Tools for supersymmetry [1]

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ABSTRACT

This is an elementary introduction to basic tools of supersymmetry: the spacetime symmetries, gauge theory and its application in gravity, spinors and superalgebras. Special attention is devoted to conformal and anti-de Sitter algebras.

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1 Introduction

In this review you will not find many general ideas. It is rather meant as a technical introduction to some basic tools for working with supersymmetric field theories. It are ingredients which I could use for various applications in the last 20 years.

In section 2, I will consider only bosonic theories. It contains an introduction to the spacetime symmetries. Also gauging of symmetries is treated, and special attention is devoted to a very useful theorem for calculating the transformation of the covariant derivative of fields.

Section 3 reviews the properties of Clifford algebras and spinors in a spacetime with arbitrary dimensions and signature. It gives an overview of Majorana, Weyl, Majorana–Weyl and symplectic Majorana–Weyl spinors. At the end several tips are given to perform complex conjugation in practice, and to manipulate gamma matrices.

Supersymmetry is introduced in section 4. I introduce especially the algebraic approach. Also the elementary facts on superalgebras are treated, including the real forms of these algebras. The aim is to come to supersymmetry algebras, as well super-Poincaré, as super-adS and superconformal algebras.

2 Bosonic spacetime symmetries

Supersymmetry is related to the structure of spacetime. Before tackling supersymmetry algebras, I will first review the bosonic spacetime symmetries and their gauging.

2.1 Coleman–Mandula result

To investigate possible spacetime symmetries, Coleman and Mandula [1] investigated how large the spacetime symmetry group can be in order that scattering amplitudes do not become trivial. They thus deal with 'visible symmetries', i.e. those which act on S-matrix elements. They restrict themselves also to 4 dimensions, and assume a finite number of different particles in a multiplet. Their research was before the advent of supersymmetry, so they were looking only at bosonic symmetries. Still the result is very significant. There are two cases to be distinguished.

1. There are massive particles. Then the symmetry algebra has at most translations P_μ, Lorentz rotations $M_{\mu\nu}$ and an algebra G of scalar (i.e. commuting with P and M) symmetries.

2. There are only massless particles. Then there is the extra possibility of having conformal symmetry (and a commuting group G).

The second possibility will be discussed in section 2.3. The first one contains a direct sum of G with the Poincaré algebra, i.e.

$$
\begin{aligned}
[M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta^{[\rho}_{[\mu}M_{\nu]}{}^{\sigma]} , \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu}P_{\rho]} , \\
[P_\mu, P_\nu] &= 0 .
\end{aligned}
\tag{2.1}
$$

For conventions, see appendix A. In supersymmetry the last commutator is often modified, giving rise to the (anti) de Sitter algebra.

2.2 Anti-de Sitter algebra and spacetime

The Poincaré algebra is an 'Inönü–Wigner' contraction of the '(anti) de Sitter algebra':

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu[\rho}M_{\sigma]\nu} - \eta_{\nu[\rho}M_{\sigma]\mu} \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu}P_{\rho]} \\
[P_\mu, P_\nu] &= \frac{1}{2R^2}M_{\mu\nu} .
\end{aligned}
\tag{2.2}
$$

With the opposite sign for the last commutator we would have the 'de Sitter algebra'. As written it is the 'anti-de Sitter (adS) algebra'. And for $R \to \infty$, we recover the Poincaré algebra. Defining $M_{d\mu} = -M_{\mu d} = R\, P_\mu$ we have generators $M_{\hat\mu\hat\nu} = -M_{\hat\nu\hat\mu}$ with $\hat\mu = 0, \ldots, d$, and defining the metric to be $\eta_{\hat\mu\hat\nu} = \mathrm{diag}(- + \ldots + -)$ the algebra can be concisely written as

$$
[M_{\hat\mu\hat\nu}, M_{\hat\rho\hat\sigma}] = \eta_{\hat\mu[\hat\rho}M_{\hat\sigma]\hat\nu} - \eta_{\hat\nu[\hat\rho}M_{\hat\sigma]\hat\mu} ,
\tag{2.3}
$$

i.e. it is the algebra $SO(d-1, 2)$. Note that every point is invariant under the rotations around this point, while not invariant under the translations. In this sense we can write

$$
adS_d = \frac{SO(d-1, 2)}{SO(d-1, 1)} .
\tag{2.4}
$$

The 'Inönü—Wigner' contraction is the statement that the $SO(d-1,2)$ algebra reduces to the Poincaré algebra when taking the limit $R \to \infty$ keeping P_μ constant in the relation $M_{d\mu} = -M_{\mu d} = R\,P_\mu$.

To obtain a space with adS metric, we start from defining it as a submanifold of a $(d+1)$-dimensional space with a flat metric of $(d-1,2)$ signature (for convenience we taken here $\mu = 0, \ldots, d-2$)

$$
\begin{array}{ccc}
ds^2 = & dX^\mu \eta_{\mu\nu} dX^\nu & - & dX^+ dX^- \\
& (d-2,1) & + & (1,1) \qquad \Rightarrow (d-1,2)
\end{array}
\tag{2.5}
$$

The adS space is the submanifold determined by the $SO(d-1,2)$-invariant equation

$$
X^\mu \eta_{\mu\nu} X^\nu - X^+ X^- + R^2 = 0 \,.
\tag{2.6}
$$

On the hypersurface one can take several sets of coordinates. E.g. the horospherical coordinates $\{x^\mu, z\}$ are defined by

$$
\begin{aligned}
X^- &= z^{-1} \\
X^\mu &= z^{-1} x^\mu \\
X^+ &= \frac{x_\mu^2 + R^2 z^2}{z} \,.
\end{aligned}
\tag{2.7}
$$

