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Two Superfields in Search of a Theory

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Abstract

We discuss maximal Supergravity and maximal supersymmetric Yang-Mills Theory and show that they can be described in terms of two constrained superfields on the light-cone, *without the use of auxiliary fields.* We show that the same superfield can be used in the maximal dimensions as well as in all those dimensions to which the two theories can be reduced. We stress the similarities between these two superfields that can be seen as masterfields for a series of theories.

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1 Introduction

When we study supersymmetric theories, the maximally supersymmetric ones, ($\mathcal{N} = 8$, $d = 4$) supergravity [1] and ($\mathcal{N} = 4$, $d = 4$) Yang-Mills Theory [2] or their eleven-dimensional or ten-dimensional versions always show up. The $\mathcal{N} = 1$ supergravity in eleven dimensions [3], is the largest supersymmetric local field theory with maximum helicity two (on reduction to $d = 4$). This theory has gained renewed prominence since its recognition as the infrared limit of M-Theory [4]. Although M-Theory casts well-defined shadows on lower-dimensional manifolds, its actual structure remains a mystery. We must therefore glean all we can from the $\mathcal{N} = 1$ supergravity theory or its dimensionally reduced versions before tackling M-Theory. $\mathcal{N} = 1$ supergravity is ultraviolet divergent in $d = 11$ but this divergence is presumably tamed by M-Theory and the hope is that an understanding of this divergent structure, will give us a window into the workings of M-Theory. Similarly the $\mathcal{N} = 1$ Yang-Mills Theory in ten dimensions [2], which is the low-energy limit of the open string theory in ten dimensions has been shown to play an important role in the AdS/CFT duality [5]. The four-dimensional version $\mathcal{N} = 4$ Yang-Mills Theory is also very special since it is free of ultraviolet divergences [6] [7].

In the normal covariant treatments of these two theories they look quite different, one being a reparametrisation invariant gravity theory, while the other is a Yang-Mills gauge theory. However, in the light-cone formulations, the so-called LC_2 formulations where all auxiliary degrees of freedom have been eliminated, [8], [9], the two superfields describing the field content of the two theories are particularly simple and and very much alike. Indeed these superfields can be regarded as master fields for a series of theories. Since they are natural partners in string theory this similarity must be quite important and much of my research in recent years has been to use this similarity and to try to use it to learn more about these theories and the underlying string theory. They are, of course, very well studied during a long time but they have consistently shown themselves to be more interesting than what meets the eye.

Writing the two theories in the LC_2 formulation has a price. We loose a lot of information from the geometry and the only guide-line will be the non-linearly realized supersymmetry algebra. However, it is important to view these very important theories from different angles and for certain question this formulation is the most adequate one.

In this talk I will start from the beginning to build up light-cone field theories and then go over to the supersymmetric ones. I will start the analysis in four dimensions of space-time and then 'oxidize' them to higher dimensions keeping the specific form of the superfields.

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2 Light-Frame Formulation of Field Theories

In his famous paper of 1949 Dirac [10] argued that for a relativitistically invariant theory any direction within the light-cone can be the evolution parameter, the "time". In particular we can use one of the light-cone directions. For this discussion we will use $x^+ = \frac{1}{\sqrt{2}}\,(x^0 + x^3)$ as the time. The coordinates and the derivatives that we will use will then be

$$x^\pm \;=\; \frac{1}{\sqrt{2}}\,(\,x^0 \pm x^3\,)\;; \qquad \partial^\pm = \frac{1}{\sqrt{2}}\,(\,-\,\partial_0 \pm \partial_3\,)\;; \tag{1}$$

$$x \;=\; \frac{1}{\sqrt{2}}\,(\,x_1\,+\,i\,x_2\,)\;; \qquad \bar\partial = \frac{1}{\sqrt{2}}\,(\,\partial_1\,-\,i\,\partial_2\,)\;; \tag{2}$$

$$\bar x \;=\; \frac{1}{\sqrt{2}}\,(\,x_1\,-\,i\,x_2\,)\;; \qquad \partial = \frac{1}{\sqrt{2}}\,(\,\partial_1\,+\,i\,\partial_2\,)\;, \tag{3}$$

so that

$$\partial^+\,x^- = \partial^-\,x^+ \;=\; -\,1\;; \qquad \bar\partial\,x \;=\; \partial\,\bar x \;=+1\;. \tag{4}$$

The derivatives are, of course, related to the momenta through the usual formula $p^\mu = -i\partial^\mu$ and we use the light-cone decomposition also for p^μ. We will only consider massless theories so we solve the condition $p^2 = 0$. We then find

$$p^- = \frac{p\bar p}{p^+}. \tag{5}$$

The generator p^- is really the Hamiltonian conjugated to the light cone time x^+ and we see that the translation generators of the Poincare algebra are written with just three operators. We will use Dirac's vocabulary that generators that involve the "time" are called dynamical (or Hamiltonians) and the others kinematical. Using light-cone notation and the complex one from above for the transverse directions, the most general form of the generators of the full Poincaré algebra at $x^+ = 0$ is then given by the four momenta

$$p^- \;=\; -\,i\frac{\partial\bar\partial}{\partial^+}\;, \qquad p^+ \;=\; -\,i\,\partial^+\;, \qquad p \;=\; -\,i\,\partial\;, \qquad \bar p \;=\; -\,i\,\bar\partial\;, \tag{6}$$

the kinematical transverse space rotation

$$j \;= j^{12} = \; x\,\bar\partial - \bar x\,\partial + \lambda, \tag{7}$$

the other kinematical generators

$$j^+ \;=\; i\,x\,\partial^+\;, \qquad \bar j^+ \;=\; i\,\bar x\,\partial^+\;, \tag{8}$$

and

$$j^{+-} \;=\; i\,x^-\,\partial^+, \tag{9}$$

as well as the dynamical boosts

$$j^- \;=\; i\,x\,\frac{\partial\bar\partial}{\partial^+} \;-\; i\,x^-\,\partial \;+ i\lambda\frac{\partial}{\partial^+}, \tag{10}$$

$$\bar j^- \;=\; i\,\bar x\,\frac{\partial\bar\partial}{\partial^+} \;-\; i\,x^-\,\bar\partial \;+ i\lambda\frac{\bar\partial}{\partial^+}. \tag{11}$$

There is one degree of freedom in the algebra, namely the parameter λ which is the helicity. At this stage it is arbitrary and checking the corresponding spin one finds, of course, that it is $|\lambda|$. Hence the algebra covers all possible free field theories. We can let the generators act on a complex field $\phi(x)$ with helicity λ, with its complex conjugate having the opposite helicity. This is the "first-quantized" version. We can also consider the fields as operators having the commutation relation.

$$[\partial^+\bar{\phi}(x), \phi(x')] = -\frac{i}{2}\delta(x-x'), \tag{12}$$

where hence the momentum field conjugate to ϕ is $\partial^+\bar{\phi}$.

We then introduce the "second-quantized" representation O in terms of the "first-quantized" representation o as $O = 2i\int d^4x \partial^+\bar{\phi}(x)\ o\ \phi(x)$. We then find that the commutator between two of the generators J_1 and J_2 is

$$[J_1, J_2] = 2i\int d^4x \partial^+\bar{\phi}(x)[j_1, j_2]\phi(x). \tag{13}$$

We can understand that P^- truly is the Hamiltonian using equ.(5)

$$P^- = 2\int d^4x \partial^+\bar{\phi}(x)\frac{\partial\bar{\partial}}{\partial^+}\phi(x). \tag{14}$$

Legendre transforming to the Lagrangian using the field momenta from equ.(12) we get the action

$$\begin{aligned} S &= \int d^4x[\partial^+\bar{\phi}(x)\partial^-\phi(x) + \partial^+\phi(x)\partial^-\bar{\phi}(x) - 2\partial^+\bar{\phi}(x)\frac{\partial\bar{\partial}}{\partial^+}\phi(x)] \\ &= \int d^4x \partial^+\bar{\phi}(x)\Box\phi(x). \end{aligned} \tag{15}$$

It is remarkable that there is a unique form of the kinematic term for any spin-λ field. We should remember though that to specify the theory we have to give all Poincaré generators, since the action via the Hamiltonian is just one of those generators. They will show what spin the field describes.

