}$ as two-dimensional world-sheet. In this case strings have tension t_0 as well as rigidity (or stiffness) s_0 that affects their behavior. This is, of course, quite different from the interpretation of $S_{\rm ext}$ as effective action for the beta functions of deforming branes, which do not seem to receive higher order curvature corrections of this type by 2-loop computations, [34]. Branes do not become stiff within the quantum theory of ordinary sigma models, and, as a result, the Willmore flow can not be interpreted as renormalization group equation. However, it might be interesting in this context to provide a systematic definition of Dirichlet sigma models for strings with rigidity and compute the effect that the world-sheet extrinsic curvature terms may have on the beta functions of embedded branes, as for the bulk. The realization of Willmore flow in quantum field theory remains an open problem.

In conclusion, we have considered the interplay of quantum field theory and geometry as it manifests in the renormalization group analysis of sigma models. Two-dimensional sigma models with world-sheet boundary allow for bulk and boundary couplings that depend on the energy scale. These theories have all the necessary geometric data that allows to connect them with the mathematical theory of geometric flows. Indeed, the corresponding beta functions realize and unite the Ricci and mean curvature flows, respectively, and offer new possibilities in mathematics by also including the effect of fluxes. It remains to be seen whether other geometric evolution equations, driven by intrinsic or extrinsic curvature terms, admit a similar realization in quantum field theory. There is plenty of room for making new contributions to this exciting area of current research that already had great impact in science.

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