The latter being the solution of (2.6) given the first two. The induced metric on the hypersurface is

$$
ds^2 = \frac{1}{z^2} \left(dx_\mu^2 + R^2 dz^2 \right) \,.
\tag{2.8}
$$

The $SO(d-1,2)$ is linearly realized in the embedding $(d+1)$-dimensional space, and these transformations, ($\hat{\mu} = \mu, +, -$ and $\Lambda^{\hat{\mu}\hat{\nu}} = -\Lambda^{\hat{\nu}\hat{\mu}}$)

$$
\delta X^{\hat{\mu}} = \Lambda^{\hat{\nu}\hat{\rho}} M_{\hat{\nu}\hat{\rho}} X^{\hat{\mu}} = -\Lambda^{\hat{\mu}}{}_{\hat{\nu}} X^{\hat{\nu}} \,.
\tag{2.9}
$$

are on the adS space distorted to[1]

$$
\begin{aligned}
\delta_{adS} x^\mu &= -\Lambda^\mu{}_- - \Lambda^{\mu\nu} x_\nu - \Lambda^+{}_+ x^\mu \\
&\quad -(x^2 + R^2 z^2)\Lambda^\mu{}_+ + 2x^\mu x_\nu \Lambda^\nu{}_+ \\
\delta_{adS} z &= -z \left(\Lambda^+{}_+ - 2x_\mu \Lambda^\mu{}_+ \right) \,.
\end{aligned}
\tag{2.10}
$$

---

[1]Note that to raise or lower indices one has to use the metric in (2.5), i.e. $\eta_{+-} = -\frac{1}{2}$, and thus $\eta^{+-} = -2$.

Suppose I just gave you the metric (2.8). Then to determine the symmetries, one should solve the 'Killing' equation (with $X^M = \{x^\mu, \phi\}$)

$$(\partial_P g_{MN})\delta X^P + 2g_{P(M}\partial_{N)}\delta X^P = 0\,, \tag{2.11}$$

which is the requirement that the Lie derivative of the metric vanishes. As an *exercise*, one may prove that the transformations (2.10) are the only solutions.

2.3 Rigid conformal symmetry

Conformal symmetry is defined as the symmetry which preserves angles. Therefore it should contain the transformations which change the metric up to a factor. That implies that the symmetries are determined by the solutions to the 'conformal Killing equation'

$$\partial_{(\mu}\xi_{\nu)} - \tfrac{1}{d}\eta_{\mu\nu}\partial_\rho\xi^\rho = 0\,. \tag{2.12}$$

In $d = 2$ with as non-zero metric elements $\eta_{z\bar{z}} = 1$, the Killing equations are reduced to $\partial_z\xi_z = \partial_{\bar{z}}\xi_{\bar{z}} = 0$ and this leads to an infinite dimensional conformal algebra (all holomorphic vectors $\xi_{\bar{z}}(z)$ and anti-holomorphic vectors $\xi_z(\bar{z})$). In dimensions $d > 2$ the conformal algebra is finite-dimensional. Indeed, the solutions are

$$\xi^\mu(x) = a^\mu + \lambda_M^{\mu\nu}x_\nu + \lambda_D x^\mu + (x^2\Lambda_K^\mu - 2x^\mu x \cdot \Lambda_K)\,. \tag{2.13}$$

Corresponding to the parameters a^μ are the translations P_μ, to $\lambda_M^{\mu\nu}$ correspond the Lorentz rotations $M_{\mu\nu}$, to λ_D are associated dilatations D, and Λ_K^μ are parameters of 'special conformal transformations' K_μ. This is expressed as follows for the full set of conformal transformations δ_C:

$$\delta_C = a^\mu P_\mu + \lambda_M^{\mu\nu}M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu\,. \tag{2.14}$$

With these transformations, one can obtain the algebra with as non-zero commutators

$$\begin{aligned}
&[M_{\mu\nu}, M^{\rho\sigma}] = -2\delta_{[\mu}^{[\rho}M_{\nu]}^{\sigma]}\,, \\
&[P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu}P_{\rho]}\,, \qquad [K_\mu, M_{\nu\rho}] = \eta_{\mu[\nu}K_{\rho]}\,, \\
&[P_\mu, K_\nu] = 2(\eta_{\mu\nu}D + 2M_{\mu\nu})\,, \\
&[D, P_\mu] = P_\mu\,, \qquad\qquad [D, K_\mu] = -K_\mu\,.
\end{aligned} \tag{2.15}$$

This is the $SO(d, 2)$ algebra[2]. Indeed one can define

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} M^{\mu\nu} & \frac{1}{4}(P^\mu - K^\mu) & \frac{1}{4}(P^\mu + K^\mu) \\ -\frac{1}{4}(P^\nu - K^\nu) & 0 & -\frac{1}{2}D \\ -\frac{1}{4}(P^\nu + K^\nu) & \frac{1}{2}D & 0 \end{pmatrix}, \qquad (2.16)$$

where indices are raised w.r.t. the rotation matrices $M^\mu{}_\nu$ with the metric $\eta = \text{diag}\,(-1, 1, ..., 1, -1)$. Note that this is the same as the anti-de Sitter algebra in $d + 1$ dimensions

$$Conf_d = adS_{d+1}, \qquad (2.17)$$

(obviously this concerns the algebras, not the spaces) which is an essential ingredient in the adS/CFT correspondence which got recently so much attention. Note in this respect the similarity between (2.10) and (2.13).

In general, fields $\phi^i(x)$ in d dimensions have the following transformations under the conformal group:

$$\begin{aligned} \delta_C \phi^i(x) &= \xi^\mu(x)\partial_\mu \phi^i(x) + \Lambda_M^{\mu\nu}(x)\, m_{\mu\nu}{}^i{}_j \phi^j(x) \\ &\quad + w_i\, \Lambda_D(x)\, \phi^i(x) + \Lambda_K^\mu\, (k_\mu \phi)^i(x)\,, \end{aligned} \qquad (2.18)$$

where the x-dependent rotation $\Lambda_{M\,\mu\nu}(x)$ and x-dependent dilatation $\Lambda_D(x)$ are given by

$$\begin{aligned} \Lambda_{M\,\mu\nu}(x) &= \partial_{[\nu}\xi_{\mu]} = \lambda_{M\,\mu\nu} - 4x_{[\mu}\Lambda_{K\,\nu]}\,, \\ \Lambda_D(x) &= \tfrac{1}{d}\partial_\rho \xi^\rho = \lambda_D - 2x \cdot \Lambda_K\,. \end{aligned} \qquad (2.19)$$