In this representation it is straightforward to try to add interaction terms to the Hamiltonian. This was done in [11]. Every dynamical generator will have interaction terms. The procedure is very painstaking and there are as far as I know no other way than trial and error to find the non-linear representation. On the other hand, once such a representation is found it represents a possible relativistically invariant interacting field theory. The result is that for every integer λ there exists a possible three-point interaction. For λ even, the unique solutions are

$$\begin{aligned} S = \int d^4x \Big\{ & \bar{\phi}(x)\Box\phi(x) \\ & + g\Big[\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\bar{\phi}(x)\partial^{+\lambda}\big(\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\lambda-n}}\phi(x)\frac{\bar{\partial}^{\lambda}}{\partial^{+\lambda}}\phi(x)\big) + c.c.\Big]\Big\} \\ & + O(g^2). \end{aligned} \tag{16}$$

For λ odd, the field $\phi(x)$ must be in the adjoint representation of an external group $\phi^a(x)$ and we have to introduce the fully antisymmetric structure constants f^{abc} in the interaction terms to find a possible term. The results is

$$\begin{aligned} S = \int d^4x \Big\{ & \bar{\phi}^a(x)\Box\phi^a(x) \\ & + gf^{abc}\Big[\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\bar{\phi}^a(x)\partial^{+\lambda}\big(\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\lambda-n}}\phi^b(x)\frac{\bar{\partial}^{\lambda}}{\partial^{+\lambda}}\phi^c(x)\big) + c.c.\Big]\Big\} \\ & + O(g^2). \end{aligned} \tag{17}$$

We note the non-locality in the interaction term in terms of inverses of ∂^+. The easiest way to understand it is to Fourier transform to momentum space. In the calculations it is really defined by the rule $\frac{1}{\partial^+}\partial^+ f(x^+) = f(x^+)$. When performing a calculation one has to specify exactly the situation of the pole in ∂^+. In an sense this is a remainder of the gauge invariance.

We can now check for special values of λ.

- *$\lambda = 0$*

The dimension of the coupling constant g is 1 (in mass units) and this is the usual ϕ^3- theory. This theory is superrenormalizable but not physical since it does not have a stable vacuum having a potential with no minimum.

- *$\lambda = 1$*

The dimension of the coupling constant g is 0 and this theory is nothing but non-abelian gauge theory in a specific gauge. If we go on we know that we need a four-point coupling to fully close the algebra. Note that the action has no local symmetry and the gauge group only appears as the external symmetry group.

- *$\lambda = 2$*

The dimension of the coupling constant g is -1 and this theory is the beginning series of a gravity theory. It is clear from the dimensions of the coupling constant that interaction terms to arbitrary order can be constructed without serious non-localities. The four-point function related to Einstein's theory is known [12]. Going beyond the four-point coupling is probably too difficult, unless powerful computer methods could be devised. We expect several solutions, of course, since we know that the Hilbert action is but the simplest of all actions consistent with the equivalence principle. Note that the action above, which is a fully gauge fixed Hilbert action expanded in the fluctuations around the Minkowski metric, has no local symmetry, no covariance and knows nothing about curved spaces. It is probably useless for discussions about global properties of space and time but can be useful in the study of quantum corrections; to understand the finiteness properties of the quantum theory.

- *$\lambda > 2$*

The dimension of the coupling constant g is < -1 and these theories are theories for higher spins. Again they are non-renormalizable in the naive sense like the the spin-2 theory above. There are strong reason to believe that these theories cannot be Poincaré invariant one by one when we go to higher orders in the coupling constant, but the result above is an indication that certain sums of such theories interacting with each other could possibly be invariant theories.

We can also find interacting solutions for λ half-integer. We can, of course, not have a three-point coupling. We will in fact not be able to find self-interacting theories but have to consider the coupling of the half-integer spin field to an integer spin field. (A quartic fermion coupling is possible, but it would not lead to a renormalizable theory so we do not pursue it here.) We then find that we can couple a spin-$\frac{1}{2}$ field to a spin-1 or a spin-0 field to recover in the first case a non-abelian gauge field coupled to a spin-$\frac{1}{2}$ field $\psi^i(x)$ in a representation characterized by i of the external group such that we can have a coupling $\bar{\psi}_i\psi^j\phi^a C^i_{ja}$, with C^i_{ja} the Clebsch-Gordan coefficient. It is interesting to note that it is only in the interacting theory that we can prove the spin-statistics theorem [13]. The formalism demands the spin-$\frac{1}{2}$ field to be of odd Grassmann type and the integer spin fields to be even. Note that there is no spinor space. The spin-$\frac{1}{2}$ field is a complex (Grassmann odd) field with no space-time index. Its equation of motion looks just like the one for a bosonic field. (Remember the free equation the follows from equ. (18).) However, the dimension of the field $\psi(x)$ is different from the one of the bosonic field, so the free action is

$$S = \int d^4x \partial^+ \bar{\psi}(x) \frac{\Box}{\partial^+} \psi(x). \tag{18}$$

The fact that we do not need to use spinors is very special for $d = 4$, since the transverse symmetry which is covariantly realized is $SO(2) \approx U(1)$, which does not distinguish spinor representations.

We have hence seen that we can find all known unitary relativistic field theories as representations of the Poincaré algebra, and we see their uniqueness and also what kind of possibilities there are for higher spin fields. In a gauge invariant formulation one can attempt to add in new terms that are gauge invariant. Invariably they lead to problems with unitarity. We do not see those terms here since the theories are unitary by construction.

3 Light-Frame Formulation of Supersymmetric Field Theories

The known extension of the Poincaré algebra is to make it into a supersymmetry algebra. This will lead to a restriction on relativistic dynamics. It is true that the world does not look supersymmetric as such, but a good working hypothesis is that at some stage supersymmetry is indeed a symmetry of the world.

Supersymmetry is an augmentation of the Poincaré algebra with a spinor generator Q_α with the anti-commutator

$$\{Q_\alpha, \bar{Q}_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu . \tag{19}$$

The spinor Q_α is four-component. It satisfies the so-called Majorana condition which makes it real in a certain representation of the γ-matrices. In the light-cone frame the spinor splits up into two two-component spinor that can be rewritten as two complex operators, which we call $Q_+ = -\frac{1}{2}\gamma_+\gamma_- Q$ and $Q_- = -\frac{1}{2}\gamma_-\gamma_+ Q$. From the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta = diag(-1,1,1,1)$ we see that $Q = Q_+ + Q_-$, and that the products $-\frac{1}{2}\gamma_+\gamma_-$ and $-\frac{1}{2}\gamma_-\gamma_+$ are projection operators. We can linearly combine the two components of the spinors into complex entities with no indices. We can also augment by letting the Q's transform as the representation $\mathbf{N}$ under $SU(N)$. The light-cone supersymmetry algebra is then

$$\{Q^m_+, \bar{Q}_{+n}\} = -\sqrt{2}\delta^m_n P^+ \tag{20}$$
$$\{Q^m_-, \bar{Q}_{-n}\} = -\sqrt{2}\delta^m_n P^- \tag{21}$$
$$\{Q^m_+, \bar{Q}_{-n}\} = -\sqrt{2}\delta^m_n P, \tag{22}$$

where all other anticommutators are zero, except for the complex conjugate of the last one. The indices m, n run from 1 to N.

The superPoincaré algebra can now be represented on a superspace with coordinates $x^\pm, x, \bar{x}, \theta^m, \bar{\theta}_n$, where the coordinates θ^m and $\bar{\theta}_n$ are complex conjugates, Grassmann odd and transform as $\mathbf{N}$ and $\bar{\mathbf{N}}$ under $SU(N)$. We will denote their derivatives as

$$\bar{\partial}_m \equiv \frac{\partial}{\partial\,\theta^m} \; ; \qquad \partial^m \equiv \frac{\partial}{\partial\,\bar{\theta}_m} \; . \tag{23}$$

The Q's are then represented as (We use the notation with lower case letters for operators that act on the field.)

$$q^m_+ = -\partial^m + \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+ \; ; \qquad \bar{q}_{+n} = \bar{\partial}_n - \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \; , \tag{24}$$

and the dynamical ones as

$$q^m_- = \frac{\bar{\partial}}{\partial^+}\,q^m_+ \; , \qquad \bar{q}_{-m} = \frac{\partial}{\partial^+}\,\bar{q}_{+m} \; . \tag{25}$$

On this space we can also represent "chiral" derivatives anticommuting with the supercharges Q.

$$d^m = -\partial^m - \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+ \; ; \qquad \bar{d}_n = \bar{\partial}_n + \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \; , \tag{26}$$

which satisfy the anticommutation relations

$$\{ d^m , \bar{d}_n \} = -i \sqrt{2} \, \delta^m{}_n \, \partial^+ \ . \tag{27}$$

To find an irreducible representation we have to impose the the chiral constraints

$$d^m \phi = 0 \ ; \qquad \bar{d}_m \bar{\phi} = 0 \ , \tag{28}$$

on a complex superfield $\phi(x^\pm, x, \bar{x}, \theta^m, \bar{\theta}_n)$. The solution is then that

$$\phi = \phi(x^+, y^- = x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m, x, \bar{x}, \theta^m). \tag{29}$$

We now have to add in θ-terms into the Lorentz generators to complete the representation of the free algebra. The result is for $\lambda = 0$

$$j = x \bar{\partial} - \bar{x} \partial + S^{12} \ , \tag{30}$$

where the little group helicity generator is

$$S^{12} = \frac{1}{2} (\theta^p \bar{\partial}_p - \bar{\theta}_p \partial^p) - \frac{i}{4\sqrt{2}\,\partial^+} (d^p \bar{d}_p - \bar{d}_p d^p). \tag{31}$$

It ensures that the chirality constraints are preserved

$$[j , d^m] = [j , \bar{d}_m] = 0 \ . \tag{32}$$

The other kinematical generators are

$$j^+ = i \, x \, \partial^+ \ , \qquad \bar{j}^+ = i \, \bar{x} \, \partial^+ \ . \tag{33}$$

The rest of the generators must be specified separately for chiral and antichiral fields. Acting on ϕ, we have

$$j^{+-} = i \, x^- \, \partial^+ - \frac{i}{2} (\theta^p \bar{\partial}_p + \bar{\theta}_p \partial^p) \ , \tag{34}$$

chosen so as to preserve the chiral combination

$$[j^{+-} , y^-] = - i \, y^- \ , \tag{35}$$

and such that its commutators with the chiral derivatives

$$[j^{+-} , d^m] = \frac{i}{2} d^m \ , \qquad [j^{+-} , \bar{d}_m] = \frac{i}{2} \bar{d}_m \ , \tag{36}$$

preserve chirality. Similarly the dynamical boosts are

$$\begin{aligned} j^- &= i \, x \frac{\partial \bar{\partial}}{\partial^+} - i \, x^- \partial + i \Big(\theta^p \bar{\partial}_p + \frac{i}{4\sqrt{2}\,\partial^+} (d^p \bar{d}_p - \bar{d}_p d^p) \Big) \frac{\partial}{\partial^+} \ , \\ \bar{j}^- &= i \, \bar{x} \frac{\partial \bar{\partial}}{\partial^+} - i \, x^- \bar{\partial} + i \Big(\bar{\theta}_p \partial^p + \frac{i}{4\sqrt{2}\,\partial^+} (d^p \bar{d}_p - \bar{d}_p d^p) \Big) \frac{\bar{\partial}}{\partial^+} \ . \end{aligned} \tag{37}$$

They do not commute with the chiral derivatives,

$$[j^- , d^m] = \frac{i}{2} d^m \frac{\partial}{\partial^+} \ , \qquad [j^- , \bar{d}_m] = \frac{i}{2} \bar{d}_m \frac{\partial}{\partial^+} \ , \tag{38}$$

but do not change the chirality of the fields on which they act. They satisfy the Poincaré algebra, in particular

$$[\,j^-\,,\,\bar{j}^+\,] \;=\; -\,i\,j^{+-} - j\;, \qquad [\,j^-\,,\,j^{+-}\,] \;=\; i\,j^-\;. \tag{39}$$

We can now follow the same path as we did in the last section to go over to a "second-quantized" version in terms of integrals over the superfield and then add interaction terms to the dynamical generators and try to close the algebra. In this way we can construct all the known supersymmetric field theories as different representations of various supersymmetry algebras with different values of λ and N. It is particularly interesting to study the cases $N = 4 \times \mathit{integer}$. For those values one can impose a further condition on the superfield ϕ namely the "inside out" condition

$$\begin{aligned} &\bar{d}_{m_1}\,\bar{d}_{m_2}\,..\bar{d}_{m_{N/2-1}}\,\bar{d}_{m_{N/2}}\,\phi \;= \\ &\quad \frac{1}{2}\epsilon_{m_1 m2\,...m_{N/2}\,...m_{N-1}m_N}\,d^{m_{N/2+1}}\,d^{m_{N/2+2}}\,...d^{m_{N-1}}\,d^{m_N}\,\bar{\phi}\;. \end{aligned} \tag{40}$$

We can now construct three-point interaction terms for any $\frac{N}{4}$ even in the dynamical generators. This is certainly a tedious exercise based on writing the most general terms in the interaction terms and then check the full algebra. The resulting action is [9]

$$\begin{aligned} S \;&=\; \int d^4x d^N\theta d^N\bar{\theta}\,\Big\{\bar{\phi}(x,\theta)\frac{\Box}{\partial^{+\frac{N}{2}}}\phi(x,\theta) \\ &+\; \frac{4g}{3}\Big[\sum_{n=0}^{\frac{N}{4}}(-1)^n\binom{\frac{N}{4}}{n}\frac{1}{\partial^{+N/2}}\bar{\phi}(x,\theta)\bar{\partial}^{\frac{N}{4}-n}\partial^{+n}\phi(x,\theta)\bar{\partial}^n\partial^{+\frac{N}{4}-n}\phi(x,\theta) \\ &+\; c.c.\Big]\Big\} + O(g^2). \end{aligned} \tag{41}$$

When $\frac{N}{4}$ is odd, again the superfield has to transform as the adjoint representation of an external group with structure constants f^{abc}. The corresponding action is then

$$\begin{aligned} S \;&=\; \int d^4x d^N\theta d^N\bar{\theta}\,\Big\{\bar{\phi}^a(x,\theta)\frac{\Box}{\partial^{+\frac{N}{2}}}\phi^a(x,\theta) \\ &+\; \frac{4g}{3}f^{abc}\Big[\sum_{n=0}^{\frac{N}{4}}(-1)^n\binom{\frac{N}{4}}{n}\frac{1}{\partial^{+N/2}}\bar{\phi}^a(x,\theta)\bar{\partial}^{\frac{N}{4}-n}\partial^{+n}\phi^b(x,\theta)\bar{\partial}^n\partial^{+\frac{N}{4}-n}\phi^c(x,\theta) \\ &+\; c.c.\Big]\Big\} + O(g^2). \end{aligned} \tag{42}$$

We note that we can construct theories with higher spin if $\frac{N}{4} > 2$. These are then very special combinations of the theories constructed in the previous section, with better quantum properties, since we know by experience that the more supersymmetry there is the better are the quantum properties.

3.1 Maximally supersymmetric Yang-Mills Theory

The case $N = 4$ is especially interesting [2]. All the physical degrees of freedom are present in the superfield which can be expanded as

$$\begin{aligned} \phi\,(y) \;&=\; \frac{1}{\partial^+}\,A\,(y) + \frac{i}{\sqrt{2}}\,\theta^m\,\theta^n\,\overline{C}_{mn}\,(y) + \frac{1}{12}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\epsilon_{mnpq}\,\partial^+\,\bar{A}\,(y) \\ &\quad + \frac{i}{\partial^+}\,\theta^m\,\bar{\chi}_m(y) + \frac{\sqrt{2}}{6}\theta^m\,\theta^n\,\theta^p\,\epsilon_{mnpq}\,\chi^q(y)\;. \end{aligned} \tag{43}$$

The fields A and $\bar{A}$ constitute the two helicities of a vector field while the antisymmetric $SU(4)$ bi-spinors C_{mn} represent six scalar fields since they satisfy

$$\overline{C}_{mn} \;=\; \frac{1}{2}\,\epsilon_{mnpq}\,C^{pq} \;. \tag{44}$$

The fermion fields are denoted by χ^m and $\bar{\chi}_m$. All have adjoint indices (not shown here), and are local fields in the modified light-cone coordinates . This is the maximal supersymmetric Yang-Mills theory. The full action is known [8]

$$\begin{aligned} \mathcal{S} \;=\; & -\int d^4x \int d^4\theta\, d^4\bar{\theta}\, \Big\{ \bar{\phi}^a\, \frac{\Box}{\partial^{+2}}\, \phi^a + \frac{4g}{3}\, f^{abc} \Big(\frac{1}{\partial^+}\, \bar{\phi}^a\, \phi^b\, \bar{\partial}\, \phi^c + \text{c.c.} \Big) \\ & -g^2 f^{abc}\, f^{ade} \Big(\frac{1}{\partial^+} (\phi^b\, \partial^+ \phi^c) \frac{1}{\partial^+}\, (\bar{\phi}^d\, \partial^+\, \bar{\phi}^e) + \frac{1}{2}\, \phi^b \bar{\phi}^c\, \phi^d\, \bar{\phi}^e \Big) \Big\}. \end{aligned} \tag{45}$$

With this action it was shown [6] that the perturbation expansion is finite. There is no need for renormalization and the theory is very special. It is one of the cornerstones of modern particle physics. From the point of this lecture it appears as a very special representation of the superPoincaré algebra.