To specify for each field ϕ^i its transformations under conformal group one has to specify:

i) transformations under the Lorentz group, encoded into the matrix $(m_{\mu\nu})^i{}_j$. The Lorentz transformation matrix $m_{\mu\nu}$ should satisfy

$$m_{\mu\nu}{}^i{}_k m_{\rho\sigma}{}^k{}_j - m_{\rho\sigma}{}^i{}_k m_{\mu\nu}{}^k{}_j = -\eta_{\mu[\rho}m_{\sigma]\nu}{}^i{}_j + \eta_{\nu[\rho}m_{\sigma]\mu}{}^i{}_j\,. \qquad (2.20)$$

The explicit form for Lorentz transformation matrices is for vectors (the indices i and j are of the same kind as μ and ν)

$$m_{\mu\nu}{}^\rho{}_\sigma = -\delta^\rho_{[\mu}\eta_{\nu]\sigma}\,, \qquad (2.21)$$

---

[2]In the 2-dimensional case $SO(2, 2) = SU(1, 1) \times SU(1, 1)$ is realised by the finite subgroup of the infinite dimensional conformal group, and is well known in terms of $L_{-1} = \frac{1}{2}(P_0 - P_1)$, $L_0 = \frac{1}{2}D + M_{10}$, $L_1 = \frac{1}{2}(K_0 + K_1)$, $\bar{L}_{-1} = \frac{1}{2}(P_0 + P_1)$, $\bar{L}_0 = \frac{1}{2}D - M_{10}$, $\bar{L}_1 = \frac{1}{2}(K_0 - K_1)$. Higher order $L_n, |n| \geq 2$ have no analogs in $d > 2$.

while for spinors, (where i and j are (unwritten) spinor indices)

$$m_{\mu\nu} = -\tfrac{1}{4}\gamma_{\mu\nu} \, . \tag{2.22}$$

ii) The Weyl weights w_i.

iii) Possible extra parts of the special conformal transformations,
apart from those connected to translations, rotations and dilatations
as in (2.13) and (2.19), $(k_\mu\phi)^i$. The Weyl weight of $(k_\mu\phi^i)$ should be
$w - 1$, and k_μ are mutually commuting operators.

In this way, the algebra (2.15) is realised on the fields as

$$[\delta_C(\xi_1), \delta_C(\xi_2)] = \delta_C \left(\xi^\mu = \xi_2^\nu\partial_\nu\xi_1^\mu - \xi_1^\nu\partial_\nu\xi_2^\mu \right) \, . \tag{2.23}$$

Note the sign difference between the commutator of matrices (2.20) and the
commutator of the generators in (2.15), which is due to the difference between
'active and passive' transformations. To understand fully the meaning of
the order of the transformations, consider in detail the calculation of the
commutator of transformations of fields. See e.g. for a field of zero Weyl
weight, and notice how the transformations act only on fields, not on explicit
spacetime points x^μ:

$$
\begin{aligned}
\lambda_D a^\mu[D, P_\mu]\phi(x) &= \left(\delta_D(\lambda_D)\delta_P(a^\mu) - \delta_P(a^\mu)\delta_D(\lambda_D)\right)\phi(x) \\
&= \delta_D(\lambda_D)a^\mu\partial_\mu\phi(x) - \delta_P(a^\mu)\lambda_D x^\mu\partial_\mu\phi(x) \\
&= a^\mu\partial_\mu\left(\lambda_D x^\nu\right)\partial_\nu\phi(x) \\
&= a^\mu\lambda_D\partial_\mu\phi(x) = \lambda_D a^\mu P_\mu\phi(x) \, .
\end{aligned}
\tag{2.24}
$$

It is important to notice that the derivative of a field of Weyl weight w
has weight $w + 1$. E.g. for a scalar of weight w (and without extra special
conformal transformations) we obtain

$$
\begin{aligned}
\delta_C\partial_\mu\phi(x) &= \xi^\nu(x)\partial_\nu\partial_\mu\phi(x) + w\,\Lambda_D(x)\,\partial_\mu\phi(x) \\
&\quad -\Lambda_{M\mu}{}^\nu(x)\partial_\nu\phi(x) + \Lambda_D(x)\partial_\mu\phi(x) - 2w\,\Lambda_{K\mu}\phi(x) \, .
\end{aligned}
\tag{2.25}
$$

With these rules the conformal algebra is satisfied. The question remains
when an action is conformal invariant. We consider local actions which can be
written as $S = \int d^d x\, \mathcal{L}(\phi^i(x), \partial_\mu\phi^i(x))$, i.e. with at most first order derivatives
on all the fields. For P_μ and $M_{\mu\nu}$ there are the usual requirements of a

covariant action. For the local dilatations we have the requirement that the weights of all fields in each term should add up to d, where ∂_μ counts also for 1, as can be seen from (2.25). Indeed, the explicit Λ_D transformations finally have to cancel with

$$\xi^\mu(x)\partial_\mu \mathcal{L} \approx -(\partial_\mu \xi^\mu(x))\mathcal{L} = -d\Lambda_D(x)\mathcal{L} \,. \tag{2.26}$$

For special conformal transformations one remains with

$$\delta_K S = 2\Lambda_K^\mu \int d^d x \, \frac{\mathcal{L}\overleftarrow{\partial}}{\partial(\partial_\nu \phi^i)}\left(-\eta_{\mu\nu}w_i\phi^i + 2m_{\mu\nu}{}^i{}_j\phi^j\right) + \Lambda_K^\mu \frac{S\overleftarrow{\delta}}{\delta\phi^i(x)}(k_\mu\phi)^i(x) \,. \tag{2.27}$$

where $\overleftarrow{\partial}$ indicates a right derivative. The first terms originate from the K-transformations contained in (2.13) and (2.19). In most cases these are sufficient to find the invariance and no $(k_\mu\phi)$ are necessary. In fact, the latter are often excluded because of the requirement that they should have Weyl weight $w_i - 1$, and in many cases there are no such fields available.