3.2 Maximal Supergravity

The next case is $N = 8$ [8]. In this case the superfield can be expanded as

$$\begin{aligned} \phi\,(y) \;=\; & \frac{1}{\partial^{+2}}\, h\,(y) \,+\, i\,\theta^m\, \frac{1}{\partial^{+2}}\, \bar{\psi}_m\,(y) \,+\, i\,\theta^{mn}\, \frac{1}{\partial^+}\, \bar{A}_{mn}\,(y) \\ & -\,\theta^{mnp}\, \frac{1}{\partial^+}\, \bar{\chi}_{mnp}\,(y) \,-\, \theta^{mnpr}\, C_{mnpr}\,(y) \,+\, i\,\tilde{\theta}^{(5)}_{mnp}\, \chi^{mnp}(y) \\ & +\, i\,\tilde{\theta}^{(6)}_{mn}\, \partial^+\, A^{mn}(y) \,+\, \tilde{\theta}^{(7)}_m\, \partial^+\, \chi^m(y) \,+\, \tilde{\theta}^{(8)}\, \partial^{+2}\, \bar{h}\,(y) \;, \end{aligned} \tag{46}$$

where

$$\theta^{m_1\,\dots\,m_n} \;\equiv\; \frac{1}{n!}\, \theta^{m_1}\, \dots\, \theta^{m_n}, \quad \tilde{\theta}^{(n)}_{n_1\,\dots\,n_{8-n}} \;\equiv\; \frac{1}{n!}\, \theta^{m_1\,\dots\,m_n}\, \epsilon_{m_1\,\dots\,m_n\,n_1\,\dots\,n_{8-n}} \;. \tag{47}$$

The helicity in the field goes from 2 to -2 and the theory has a spectrum comprised of a metric, twenty-eight vector fields, seventy scalar fields, fifty-six spin one-half fields and eight spin three-half fields. This theory is the maximal supergravity theory in $d = 4$. The action can be simplified [18] to

$$S = \int d^4x d^8\theta d^8\bar{\theta} \Big\{ \bar{\phi}(x,\theta) \frac{\Box}{\partial^{+4}} \phi(x,\theta) \,+\, \frac{3}{2}\, g\, \frac{1}{\partial^{+2}}\, \bar{\phi}\;\; \bar{\partial}\, \phi\, \bar{\partial}\, \phi + c.c. \Big] \Big\} + O(g^2). \tag{48}$$

The four-point coupling was finally found last year [15]. It is however quite complicated and reflects the fact that the ∂^+-derivatives can be sprinkled out in very many ways. It is remarkable though, that the actions for the maximally supersymmetric Yang-Mills Theory and Supergravity Theory are so similar. In some sense the Supergravity Theory is just an extension of the Yang-Mills one. In the modern particle physics these two theories are very intimately connected even though the direct physical consequences of them look quite different.

3.3 The Hamiltonian as a Quadratic Form

The two theories share also another unique property. We note that the free Hamiltonian for the ($\mathcal{N} = 4$, $d = 4$) Yang-Mills Theory

$$H^0 \;=\; \int d^4x\, d^4\theta\, d^4\bar{\theta}\, \bar{\phi}^a\, \frac{2\,\partial\bar{\partial}}{\partial^{+2}}\, \phi^a \;, \tag{49}$$

can be rewritten as a quadratic form

$$H^0 \;=\; \frac{1}{2\sqrt{2}}\,(\,\mathcal{W}_0\,,\,\mathcal{W}_0\,)\;, \tag{50}$$

using the inner product notation

$$(\,\phi\,,\,\xi\,) \;\equiv\; 2i\int d^4x\,d^4\theta\,d^4\,\bar\theta\;\bar\phi\,\frac{1}{\partial^+}\xi\;, \tag{51}$$

where ϕ and ξ are chiral superfields and

$$\mathcal{W}_0^a \;=\; \frac{\partial}{\partial^+}\bar q_+\phi^a\;, \tag{52}$$

is a fermionic superfield, the *free* dynamical supersymmetry variation of the superfield ($SU(4)$ spinor indices are summed over). The proof is straightforward, and requires integration by parts and the use of of the inside-out property of the superfields.

Also the fully interacting Hamiltonian [14] can be expressed as a quadratic form

$$H \;=\; \frac{1}{2\sqrt{2}}\,(\,\mathcal{W}^a\,,\,\mathcal{W}^a\,)\;, \tag{53}$$

where now

$$\mathcal{W}^a \;=\; \frac{\partial}{\partial^+}\bar q_+\phi^a - gf^{abc}\frac{1}{\partial^+}(\bar d\phi^b\partial^+\phi^c)\;, \tag{54}$$

is the complete (classical) dynamical supersymmetry variation. The power of supersymmetry allows for this simple rewriting of the fully interacting Hamiltonian.

The same can be shown to be true also for the ($\mathcal{N}=8$, $d=4$) supergravity [15] and was an important clue to find the four-point coupling. We have not found any other theory with this property which again renders the two models to behave very similarly and to have unique properties.

Note that this is not the same as the statement that the Hamiltonian satisfies

$$H = -2\sqrt{2}\{Q_-^m, \bar Q_{-m}\}. \tag{55}$$

4 Light-Frame Formulations of Higher Dimensional Theories

The procedure to find representations of the Poincaré algebra that we have followed in the previous section can, of course, be extended to field theories in dimensions of space-time higher than four. The covariant subalgebra which will be linearly realized is then $SO(d-2)$, so the physical fields will be representations of this algebra and hence characterized by these representations like we used helicity to distinguish the physical fields in four dimensions. If we just implement Poincaré invariance as in sect.2 we can, in principle, find all the possible field theories. However, the procedure gets easily tedious and furthermore there are few interesting quantum field theories in higher dimensions because of the renormalization problems. The only ones that are discussed are supersymmetric field theories since they are connected to the Superstring Theory. The ones that we have been interested in are the ones which lead to interesting field theories when compactified to four dimensions, so let us concentrate on those. The ones I will discuss here are ten-dimensional SuperYang-Mills and eleven-dimensional Supergravity which under compactification leads to the maximal theories discussed above.

4.1 Ten-Dimensional SuperYang-Mill Theory

The physical degrees of freedom of this theory are $\mathbf{8_v}$ and an $\mathbf{8_s}$. If we insist that the superfield should be a representation of the transverse $SO(8)$ it must be in one of the representations above. Since the natural spinor coordinate will also be an $\mathbf{8_s}$, such a superfield must include 8×2^8 components and must hence be very strongly restricted. Such a formalism has been developed [16], but it is not clear that the formalism is useful. Also it is not easily generalizable to the eleven-dimensional case. Instead I will describe a recent procedure developed in [17].

The idea is to use the same superfield as in four dimensions. In order to do that we have to sacrifice the explicit covariance under $SO(8)$ and use the decomposition

$$SO(8) \supset \; SO(2) \; \times \; SO(6) \; . \tag{56}$$

Since $SO(6) \sim SU(4)$ we can identify the $SU(4)$ as the external symmetry group in the superfield equ. (46). The remaining symmetry $SO(8)/(SO(6) \times SO(2))$ will transform among the components of the superfield. First of all, the transverse light-cone space variables need be generalized to eight. We stick to the represention used in the superfield, and introduce the six extra coordinates and their derivatives as antisymmetric bi-spinors

$$x^{m\,4} \;=\; \frac{1}{\sqrt{2}}\,(\,x_{m+3} \,+\, i\,x_{m+6}\,)\;, \qquad \partial^{m\,4} \;=\; \frac{1}{\sqrt{2}}\,(\,\partial_{m+3} \,+\, i\,\partial_{m+6}\,)\;, \tag{57}$$

for $m \neq 4$, and their complex conjugates

$$\bar{x}_{pq} \;=\; \frac{1}{2}\,\epsilon_{pqmn}\,x^{mn}\;; \qquad \bar{\partial}_{pq} \;=\; \frac{1}{2}\,\epsilon_{pqmn}\,\partial^{mn}\;. \tag{58}$$

Their derivatives satisfy

$$\bar{\partial}_{mn}\,x^{pq} \;=\; (\,\delta_m{}^p\,\delta_n{}^q \,-\, \delta_m{}^q\,\delta_n{}^p\,)\;; \qquad \partial^{mn}\,\bar{x}_{pq} \;=\; (\,\delta^m{}_p\,\delta^n{}_q \,-\, \delta^m{}_q\,\delta^n{}_p\,)\;, \tag{59}$$

and

$$\partial^{mn}\,x^{pq} \;=\; \frac{1}{2}\,\epsilon^{pqrs}\,\partial^{mn}\,\bar{x}_{rs} \;=\; \epsilon^{mnpq}\;. \tag{60}$$

There are then no modifications to be made to the chiral superfield, except for the dependence on the extra coordinates

$$A(y) \;=\; A(x, \bar{x}, x^{mn}, \bar{x}_{mn}, y^-)\;, \quad etc...\;\; . \tag{61}$$

These extra variables will be acted on by new operators that generate the higher-dimensional symmetries.

4.2 The SuperPoincaré Algebra in 10 Dimensions

The SuperPoincaré algebra needs to be generalized from the form in four dimensions. One starts with the construction of the $SO(8)$ little group using the decomposition $SO(8) \supset SO(2) \times SO(6)$. The $SO(2)$ generator is the same; the $SO(6) \sim SU(4)$ generators are given by

$$\begin{aligned} j^m{}_n \;=\;& \frac{1}{2}\,(\,x^{mp}\,\bar{\partial}_{pn} \,-\, \bar{x}_{pn}\,\partial^{mp}\,) \,-\, \theta^m\,\bar{\partial}_n \,+\, \bar{\theta}_n\,\partial^m \,+\, \frac{1}{4}\,(\,\theta^p\,\bar{\partial}_p \,-\, \bar{\theta}_p\,\partial^p\,)\,\delta^m{}_n \\ &+\frac{i}{2\sqrt{2}\,\partial^+}\,(\,d^m\,\bar{d}_n - \bar{d}_n\,d^m\,) + \frac{i}{8\sqrt{2}\,\partial^+}\,(\,d^p\,\bar{d}_p - \bar{d}_p\,d^p\,)\,\delta^m{}_n\;. \end{aligned} \tag{62}$$

Note that we use the same spinors as in 4 dimensions because of the decomposition $SO(8) \supset SO(2) \times SO(6)$, where $SO(6) \sim SU(4)$. The extra terms with the d and $\bar{d}$ operators are not necessary for closure of the algebra. However they insure that the generators commute with the chiral derivatives. They satisfy the commutation relations

$$\Big[\, j\,,\, j^m{}_n \Big] \;=\; 0\;, \qquad \Big[\, j^m{}_n\,,\, j^p{}_q \Big] \;=\; \delta^m{}_q\, j^p{}_n - \delta^p{}_n\, j^m{}_q\;. \tag{63}$$

The remaining $SO(8)$ generators lie in the coset $SO(8)/(SO(2)\times SO(6))$

$$\begin{aligned}
j^{pq} \;&=\; x\,\partial^{pq} - x^{pq}\,\partial + \frac{i}{\sqrt{2}}\,\partial^+\,\theta^p\,\theta^q - i\,\sqrt{2}\,\frac{1}{\partial^+}\,\partial^p\,\partial^q + \frac{i}{\sqrt{2}\,\partial^+}\,d^p\,d^q\;,\\
\bar{j}_{mn} \;&=\; \bar{x}\,\bar{\partial}_{mn} - \bar{x}_{mn}\,\bar{\partial} + \frac{i}{\sqrt{2}}\,\partial^+\,\bar{\theta}_m\,\bar{\theta}_n - i\,\sqrt{2}\,\frac{1}{\partial^+}\,\bar{\partial}_m\,\bar{\partial}_n + \frac{i}{\sqrt{2}\,\partial^+}\,\bar{d}_m\,\bar{d}_n\;.
\end{aligned} \tag{64}$$

All $SO(8)$ transformations are specially constructed so as not to mix chiral and antichiral superfields,

$$[\, j^{mn}\,,\, \bar{d}_p\,] \;=\; 0\;; \qquad [\, \bar{j}_{mn}\,,\, d^p\,] \;=\; 0\;, \tag{65}$$

and satisfy the $SO(8)$ commutation relations

$$\begin{aligned}
\Big[\, j\,,\, j^{mn} \Big] \;&=\; j^{mn}\;, \qquad \Big[\, j\,,\, \bar{j}_{mn} \Big] \;=\; -\,\bar{j}_{mn}\;,\\
\Big[\, j^m{}_n\,,\, j^{pq} \Big] \;&=\; \delta^q{}_n\, j^{mp} \,-\, \delta^p{}_n\, j^{mq}\;, \qquad \Big[\, j^m{}_n\,,\, \bar{j}_{pq} \Big] \;=\; \delta^m{}_q\,\bar{j}_{np} \,-\, \delta^m{}_p\,\bar{j}_{nq}\;,\\
\Big[\, j^{mn}\,,\, \bar{j}_{pq} \Big] \;&=\; \delta^m{}_p j^n{}_q \,+\, \delta^n{}_q j^m{}_p \,-\, \delta^n{}_p j^m{}_q \,-\, \delta^m{}_q j^n{}_p \,-\, (\,\delta^m{}_p\,\delta^n{}_q \,-\, \delta^n{}_p\,\delta^m{}_q\,)\, j\;.
\end{aligned}$$

Rotations between the 1 or 2 and 4 through 9 directions induce on the chiral fields the changes

$$\delta\,\phi \;=\; \Big(\frac{1}{2}\,\omega_{mn}\, j^{mn} + \frac{1}{2}\,\bar{\omega}^{mn}\,\bar{j}_{mn}\Big)\,\phi\;, \tag{66}$$

where complex conjugation is like duality

$$\bar{\omega}_{pq} \;=\; \frac{1}{2}\,\epsilon_{mnpq}\,\omega^{mn}\;. \tag{67}$$

For example, a rotation in the $1-4$ plane through an angle θ corresponds to taking $\theta=\omega_{14}=\omega_{23}$ ($=\omega^{23}=\omega^{14}$ by reality), all other components being zero. Finally, we verify that the kinematical supersymmetries are duly rotated by these generators

$$[\, j^{mn}\,,\, \bar{q}_{+\,p}\,] \;=\; \delta^n{}_p\, q^m_+ \,-\, \delta^m{}_p\, q^n_+\;; \qquad [\, \bar{j}_{mn}\,,\, q^p_+\,] \;=\; \delta_n{}^p\,\bar{q}_{+\,m} \,-\, \delta_m{}^p\,\bar{q}_{+\,n}\;. \tag{68}$$

We now use the $SO(8)$ generators to construct the SuperPoincaré generators

$$\begin{aligned}
j^+ \;&=\; i\,x\,\partial^+\;; \qquad \bar{j}^+ \;=\; i\,\bar{x}\,\partial^+\\
j^{+\,mn} \;&=\; i\,x^{mn}\,\partial^+\;; \qquad \bar{j}^+{}_{mn} \;=\; i\,\bar{x}_{mn}\,\partial^+\;.
\end{aligned} \tag{69}$$

The dynamical boosts are now

$$\begin{aligned}
j^- \;=\; i\,x\,\frac{\partial\bar{\partial} \,+\, \frac{1}{4}\,\bar{\partial}_{pq}\,\partial^{pq}}{\partial^+} \;&-\; i\,x^-\,\partial + i\,\frac{\partial}{\partial^+}\Big\{\,\theta^m\,\bar{\partial}_m \,+\, \frac{i}{4\sqrt{2}\,\partial^+}\,(d^p\,\bar{d}_p - \bar{d}_p\,d^p)\,\Big\} -\\
&-\; \frac{1}{4}\,\frac{\bar{\partial}_{pq}}{\partial^+}\,\Big\{\,\frac{\partial^+}{\sqrt{2}}\,\theta^p\,\theta^q \,-\, \frac{\sqrt{2}}{\partial^+}\,\partial^p\,\partial^q + \frac{1}{\sqrt{2}\partial^+} d^p\,d^q\,\Big\}\;,
\end{aligned} \tag{70}$$

and its conjugate

$$
\begin{aligned}
\bar{j}^{\,-} \;=\; & i\,\bar{x}\,\frac{\partial\bar{\partial}\,+\,\frac{1}{4}\,\bar{\partial}_{pq}\,\partial^{pq}}{\partial^+} \;-\; i\,x^-\,\bar{\partial} \;+\; i\,\frac{\bar{\partial}}{\partial^+}\Big\{\,\bar{\theta}_m\,\partial^m \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,(d^p\,\bar{d}_p-\bar{d}_p\,d^p)\,\Big\}\;- \\
& -\;\frac{1}{4}\,\frac{\partial^{pq}}{\partial^+}\,\Big\{\,\frac{\partial^+}{\sqrt{2}}\,\bar{\theta}_p\,\bar{\theta}_q\;-\;\frac{\sqrt{2}}{\partial^+}\,\bar{\partial}_p\,\bar{\partial}_q+\frac{1}{\sqrt{2}\partial^+}\bar{d}_p\,\bar{d}_q\,\Big\}\;. \qquad (71)
\end{aligned}
$$

The others are obtained by using the $SO(8)/(SO(2)\times SO(6))$ rotations

$$
j^{-\,mn}\;=\;[\,j^-\,,\,j^{mn}\,]\;;\qquad \bar{j}^{\,-}{}_{mn}\;=\;[\,\bar{j}^{\,-}\,,\,\bar{j}_{mn}\,]\;. \qquad (72)
$$

We do not show their explicit forms as they are too cumbersome. The four supersymmetries in four dimensions turn into one supersymmetry in ten dimensions. In our notation, the kinematical supersymmetries q^n_+ and $\bar{q}_{+n}$, are assembled into one $SO(8)$ spinor. The dynamical supersymmetries are obtained by boosting

$$
i\,[\,\bar{j}^{\,-}\,,\,q^m_+\,]\;\equiv\;\mathcal{Q}^m\;,\qquad i\,[\,j^-\,,\,\bar{q}_{+\,m}\,]\;\equiv\;\overline{\mathcal{Q}}_m\;, \qquad (73)
$$

where

$$
\begin{aligned}
\mathcal{Q}^m \;&=\; \frac{\bar{\partial}}{\partial^+}\,q_+{}^m\;+\;\frac{1}{2}\,\frac{\partial^{mn}}{\partial^+}\,\bar{q}_{+\,n}\;,\\
\overline{\mathcal{Q}}_m \;&=\; \frac{\partial}{\partial^+}\,\bar{q}_{+\,m}\;+\;\frac{1}{2}\,\frac{\overline{\partial}_{mn}}{\partial^+}\,q_+{}^n\;. \qquad (74)
\end{aligned}
$$