Although we will show that this condition is satisfied for many dilatational invariant theories, it is non-trivial. As a counterexample we give the action of the scalars ϕ^1 and ϕ^2 (with Weyl weights $(\frac{d}{2} - 1)$)

$$\mathcal{L} = \left(1 + \frac{\phi^1}{\phi^2}\right)(\partial_\mu\phi^1)(\partial^\mu\phi^2) \,. \tag{2.28}$$

Exercise.

There are typical cases in which (2.27) does not receive any contributions. Check the following ones

1. scalars with Weyl weight 0.

2. spinors appearing as $\partial\!\!\!/\lambda$ if their Weyl weight is $(d-1)/2$. This is also the appropriate weight for actions as $\bar\lambda \, \partial\!\!\!/\lambda$.

3. Vectors or antisymmetric tensors whose derivatives appear only as field strengths $\partial_{[\mu_1}B_{\mu_2\ldots\mu_p]}$ if their Weyl weight is $p-1$. This value of the Weyl weight is what we need also in order that their gauge invariances and their zero modes commute with the dilatations. Then scale invariance of the usual square of the field strengths will fix $p = \frac{d}{2}$.

4. Scalars X^i with Weyl weight $\frac{d}{2} - 1$ and

$$\mathcal{L} = (\partial_\mu X^i) A_{ij} (\partial^\mu X^j) \,, \tag{2.29}$$

where A_{ij} are constants.

2.4 Gauge theory and gravity

From the theory of general relativity we know how to construct actions invariant under the local Poincaré group. However, I will now consider the constructions from an algebraic viewpoint, i.e. as a gauge theory of the Poincaré group. First, recall the general equations for gauge theories with infinitesimal transformations

$$\delta = \epsilon^A T_A \,, \tag{2.30}$$

where A thus labels all the symmetries, and ϵ^A are all the parameters. For every symmetry one introduces a gauge field h_μ^A, and if the algebra is

$$[\delta_A(\epsilon_1^A), \delta_B(\epsilon_2^B)] = \delta_C \left(\epsilon_2^B \epsilon_1^A f_{AB}{}^C \right) \,, \tag{2.31}$$

then these transform as

$$\delta(\epsilon) h_\mu^A = \partial_\mu \epsilon^A + \epsilon^C h_\mu^B f_{BC}{}^A \,. \tag{2.32}$$

Covariant derivatives are defined as

$$\nabla_\mu = \partial_\mu - \delta_A(h_\mu^A) \,, \tag{2.33}$$

and their commutators are new transformations with as parameters the curvatures:

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] &= -\delta_A(R_{\mu\nu}^A) \\
R_{\mu\nu}^A &= 2\partial_{[\mu} h_{\nu]}^A + h_\nu^C h_\mu^B f_{BC}{}^A \,,
\end{aligned} \tag{2.34}$$

which transform 'covariantly' as

$$\delta R_{\mu\nu}^A = \epsilon^C R_{\mu\nu}^B f_{BC}{}^A \,. \tag{2.35}$$

I now apply this to the Poincaré group with gauge fields

$$h_\mu^A T_A = e_\mu^a P_a + \omega_\mu{}^{ab} M_{ab} \,. \tag{2.36}$$

The indices $\mu, \nu, \ldots$ of the algebra of the previous sections should now be replaced by flat indices $a, b, \ldots$. The curvatures are then (for later convenience I put an indication P for the Poincaré curvatures)

$$
\begin{aligned}
R_{\mu\nu}^{P}(P^a) &= 2\partial_{[\mu}e_{\nu]}^{a} + 2\omega_{[\mu}{}^{ab}e_{\nu]b} \\
R_{\mu\nu}^{P}(M^{ab}) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{\mu}^{c[a}\omega_{\nu}{}^{b]}{}_{c}.
\end{aligned} \tag{2.37}
$$

Usually the spin connection is not considered as an independent field. One can obtain the relation between the spin connection and the vierbeins by imposing a constraint

$$
R_{\mu\nu}^{P}(P^a) = 0. \tag{2.38}
$$

As the 'vielbein' e_{μ}^{a} is assumed to be invertible, this can be solved for the 'spin connection'

$$
\omega_{\mu}{}^{ab} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{a\rho}e^{b\sigma}e_{\mu c}\partial_{[\rho}e_{\sigma]}^{c}. \tag{2.39}
$$

Such a constraint, which can be solved for a field, is called *a conventional constraint*. Similar constraints are often used in the superspace approach.

The constraint is not invariant under all the symmetries. Therefore the theory with the constraint, and thus a dependent spin connection has an algebra that is different from (2.1). Essentially the translations P_a are replaced by 'covariant general coordinate transformations'

$$
\delta_{cgct}(\xi) = \delta_{gct}(\xi) - \delta_{I}(\xi^{\mu}h_{\mu}^{I}), \tag{2.40}
$$

where I stands for all transformations except the translations. In fact, one can check that the translations P_a on the vierbein take the form of general coordinate transformations, due to the constraint (2.38).

From now on we replace translations by these covariant general coordinate transformations. On non-gauge fields we have (and we write these equations again more general than for the pure Poincaré algebra, as we want to use it later for larger algebras containing translations)

$$
\nabla_{\mu}\phi = \partial_{\mu}\phi - e_{\mu}^{a}P_a\phi - \delta_{I}(h_{\mu}^{I})\phi = 0, \tag{2.41}
$$

which, solved for P_a, gives what is now called the covariant derivative

$$
P_a\phi = D_a\phi \equiv e_a^{\mu}\left(\partial_{\mu}\phi - \delta_{I}(h_{\mu}^{I})\phi\right). \tag{2.42}
$$

While in the original algebra translations did commute, (covariant) general coordinate transformations do not commute, i.e. $f_{ab}^A \neq 0$. If one takes this into account in (2.34), thus adding a new term with $f_{ab}^A e_\mu^a e_\nu^b$, then the full curvature vanishes, consistent with (2.41). Then one can solve this again for f_{ab}^A, and one obtains

$$f_{ab}^A = -e_a^\mu e_b^\nu R_{\mu\nu}^A \equiv -R_{ab}^A, \tag{2.43}$$

such that still

$$[D_a, D_b] = -\delta_I(R_{ab}^I). \tag{2.44}$$

In the curvature formula (2.34) one thus takes $f_{ab}^I = 0$, as in the original algebra, but the other sums over symmetries include also the translations.