They satisfy the supersymmmetry algebra

$$
\{\,\mathcal{Q}^m\,,\,\overline{\mathcal{Q}}_n\,\}\;=\;i\,\sqrt{2}\,\delta^m_{\ n}\,\frac{1}{\partial^+}\,\Big(\partial\,\bar{\partial}\;+\;\frac{1}{4}\,\bar{\partial}_{pq}\,\partial^{pq}\Big)\;, \qquad (75)
$$

and can be obtained from one another by $SO(8)$ rotations, as

$$
\frac{1}{2}\,\epsilon_{pqmn}\,[\,j^{pq}\,,\,\mathcal{Q}^m\,]\;=\;4\,\overline{\mathcal{Q}}_n\;, \qquad (76)
$$

while

$$
[\,\bar{j}_{pq}\,,\,\mathcal{Q}^m\,]\;=\;0\;. \qquad (77)
$$

Note also that

$$
\{\,\mathcal{Q}^m\,,\,q^n_+\,\}\;=\;\frac{i}{\sqrt{2}}\,\partial^{mn}\;, \qquad (78)
$$

4.3 The Generalized Derivatives

The cubic interaction in the $N=4$ Lagrangian contains explicitly the derivative operators ∂ and $\bar{\partial}$. To achieve covariance in ten dimensions, these must be generalized. We propose the following operator

$$
\overline{\nabla}\;\equiv\;\bar{\partial}\,+\,\frac{i\,\alpha}{4\,\sqrt{2}\,\partial^+}\,\bar{d}_p\,\bar{d}_q\,\partial^{pq}\;, \qquad (79)
$$

which naturally incorporates the rest of the derivatives ∂^{pq}, with α as an arbitrary parameter. After some algebra, we find that $\overline{\nabla}$ is covariant under $SO(8)$ transformations. We define its rotated partner as

$$
\nabla^{mn}\;\equiv\;\Big[\,\overline{\nabla}\,,\,j^{mn}\,\Big]\;, \qquad (80)
$$

where

$$\nabla^{mn} \;=\; \partial^{mn} \;-\; \frac{i\,\alpha}{4\,\sqrt{2}\,\partial^+}\,\bar{d}_r\,\bar{d}_s\,\epsilon^{mnrs}\,\partial\;. \tag{81}$$

If we apply to it the inverse transformation, it goes back to the original form

$$\Big[\,\bar{j}_{pq}\,,\,\nabla^{mn}\,\Big] \;=\; (\,\delta_p{}^m\,\delta_q{}^n\;-\delta_q{}^m\,\delta_p{}^n\,)\,\overline{\nabla}\;, \tag{82}$$

and these operators transform under $SO(8)/(SO(2)\times SO(6))$, and $SO(2)\times SO(6)$ as the components of an 8-vector.

We introduce the conjugate operator ∇ by requiring that

$$\nabla\,\bar{\phi} \;\equiv\; \overline{(\overline{\nabla}\,\phi)}\;, \tag{83}$$

with

$$\nabla \;\equiv\; \partial \;+\; \frac{i\,\alpha}{4\,\sqrt{2}\,\partial^+}\,d^p\,d^q\,\bar{\partial}_{pq}\;. \tag{84}$$

Define

$$\overline{\nabla}_{\,mn} \equiv \Big[\,\nabla\,,\,\bar{j}_{mn}\,\Big]\;, \tag{85}$$

which is given by

$$\overline{\nabla}_{mn} \;=\; \bar{\partial}_{mn} \;-\; \frac{i\,\alpha}{4\,\sqrt{2}\,\partial^+}\,d^r\,d^s\,\epsilon_{mnrs}\,\bar{\partial}\;. \tag{86}$$

We then verify that

$$\Big[\,j^{mn}\,,\,\overline{\nabla}_{pq}\,\Big] \;=\; (\,\delta_p{}^m\,\delta_q{}^n\;-\delta_q{}^m\,\delta_p{}^n\,)\,\nabla\;. \tag{87}$$

The kinetic term is trivially made $SO(8)$-invariant by including the six extra transverse derivatives in the d'Alembertian. The quartic interactions are obviously invariant since they do not contain any transverse derivative operators. Hence we need only consider the cubic vertex. In the paper [17] it is shown that to achieve covariance in ten dimensions, it suffices indeed to replace the transverse ∂ and $\bar{\partial}$ by ∇ and $\overline{\nabla}$, respectively. This is done by checking the invariance under the little group $SO(8)$. Together with the result from four dimensions this is enough to warrant invariance under the full superPoincaré group in ten dimensions. In this process the parameter α is determined to be 1. The full action is then

$$\begin{aligned}\mathcal{S} \;=\; &-\int d^4x\int d^4\theta\,d^4\bar{\theta}\,\Big\{\bar{\phi}^a\,\frac{\Box}{\partial^{+2}}\,\phi^a+\frac{4g}{3}\,f^{abc}\,\Big(\frac{1}{\partial^+}\,\bar{\phi}^a\,\phi^b\,\bar{\nabla}\,\phi^c+\text{c.c.}\Big)\\ &-g^2\,f^{abc}\,f^{ade}\Big(\,\frac{1}{\partial^+}(\phi^b\,\partial^+\phi^c)\frac{1}{\partial^+}\,(\bar{\phi}^d\,\partial^+\,\bar{\phi}^e)+\frac{1}{2}\,\phi^b\bar{\phi}^c\,\phi^d\,\bar{\phi}^e\Big)\,\Big\}.\end{aligned} \tag{88}$$

This action is suitable in order to investigate the perturbative properties of the theory. It is, of course, non-renormalizable but has still remarkable properties that Nature might use. One can also study possible higher symmetries of this action.

4.4 Eleven-Dimensional Supergravity

$N=1$ Supergravity in eleven dimensions, contains three different massless fields, two bosonic (gravity and a three-form) and one Rarita-Schwinger spinor. Its physical degrees of freedom are classified in terms of the transverse little group, $SO(9)$, with the graviton $G^{(\,MN\,)}$, transforming as a symmetric second-rank tensor, the three-form $B^{[\,MNP\,]}$ as an anti-symmetric third-rank tensor and the Rarita-Schwinger field as a spinor-vector, Ψ^M ($M, N, \ldots$ are $SO(9)$ indices). This theory on reduction to four dimensions leads to the maximally supersymmetric $N=8$ theory.

In order to use the formalism and especially the superfield equ. (49) developed in four dimensions for the maximally supersymmetric $N = 8$ theory we have to decompose

$$SO(9) \supset \; SO(2) \;\times\; SO(7) \; . \tag{89}$$

The $SO(7)$ symmetry can in fact be upgraded to an $SU(8)$ symmetry. However, it is important to remember that it is really the $SO(7)$ which is relevant when we "oxidize" the theory to $d = 11$ and the coordinates θ^m and $\bar{\theta}_n$ used in the four-dimensional case will now be interpreted as spinors under $SO(7) \times SO(2)$. To distinguish this we will change the notation m, n to α, β for the spinors and use the notation a, b for the vector indices of $SO(7)$.

The first step is to generalize the transverse variables to nine. In the Yang-Mills case, the compactified $SO(6)$ was easily described by $SU(4)$ parameters and we made use of the convenient bi-spinor notation. In the present case, the compactified $SO(7)$ has no equivalent unitary group so we simply introduce additional real coordinates, x^a and their derivatives ∂^a(where a runs from 4 through 10). The chiral superfield remains unaltered, except for the added dependence on the extra coordinates

$$h(y) \;=\; h(x, \bar{x}, x^a, y^-) \; , \quad etc... \; . \tag{90}$$

These extra variables will be acted on by new operators that will restore the higher-dimensional symmetries.

4.5 The SuperPoincaré Algebra in 11 Dimensions

The SuperPoincaré algebra needs to be generalized from its four-dimensional version. The $SO(2)$ generators stay the same and we propose generators of the coset $SO(9)/(SO(2) \times SO(7))$, of the form,

$$\begin{aligned} j^a \;\; &= \;\; -\,i\,(\,x\,\partial^a \,-\, x^a\,\partial\,) + \frac{i}{2\,\sqrt{2}}\,\partial^+\,\theta^\alpha\,(\gamma^a)_{\alpha\,\beta}\,\theta^\beta \,-\, \frac{i}{\sqrt{2}\,\partial^+}\,\partial^\alpha\,(\gamma^a)_{\alpha\,\beta}\,\partial^\beta \\ &\quad + \frac{i}{2\,\sqrt{2}\,\partial^+}\,d^\alpha\,(\gamma^a)_{\alpha\,\beta}\,d^\beta \end{aligned} \tag{91}$$

$$\begin{aligned} \bar{j}^{\,b} \;\; &= \;\; -\,i\,(\,\bar{x}\,\partial^b \,-\, x^b\,\bar{\partial}\,) + \frac{i}{2\,\sqrt{2}}\,\partial^+\,\bar{\theta}_\alpha\,(\gamma^b)^{\alpha\,\beta}\,\bar{\theta}_\beta \,-\, \frac{i}{\sqrt{2}\,\partial^+}\,\bar{\partial}_\alpha\,(\gamma^b)^{\alpha\,\beta}\,\bar{\partial}_\beta \\ &\quad + \frac{i}{2\,\sqrt{2}\,\partial^+}\,\bar{d}_\alpha\,(\gamma^b)^{\alpha\,\beta}\,\bar{d}_\beta \end{aligned} \tag{92}$$