The Ricci tensor is a contraction of the M-curvature

$$R_{\mu\nu} = R_{\mu\rho}^P(M^{ba})e_b{}^\rho e_{\nu a}, \qquad R = R_{\mu\nu}g^{\mu\nu}, \tag{2.45}$$

and the field equation of the usual pure Poincaré action is

$$\frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g}\, R = \sqrt{-g}\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right). \tag{2.46}$$

In the rigid theory, a scalar does transform under Lorentz rotations. Indeed, see (2.18) where $\xi(x)$ contains a Lorentz rotation according to (2.13). This is not the case in the local theory. Here, $\xi(x)$ is the local parameter for general coordinate transformations. The former rotation is thus part of the general coordinate transformation. In the local theory, scalars are invariant under Lorentz rotations. For other fields, the rotations exclusively come from the part in m_{ab} (and in a conformal theory, the full $\Lambda_M(x)$ is the local parameter, and the special conformal transformations are not part of it, but are exclusively contained in $(k_\mu\phi)$). E.g. for a spinor, ψ, the covariant derivative is (if they do not transform under any other symmetry)

$$D_a\psi = e_a^\mu\left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\right)\psi. \tag{2.47}$$

Further it is important to realize that the rotations M^{ab} act on vectors V_a, but a local spacetime vector V_μ is invariant. On the other hand, the general coordinate transformation on V_μ has not just the $\xi^\nu\partial_\nu V_\mu$ term, but also the term $V_\nu\partial_\mu\xi^\nu$.

The following theorem is extremely useful when calculating transformations [2, 3].

Theorem on covariant derivatives. *We define a covariant quantity as one whose transformation has no derivative on a parameter. In particular, it has to be a world scalar such that general coordinate transformations on the field do not contain a derivative on ξ^μ. If a covariant quantity transforms only into covariant quantities, then a covariant derivative (2.42) on such a covariant quantity is a new covariant quantity.*

This implies that transformations of a covariant quantity contain only derivatives as covariant derivatives D_a on other covariant quantities, or in the form of curvatures with Lorentz indices R_{ab}.

Proof. In this proof we will allow the algebra to be 'open', see below. First, we give the conditions again in formulas, which may clarify some of the sentences in the statement of the proof. Recall that the full set of generators (which correspond to indices $A, B, \ldots$), contain translations (indices $a, \ldots$) and the remaining transformations are indicated by indices I. Gauge fields of these other symmetries may have transformation laws with other terms than in (2.32), in particular with matter fields. Their transformations under the (non-translation) symmetries are

$$\delta h_\mu^I = \partial_\mu \epsilon^I + \epsilon^J h_\mu^A f_{AJ}{}^I + \delta_m h_\mu^I \,. \tag{2.48}$$

The last term can contain 'covariant' matter fields, but not explicit gauge field (apart from e_μ^a). Consider a matter field ϕ that transforms as

$$\delta \phi = \epsilon^I (T_I \phi) \,, \tag{2.49}$$

which has thus no derivative on ϵ^I. The requirement that a covariant quantity transforms in a covariant quantity is the statement that

$$\delta(T_I \phi) = \epsilon^J (T_J T_I \phi) \,. \tag{2.50}$$

First, consider the transformation of the covariant derivative

$$D_\mu \phi = \partial_\mu \phi - h_\mu^I T_I \phi \,. \tag{2.51}$$

The algebra is

$$[\delta(\epsilon_1), \delta(\epsilon_2)] \phi = \epsilon_2^J \epsilon_1^I [T_I, T_J\} \phi = \epsilon_2^J \epsilon_1^I \left(f_{IJ}{}^K T_K \phi + f_{IJ}{}^a D_a \phi + \eta_{IJ} \right) \,. \tag{2.52}$$

Remark that all formulas have been written in a way that they are also applicable to fermionic transformations (thus the order in which fields and parameters are written, is carefully chosen). The symbol $[.,.\}$ will be introduced

later in (4.3) The commutator can include general coordinate transformations, which act on the ϕ as $D_a\phi = e_a^\mu D_\mu\phi$. Furthermore, η is a possible non-closure term. The previous formulas imply

$$
\begin{aligned}
\delta D_\mu\phi &= \epsilon^I\partial_\mu T_I\phi - \epsilon^J h_\mu^A f_{AJ}{}^I T_I\phi - (\delta_m h_\mu^I)T_I\phi - h_\mu^I\epsilon^J(T_J T_I\phi)\\
&= \epsilon^I\left(\partial_\mu - h_\mu^J T_J\right)T_I\phi - \epsilon^J e_a^\mu f_{aJ}{}^I T_I\phi\\
&\quad - h_\mu^I\epsilon^J f_{JI}{}^a D_a\phi - h_\mu^I\epsilon^J\eta_{JI} - (\delta_m h_\mu^I)T_I\phi\,.
\end{aligned}
\tag{2.53}
$$

Consider now the transformation of (2.42), using

$$
\delta e_a^\mu = -e_b^\mu e_a^\nu\delta e_\nu^b = -\epsilon^I h_a^A f_{AI}{}^b e_b^\mu\,.
\tag{2.54}
$$

Then the first term on the last line of (2.53) is cancelled and we obtain

$$
\delta D_a\phi = \epsilon^I D_a T_I\phi - \epsilon^I f_{aI}{}^A T_A\phi + \epsilon^I h_a^J\eta_{JI} - (\delta_m h_a^I)T_I\phi\,.
\tag{2.55}
$$

∎

Consider the final result. In the second term the index A is used, which may take the value a, with $T_a = D_a$. Otherwise, this term appears whenever a gauge field h_μ^I has in its transformation law a term which is proportional to the vielbein e_μ^a. As in our terminology on covariant quantities we did not consider the vierbein as a gauge field, this term can be included in $\delta_m h_\mu^I$. Thus for closed algebras ($\eta_{JI} = 0$) the result is as follows: in the transformation of the covariant derivative, the derivative should not work on the parameter, and further terms appear only from the transformation of gauge fields that are not proportional to gauge fields. Thus gauge fields never appear 'naked' in such transformations. One only has to be careful here that the vielbein is not considered as a gauge field itself.

E.g. consider the calculation of the transformation of (2.47). First of all, remark that this would not work for $D_\mu\psi$, which is not a covariant quantity. Then, the transformations of the spin connection ω_μ^{bc} can be neglected as this field transforms only in the derivative of a parameter or in an explicit gauge field, and the latter do not appear in the final result due to the theorem. We do have to take into account that the vielbein transforms as

$$
\delta e_\mu^a = \xi^\nu\partial_\nu e_\mu^a - \lambda_M^{ab}e_{\mu b} + \dots\,,
\tag{2.56}
$$

where the $\dots$ stand for e.g. a term $e_\nu^a\partial_\mu\xi^\nu$, which can be neglected because it contains a derivative on the parameter. The final result is thus

$$
\begin{aligned}
\delta D_a\psi &= \left(\xi^\nu\partial_\nu e_a^\mu - \lambda_{M\,ab}e_b^\mu\right)\partial_\mu\psi + e_\alpha^\mu\partial_\mu\left(\xi^\nu\partial_\nu - \tfrac{1}{4}\lambda_M^{bc}\gamma_{bc}\right)\psi + \dots\\
&= \xi^b D_b D_a\psi - \lambda_M^{ab}D_b\psi - \tfrac{1}{4}\lambda_M^{bc}\gamma_{bc}D_a\psi\,.
\end{aligned}
\tag{2.57}
$$

This shows the usefulness of the theorem: although in intermediate steps we omitted some terms, at the end we obtain the full result.

2.5   Local conformal transformations

We now consider the gauging of the conformal algebra, and show how Poincaré gravity is recovered in a gauge for dilatations and special conformal transformations. The gauge fields are introduced as

$$h_\mu^A T_A = e_\mu^a P_a + \omega_\mu{}^{ab} M_{ab} + b_\mu D + f_\mu^a K_a \,. \tag{2.58}$$

The curvatures are

$$
\begin{aligned}
R_{\mu\nu}(P^a) &= R_{\mu\nu}^P(P^a) + 2b_{[\mu} e_{\nu]}^a \\
R_{\mu\nu}(M^{ab}) &= R_{\mu\nu}^P(M^{ab}) + 8f_{[\mu}^{[a} e_{\nu]}^{b]} \\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]} - 4f_{[\mu}^a e_{\nu]a} \\
R_{\mu\nu}(K^a) &= 2\partial_{[\mu} f_{\nu]}^a + 2\omega_{[\mu}{}^{ab} f_{\nu]b} - 2b_{[\mu} f_{\nu]}^a \,.
\end{aligned}
\tag{2.59}
$$

The conventional constraint (2.38) could be imposed because the gauge field ω_μ^{ab} appears in the curvature multiplied by the invertible vielbein. Considering the curvatures, one can see that also the gauge field f_μ^a appears in such a way in the curvatures of D and M. It can be solved if we impose as second conventional constraint:

$$
\begin{aligned}
R_{\mu\nu}(P^a) &= 0 \\
R_{\mu\nu}(M^{ab}) e_b^\nu &= 0 \,.
\end{aligned}
\tag{2.60}
$$

Therefore as well ω_μ^{ab} as f_μ^a are dependent fields. In the first one there is a modification to (2.39) due to the term proportional to b_μ. The solution for the K gauge field is (with the definitions as in (2.45))

$$f_\mu^a = \tfrac{1}{4} R_\mu{}^a - \tfrac{1}{24} R e_\mu^a \,. \tag{2.61}$$

Because of the theorem on covariant derivatives it is useful to give the part of the transformations of gauge fields proportional to vielbeins. These are (with the dots representing the transformations in other explicit gauge fields)

$$
\begin{aligned}
\delta e_\mu^a &= -\Lambda_D e_\mu^a - \Lambda_M^{ab} e_{\mu b} + \dots \\
\delta b_\mu &= 2\Lambda_K^a e_{\mu a} + \dots \\
\delta \omega_\mu^{ab} &= 4\Lambda_K^{[a} e_\mu^{b]} + \dots \,.
\end{aligned}
\tag{2.62}
$$

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Exercise

It is interesting to compare transformations of derivatives of fields in the rigid theory with those of covariant derivatives in conformal gravity. In (2.25) the second line came from the derivatives of ξ (first two terms) and the derivative of Λ_D. In the gauge theory, the full x-dependent $\xi_\mu(x)$ is the parameter of general coordinate transformations. Show, with the help of the theorem on covariant derivatives, how one re-obtains (2.25) with all derivatives replaced by covariant ones (and changing to a-indices).

Show that also in the local theory, (2.27) is the rule to determine whether the action with ordinary derivatives replaced by covariant derivatives, is an invariant. ∎

The transformation law of a covariant derivative determines the covariant box

$$
\begin{aligned}
\Box^C \phi \;&\equiv\; \eta^{ab} D_b D_a \phi = e^{a\mu}\left(\partial_\mu D_a\phi - (w+1)b_\mu D_a\phi + \omega_{\mu\,ab}D^b\phi + 2w f_{\mu a}\right)\\
&=\; e^{-1}\left(\partial_\mu - (w+2-d)b_\mu\right)eg^{\mu\nu}\left(\partial_\nu - wb_\nu\right)\phi + w\frac{6-d}{12}R\,\phi\,. \quad (2.63)
\end{aligned}
$$

The latter is the well-known $R/6$ term in $d=4$. In fact, choosing $w = \frac{d}{2}-1$, one has a conformal invariant scalar action

$$
I = \int d^d x\, e\phi\Box^C\phi\,. \tag{2.64}
$$

Exercise: show that $\int d^d x\, eD_a\phi D^a\phi$ is not invariant. How does the difference between this statement and the one in (2.29) occur?

Consider now that we want a Poincaré invariant action. Then we have to break dilatations and special conformal transformations, as these are not part of the Poincaré algebra. Considering (2.62) it is clear that the latter can be broken by a gauge choice

$$
K - \text{gauge}: \qquad b_\mu = 0\,. \tag{2.65}
$$

Therefore, of the 'Weyl multiplet' (the multiplet of fields with the gauge fields of the conformal algebra), only the vielbein remains. One could take as gauge choice for dilatations a fixed value of a scalar ϕ. And one easily checks that then the action (2.64) reduces to the Poincaré gravity action.