which satisfy the $SO(9)$ commutation relations,

$$\begin{aligned} \Big[\,j\,,\,j^a\,\Big] \;&=\; j^a \;, \qquad \Big[\,j\,,\,\bar{j}^b\,\Big] \;=\; -\,\bar{j}^b \\ \Big[\,j^{cd}\,,\,j^a\,\Big] \;&=\; \delta^{ca}\,j^d \,-\, \delta^{da}\,j^c \\ \Big[\,j^a\,,\,\bar{j}^b\,\Big] \;&=\; i\,j^{ab} \,+\, \delta^{ab}\,j, \end{aligned} \tag{93}$$

where j is the same as before, and the $SO(7)$ generators read,

$$\begin{aligned} j^{ab} \;\; &= \;\; -\,i\,(\,x^a\,\partial^b \,-\, x^b\,\partial^b\,) \,+\, \theta^\alpha\,(\gamma^a)^{\alpha\,\beta}\,(\gamma^b)^{\beta\,\sigma}\,\bar{\partial}_\sigma \\ &\quad + \bar{\theta}_\alpha\,(\gamma^a)^{\alpha\,\beta}\,(\gamma^b)^{\beta\,\sigma}\,\partial^\sigma \,-\, \frac{1}{\sqrt{2}\,\partial^+}\,d^\alpha\,(\gamma^a)^{\alpha\,\beta}\,(\gamma^b)^{\beta\,\sigma}\,\bar{d}_\sigma \; . \end{aligned} \tag{94}$$

The full $SO(9)$ transverse algebra is generated by j , j^{ab} , j^a and $\bar{j}^b$. All rotations are specially constructed to preserve chirality. For example,

$$[\,j^a\,,\,\bar{d}_\alpha\,] \;=\; 0 \; ; \qquad [\,\bar{j}^{\,b}\,,\,d^\alpha\,] \;=\; 0 \; . \tag{95}$$

The remaining kinematical generators do not get modified,

$$j^{+} \;=\; j^{+}\;, \qquad j^{+-} \;=\; j^{+-}\;, \tag{96}$$

while new kinematical generators appear,

$$j^{+\,a} \;=\; i\,x^a\,\partial^+\;; \qquad \bar{j}^{+\,b} \;=\; i\,\bar{x}^b\,\partial^+\;. \tag{97}$$

We generalize the linear part of the dynamical boosts to,

$$\begin{aligned} j^- \;=\; & i\,x\,\frac{\partial\bar\partial\,+\,\frac{1}{2}\,\partial^a\,\partial^a}{\partial^+}\;-\;i\,x^-\,\partial + i\,\frac{\partial}{\partial^+}\Big\{\theta^\alpha\,\bar\partial_\alpha\;+\;\frac{i}{4\sqrt{2}\,\partial^+}\,(d^\alpha\,\bar d_\alpha - \bar d_\alpha\,d^\alpha)\Big\} \\ & -\frac{1}{4}\,\frac{\partial^a}{\partial^+}\Big\{\partial^+\,\theta^\alpha\,(\,\gamma^a)_{\alpha\,\beta}\,\theta^\beta\;-\;\frac{2}{\partial^+}\;\partial^\alpha\,(\,\gamma^a)_{\alpha\,\beta}\,\partial^\beta\;+\;\frac{1}{\partial^+}\;d^\alpha\,(\,\gamma^a)_{\alpha\,\beta}\,d^\beta\Big\}. \end{aligned} \tag{98}$$

The other boosts may be obtained by using the $SO(9)/(SO(2)\times SO(7))$ rotations,

$$j^{-\,a} \;=\; [\,j^-\,,\,j^a\,]\;; \qquad \bar{j}^{-\,b} \;=\; [\,\bar{j}^-\,,\,\bar{j}^b\,]\;. \tag{99}$$

We do not show their explicit forms as they are too cumbersome. The dynamical supersymmetries are obtained by boosting

$$\begin{aligned} [\,j^-\,,\,\bar q_{+\,\eta}\,] \;\equiv\; \overline{\mathcal{Q}}_\eta \;&=\; -\,i\,\frac{\partial}{\partial^+}\,\bar q_{+\,\eta}\;-\;\frac{i}{\sqrt{2}}\,(\,\gamma^b\,)_{\eta\,\rho}\,q_+^{\;\rho}\,\frac{\partial^b}{\partial^+}\;, \\ [\,\bar{j}^-\,,\,q_+^\alpha\,] \;\equiv\; \mathcal{Q}^\alpha \;&=\; i\,\frac{\bar\partial}{\partial^+}\,q_+{}^\alpha\;+\;\frac{i}{\sqrt{2}}\,(\,\gamma^a\,)^{\alpha\,\beta}\,\bar q_{+\,\beta}\,\frac{\partial^a}{\partial^+}\;. \end{aligned} \tag{100}$$

They satisfy,

$$\{\,\mathcal{Q}^\alpha\,,\,q_+^\eta\,\} \;=\; -\,(\,\gamma^a\,)^{\alpha\,\eta}\,\partial^a\;, \tag{101}$$

and the supersymmetry algebra,

$$\{\,\mathcal{Q}^\alpha\,,\,\overline{\mathcal{Q}}_\eta\,\} \;=\; i\,\sqrt{2}\;\delta^\alpha{}_\eta\,\frac{1}{\partial^+}\Big(\partial\,\bar\partial\;+\;\frac{1}{2}\,\partial^a\,\partial^a\Big)\;. \tag{102}$$

Having constructed the free $N=1$ SuperPoincaré generators in eleven dimensions which act on the chiral superfield, we turn to building the interacting theory.

4.6 The Generalized Derivatives

The cubic interaction in the $N=8$ Lagrangian explicitly contains the transverse derivative operators ∂ and $\bar\partial$. To achieve covariance in eleven dimensions, we proceed to generalize these operators as we did for $N=4$ Yang-Mills. We propose the generalized derivative

$$\overline{\nabla} \;=\; \bar\partial\;+\;\frac{\sigma}{16}\,\bar d_\alpha\,(\,\gamma^a\,)^{\alpha\,\beta}\,\bar d_\beta\,\frac{\partial^a}{\partial^+}\,, \tag{103}$$

which naturally incorporates the coset derivatives ∂^m. Here σ is a parameter, still to be determined. We use the coset generators to produce its rotated partner $\overline{\nabla}$ by,

$$[\,\overline{\nabla}\,,\,j^a\,] \equiv \nabla^a \;=\; -\,i\,\partial^a\;+\;\frac{i\,\sigma}{16}\,\bar d_\alpha\,(\,\gamma^a\,)^{\alpha\,\beta}\,\bar d_\beta\,\frac{\partial}{\partial^+}\,. \tag{104}$$

It remains to verify that the original derivative operator is reproduced by undoing this rotation; indeed we find the required closure,

$$[\,\nabla^a\,,\,\bar{j}^{\,b}\,]\;=\;\delta^{a\,b}\;\overline{\nabla}$$

The new derivative ($\overline{\nabla}$, ∇^a), thus transforms as a 9-vector under the little group in eleven dimensions. We note that σ is not determined by these algebraic requirements. Instead, its value will be fixed by requiring that our generalized vertex satisfy the correct invariance requirements. We define the conjugate derivative ∇, by requiring that

$$\nabla\,\bar{\phi}\;\equiv\;\overline{(\overline{\nabla}\,\phi)}\;. \tag{105}$$

This tells us that,

$$\nabla\;\equiv\;\;\partial\,+\,\frac{\sigma^*}{16}\,d^\alpha\,(\,\gamma^b\,)^{\alpha\,\beta}\,d^\beta\,\frac{\partial^b}{\partial^+} \tag{106}$$

This construction is akin to that for the $N=4$ Yang-Mills theory, but this time it applies to the "oxidation" of the ($N=8$, $d=4$) theory to ($N=1$, $d=11$) Supergravity. This points to remarkable algebraic similarities between the two theories, with possibly profound physical consequences. It remains to show that the simple replacement of the transverse derivatives $\partial,\bar{\partial}$ by $\nabla,\overline{\nabla}$ in the ($N=8$, $d=4$) interacting theory yields the fully covariant Lagrangian in eleven dimensions.