3 Spinors in arbitrary dimensions

I now present information about the Clifford algebra and spinors in arbitrary dimensions. I will discuss arbitrary signatures of spacetime. Different from many other treatments, I will not consider arguments of field theory. E.g. in many other papers one uses that spinors are solutions of the Dirac equation. That is also the case in [4], which is however the first reference for the facts that I will recall.

3.1 Gamma matrices and their symmetry

I consider arbitrary spacetime dimensions $d = t + s$, with t timelike directions and s spacelike directions, and consider the general facts of Clifford algebras and spinors, as has been done first in [5].

The Clifford algebra is

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\eta_{ab} \,, \tag{3.1}$$

where $\eta = \mathrm{diag}(- \ldots - + \ldots +)$, writing first the timelike directions and then the spacelike ones.

I first give a representation of the Clifford algebra for signature $(0, d)$ (only spacelike) in terms of σ matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.2}$$

The only relevant properties are $\sigma_1 \sigma_2 = \mathrm{i}\sigma_3$ and cyclic, and that they square to $\mathbb{1}_2$ and are hermitian.

$$
\begin{aligned}
\Gamma_1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\Gamma_2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \ldots \\
\Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \ldots \\
\Gamma_5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots \\
\ldots &= \ldots .
\end{aligned}
\tag{3.3}
$$

This is for even dimensions a representation of dimension $2^{d/2}$. For odd dimensions, e.g. $d = 5$, one does not need the last σ_1 factor in Γ_5, and the

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dimension is still $2^{(d-1)/2}$. These matrices are all hermitian. If there are t timelike directions, we just have to multiply the first t matrices in the representation above by i. E.g. for the Minkowski case $\Gamma_1 = \mathrm{i}\sigma_1 \otimes \mathbb{1} \otimes \dots$. Thus we get as hermiticity property for timelike and spacelike directions

$$\Gamma_t^\dagger = -\Gamma_t\,, \qquad \Gamma_s^\dagger = \Gamma_s\,, \tag{3.4}$$

or in general

$$\Gamma_a^\dagger = (-)^t A \Gamma_a A^{-1}\,, \qquad A = \Gamma_1 \dots \Gamma_t\,. \tag{3.5}$$

Of course the realization given above is not unique. One preserves (3.1) with

$$\Gamma' = U^{-1}\Gamma U\,. \tag{3.6}$$

To respect also (3.4), U has to be unitary. A further remark on the uniqueness will follow in footnote 5.

Another important fact is that **in even dimensions** a complete set of $2^{d/2} \times 2^{d/2}$ matrices is provided by $\{\Gamma^{(n)}\}$ with $n = 0, 1, \dots d$, and

$$\Gamma^{(n)} = \Gamma_{a_1 \dots a_n} \equiv \Gamma_{[a_1} \Gamma_{a_2} \dots \Gamma_{a_n]}\,. \tag{3.7}$$

The last matrix $\Gamma^{(d)}$, we write as[3]

$$\Gamma_* = (-\mathrm{i})^{d/2+t}\Gamma_1 \dots \Gamma_d\,, \qquad \Gamma_*\Gamma_* = 1\,, \tag{3.8}$$

where the normalization is chosen such that the last equation holds in any case. Γ_* is independent on the signature of spacetime, in the sense that when I introduced first the Γ matrices in Euclidean signature, and then went to an arbitrary signature, I multiplied the first t gamma matrices by i. This drops out again by the multiplication by $(-\mathrm{i})^t$ in (3.8). In the representation (3.3), Γ_* is

$$\Gamma_* = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \dots\,. \tag{3.9}$$

For even dimensions we have (the Levi–Civita tensor ε is defined in appendix A)

$$\Gamma_{a_1 \dots a_n} = \frac{1}{(d-n)!}\varepsilon_{a_1 \dots a_d}\mathrm{i}^{d/2+t}\Gamma_*\Gamma^{a_d \dots a_{n+1}}\,. \tag{3.10}$$

---

[3]If 0 is used as label of the time direction in Minkowski space, then we have $\Gamma_* = (-\mathrm{i})^{d/2+1}\Gamma_0\Gamma_1 \dots \Gamma_{d-1}$.

Note that Γ_* anticommutes with Γ_a and can thus be used as Γ_{d+1} in the next, **odd dimension**. In odd dimensions the product of all Γ matrices gives ± 1 or $\pm i$, and a basis is formed by $\Gamma^{(n)}$ with $n = 0, 1, \ldots, (d-1)/2$.

There is always a '**charge conjugation matrix**' $\mathcal{C}$, such that

$$\mathcal{C}^T = -\varepsilon \mathcal{C}, \qquad \Gamma_a^T = -\eta \mathcal{C} \Gamma_a \mathcal{C}^{-1}, \qquad (3.11)$$

for $\varepsilon = \pm 1$ and $\eta = \pm 1$. A formal proof can be found in [4, 5], but I prove it here by giving two possibilities in the representation (3.3) for even dimensions

$$
\begin{aligned}
\mathcal{C}_+ &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots & \eta = +1 \\
\mathcal{C}_- &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots \propto \mathcal{C}_+ \Gamma_* & \eta = -1.
\end{aligned}
\qquad (3.12)
$$

Note that they satisfy $\mathcal{C} = \mathcal{C}^\dagger = \mathcal{C}^{-1}$. For odd dimensions, only one of the two can be used, see table 1 below. If one changes the representation as in (3.6), then, to preserve (3.11), the charge conjugation matrix should transforms as

$$\mathcal{C}' = U^T \mathcal{C} U. \qquad (3.13)$$

Therefore you see that the unitarity, $\mathcal{C}^\dagger = \mathcal{C}^{-1}$, is true in any representation, but the charge conjugation not necessarily squares to $\mathbb{1}$.