This can be done by checking the invariance under the little group $SO(9)$. This is a very tedious exercise which was done in paper [18]. Indeed it is possible to show that the three-point coupling is invariant for the specific choice of $\sigma=-\sqrt{2}$ and the eleven-dimensional supergravity theory can be written as

$$S=\int d^{10}xd^8\theta d^8\bar{\theta}\left\{\bar{\phi}(x,\theta)\frac{\Box}{\partial^{+4}}\phi(x,\theta)\,+\,\frac{3}{2}\,g\,\frac{1}{\partial^{+2}}\,\overline{\phi}\;\;\overline{\nabla}\,\phi\,\overline{\nabla}\,\phi+c.c.\Big]\right\}+O(g^2). \tag{107}$$

The four-point coupling for $d=4$ was computed as said above in [15]. It should be straightforward to show that it can be taken up to $d=11$ as was done for the three-point coupling. The trouble though is that it contains so many terms. With it one can in principle study various properties of this theory, such as the one-loop graphs. They will diverge but there might be ways to add more fields to get convergent answer. This is a long term goal of this project. One can also study the symmetries of the action. It is clear that the action is quite unique and has a profound rôle in modern particle physics and any symmetry that can be found for this action is a genuine physical symmetry. This theory is also the low-energy limit of the mystic M-theory which is supposed to be the underlying theory to all string theories. This theory is shrouded in mystery and any attempt to better understand the supergravity theory can help us eventually understand M-theory.

5 Oxidations and Reductions

We have seen how the two superfields can be used as master fields. In the case of the $\mathcal{N}=4$ we have used the superfield for both $d=4$ and for $d=10$. Similarly the $d=6$ theory can be found [19]. We can reduce the theory also to $d=3$.The 10-dimensional case saturates the algebra and the superfield. Similarly we can consider the superfield for $\mathcal{N}=8$. We have shown how to use it for $d=4$ and $d=11$. We can again use the superfield also for the other dimensions between 11 and 3. However, in this case we can use the superfield also for higher dimensions so let me describe some results in $d=18$.

5.1 Particle Dynamics and Supersymmetry in 18 dimensions

If we consider the linear parts of the supersymmetry algebras we can consider them as part of a description of particle dynamics.. We describe an $N = n$ superparticle by its coordinates x^i, x^+, x^- θ^m, $\bar{\theta}_m$, where $m = 1, ...n$. The action for the free particle is

$$S = \int d\tau (\frac{1}{2}\dot{x}^{i2} + \theta^m \dot{\bar{\theta}}_m) \tag{108}$$

It has to be supplemented with the full supersymmetry algebra to have a fully invariant theory. If we instead change θ^m to a transverse fermionic vector coordinate λ^i, we will describe a spinning particle. For $d = 4$, i.e. $i = 1, 2$ it is a Dirac particle.

For the $\mathcal{N} = 4$ theory in $d = 10$ there exist both a spinning particle and a superparticle solution. This is a consequence of the triality of $SO(8)$. For the ($\mathcal{N} = 8$ we can push the algebra up to $d = 18$

The 18-dimensional superparticle should describe a **128** boson and a **128** fermion under the little group of $SO(16)$. The kinematical light-cone supersymmetry transforms as a 16, i.e. is vectorial. We will break the $SO(1.17)$ covariance further down to $SU(8) \times U(1) \times \frac{SO(16)}{SU(8)\times U(1)} \times \frac{SO(1,17)}{SO(16)}$ with the generators identified as $153 = 63 + 1 + 28 + \overline{28} + 8 + \bar{8} + 8 + \bar{8} + 1$ in terms of representations of $SU(8)$. We will introduce the generators with the notation j^m_n , j , j^{mn} , $\bar{j}_{mn}$, j^{+m} , $\bar{j}^+{}_m$, j^{-m} , $\bar{j}{-}_m$, j^{+-} with $m = 1, ...8$.

Let us start with the bosonic part. We will describe the phase space in terms of x^+, x^-, x^m and $\bar{x}_m$ and the corresponding ones for the momenta. The coordinates satisfy $\partial^m_x \bar{x}_n = \delta^m_n$. $p^2 = -2p^+p^- + 2p\bar{p}$.

The orbital generators are straightforward to construct. We take $x^+ = 0$

$$\begin{aligned}
l^m{}_n &= x^m \bar{\partial}_{xn} - \bar{x}_n \partial_x{}^m - \frac{\delta^m_n}{8}(x^p \bar{\partial}_{xp} - \bar{x}_p \partial_x{}^p), && (109)\\
l &= x^p \bar{\partial}_{xp} - \bar{x}_p \partial_x{}^p, && (110)\\
l^{mn} &= x^m \partial_x{}^n - x^n \partial_x{}^m, && (111)\\
\bar{l}_{mn} &= \bar{x}_m \bar{\partial}_{xn} - \bar{x}_n \bar{\partial}_{xm}, && (112)\\
l^{+m} &= i x^m \partial^+, && (113)\\
\bar{l}^+{}_m &= -i \bar{x}_m \partial^+, && (114)\\
l^{-m} &= i(x^m \frac{\partial^p_x \bar{\partial}_{xp}}{\partial^+} - x^- \partial^m_x), && (115)\\
\bar{l}^-{}_m &= -i(\bar{x}_m \frac{\partial^p_x \bar{\partial}_{xp}}{\partial^+} - x^- \bar{\partial}_{xm}), && (116)\\
l^{+-} &= i x^- \partial^+. && (117)
\end{aligned}$$

Let us now try to construct the spin parts s. We will use the supersymmetry generators in order to get chiral operators. The kinematical ones (we write them in terms of q's instead of θ's and d's as we have done before, which is slightly simpler.)

$$q^m_+ = -\partial^m + \frac{i}{\sqrt{2}} \theta^m \partial^+ \; ; \qquad \bar{q}_{+n} = \bar{\partial}_n - \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+ \; , \tag{118}$$

satisfy

$$\{ q^m_+ \, , \, \bar{q}_{+n} \} = i \sqrt{2} \, \delta^m{}_n \, \partial^+ \; . \tag{119}$$

With these we can construct

$$
\begin{aligned}
s^m{}_n &= -i\frac{1}{\sqrt{2}\partial^+}q_+^m\bar{q}_{+n} + i\frac{\delta_n^m}{8\sqrt{2}\partial^+}q_+^p\bar{q}_{+p}, && (120)\\
s &= -i\frac{1}{\sqrt{2}\partial^+}q_+^p\bar{q}_{+p}, && (121)\\
s^{mn} &= -i\frac{1}{\sqrt{2}\partial^+}q_+^m q_+^n, && (122)\\
\bar{s}_{mn} &= -i\frac{1}{\sqrt{2}\partial^+}\bar{q}_{+m}\bar{q}_{+n}, && (123)\\
& && (124)
\end{aligned}
$$

These ones close to the $SO(16)$ algebra. The remaining coset generators have to be guessed. We start with the

$$
\begin{aligned}
s^{+m} &= 0, && (125)\\
\bar{s}^+{}_m &= 0, && (126)\\
s^{+-} &= -\frac{i}{2}(\theta^p\bar{\partial}_p + \bar{\theta}_p\partial^p). && (127)
\end{aligned}
$$

Note that the l^{+-} is not written in terms of the q_+'s. It commutes with the covariant derivatives to a covariant derivative which is good enough to warrant chirality. By making an ansatz for j^{-m} and $\bar{j}_m^-$ one can derive a possible form by using the commutators

$$
\begin{aligned}
[J^{+m}, J^{-n}] &= -J^{mn}, && (128)\\
[\bar{J}^+{}_m, J^{-n}] &= -J^n{}_m - i\delta^n{}_m J^{+-} - \frac{\delta^n{}_m}{8}J. && (129)
\end{aligned}
$$

A solution to these commutators is

$$
\begin{aligned}
s^{-m} &= \frac{i}{2}\frac{\partial_x{}^m}{\partial^+}(\theta^p\bar{\partial}_p + \bar{\theta}_p\partial^p) + \frac{1}{\sqrt{2}}\frac{\partial_x{}^p}{\partial^{+2}}q_+^m\bar{q}_{+p}\\
&+ \frac{1}{\sqrt{2}}\frac{\bar{\partial}_{xp}}{\partial^{+2}}q_+^m q_+^p, && (130)\\
\bar{s}^-{}_m &= -\frac{i}{2}\frac{\partial_x{}^m}{\partial^+}(\theta^p\bar{\partial}_p + \bar{\theta}_p\partial^p) - \frac{1}{\sqrt{2}}\frac{\bar{\partial}_{xp}}{\partial^{+2}}\bar{q}_{+m}\, q_+^p\\
&- \frac{1}{\sqrt{2}}\frac{\partial_x^p}{\partial^{+2}}\bar{q}_{+m}\bar{q}_{+p}. && (131)
\end{aligned}
$$

We can now check the full Poincaré algebra $SO(1,17)$ closes. What kind of algebra does the full set of generators including the supersymmetry generators q_+^m close to? By commuting q_+^m with J^{-n} we will generate the dynamical supersymmetry generators. However, unlike the previous cases we have studied this process does not close. The algebra becomes an open algebra. Noting that q_+^m and $\bar{q}_{+m}$ represent 16 components we can see that an alternative way to describe them is a an $SO(16)$ vector and we hence have an 18-dimensional spinning particle. However, this study is interesting since $d = 18$ is the dimension of the "missing string theory". We know since many years that the string theory algebras demand $d = 26$ or $d = 10$. It has been been conjectured by many that there should be some kind of consistent structure also in $d = 18$. It has be be more general than the straight attempt above but it should contain it. If we could find a new structure in $d = 18$ it could give us a lot of hints to the true nature of M-theory.

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