We will now deduce which signs of the two variables ϵ and η are allowed for each value of s and t. First remark that

$$\left(\mathcal{C} \Gamma^{(n)} \right)^T = -\epsilon (-)^{n(n-1)/2} (-\eta)^n \mathcal{C} \Gamma^{(n)}. \qquad (3.14)$$

This has a periodicity of $n \to n+4$. This periodicity implies that the number of symmetric matrices should be the sum of lines of the following formulae

$$
\begin{aligned}
\binom{d}{0} + \binom{d}{4} + \ldots &= 2^{d-2} + 2^{d/2-1} \cos \frac{d\pi}{4} \\
\binom{d}{1} + \binom{d}{5} + \ldots &= 2^{d-2} + 2^{d/2-1} \sin \frac{d\pi}{4} \\
\binom{d}{2} + \binom{d}{6} + \ldots &= 2^{d-2} - 2^{d/2-1} \cos \frac{d\pi}{4} \\
\binom{d}{3} + \binom{d}{7} + \ldots &= 2^{d-2} - 2^{d/2-1} \sin \frac{d\pi}{4}.
\end{aligned}
\qquad (3.15)
$$

On the other hand we can count that there should be

$$2^{\left[\text{Int } \frac{d}{2}\right]-1}\left(2^{\left[\text{Int } \frac{d}{2}\right]}+1\right) \qquad \text{symmetric matrices}$$

$$2^{\left[\text{Int } \frac{d}{2}\right]-1}\left(2^{\left[\text{Int } \frac{d}{2}\right]}-1\right) \qquad \text{antisymmetric matrices.} \qquad (3.16)$$

For odd dimensions, there is a unique solution. For even dimensions, two of the lines in (3.15) are equal, and there are thus two possibilities for which matrices are symmetric. This corresponds to the two possibilities for the charge conjugation matrix in that case. The result is given in table 1.

d (mod 8)	S	A	ϵ	η
0	0,3	2,1	-1	$+1$
	0,1	2,3	-1	-1
1	0,1	2,3	-1	-1
2	1,0	3,2	-1	-1
	1,2	3,0	$+1$	$+1$
3	1,2	0,3	$+1$	$+1$
4	2,1	0,3	$+1$	$+1$
	2,3	0,1	$+1$	-1
5	2,3	0,1	$+1$	-1
6	3,2	1,0	$+1$	-1
	3,0	1,2	-1	$+1$
7	0,3	1,2	-1	$+1$

Table 1: *For all dimensions (modulo 8) one finds which $\mathcal{C}\Gamma^{(n)}$ should be symmetric (S) and antisymmetric (A). These values of n are modulo 4). This determines signs of ϵ and η.*

Exercise

Check with the explicit representation the signs of η and ϵ for the two possible charge conjugation matrices in 4 dimensions. Then consider the fifth gamma matrix, and see that the (3.11) only holds for one choice of $\mathcal{C}$.

3.2 Irreducible spinors

We still do not know whether the spinor is irreducible. There are two types of projections that one can envisage. The first is only for even dimensions,

where we have Γ_* available as in (3.8). We can thus define left and right spinors. For definiteness we take

$$\lambda_L = \tfrac{1}{2}\left(1 + \Gamma_*\right)\lambda\,, \qquad \lambda_R = \tfrac{1}{2}\left(1 - \Gamma_*\right)\lambda\,. \tag{3.17}$$

E.g. in four dimensional Minkowski space ($s = 3, t = 1$) we define[4]

$$\gamma_5 = \Gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3\,, \qquad \lambda_L = \tfrac{1}{2}(1 + \gamma_5)\lambda\,, \qquad \lambda_R = \tfrac{1}{2}(1 - \gamma_5)\lambda\,. \tag{3.18}$$

The second possible projection is a *reality condition*. Consider first the reality properties of the Γ matrices. Combining (3.5) and (3.11), we have[5]

$$\Gamma_a^* = -\eta(-)^t B\Gamma_a B^{-1}\,, \qquad B^T = \mathcal{C}A^{-1}\,. \tag{3.19}$$

As A and $\mathcal{C}$ are unitary, we find that also B is unitary. One obtains

$$B^* B = -\epsilon\eta^t(-)^{t(t+1)/2}\,. \tag{3.20}$$

To prove this equation, see first that internal consistency of (3.5) implies that

$$A^T = \eta^t \mathcal{C}A^{-1}\mathcal{C}^{-1} = (-)^{t(t+1)/2}\eta^t \mathcal{C}A\mathcal{C}^{-1}\,. \tag{3.21}$$

Let us now try to put a reality condition on spinors

$$\lambda^* = \tilde{B}\lambda\,, \tag{3.22}$$

for some matrix $\tilde{B}$. First of all we want consistency with Lorentz transformations, i.e. taking the Lorentz transformation of both sides

$$\begin{aligned}
\left(-\tfrac{1}{4}\Gamma_{ab}\lambda\right)^* &= -\tfrac{1}{4}\tilde{B}\Gamma_{ab}\lambda \qquad \Rightarrow \\
B\Gamma_{ab}B^{-1}\tilde{B} &= \tilde{B}\Gamma_{ab}\,,
\end{aligned} \tag{3.23}$$

and therefore we take

$$\tilde{B} = \alpha B\,. \tag{3.24}$$

---

[4] For 4 dimensional Minkowski space we write $\gamma_0, \ldots, \gamma_3$ in stead of $\Gamma_1, \ldots, \Gamma_4$.

[5] Here we can give some remark on the uniqueness of the Clifford algebra representation. It is clear that $\pm\Gamma^*$ can also be used as representations of the Clifford algebra. The equation (3.19), which for even dimensions holds for two possibilities of η, implies that in even dimensions these representations are unitary equivalent with the original one. The fact that for odd dimensions only one sign of η can be used implies that in odd dimensions there are two unitary inequivalent representations.

For consistency, $\lambda^{**} = \lambda$, (3.22) implies that $\tilde{B}^* \tilde{B} = 1$. Combining this with (3.20) gives $|\alpha| = 1$ and **the right hand side of (3.20) should be 1**. It turns out that this condition is satisfied for

$$
\begin{aligned}
s - t &= 0, 1, 7 \text{ mod } 8 \\
s - t &= 2 \text{ mod } 8 \ \text{ with } \eta = +1 \\
s - t &= 6 \text{ mod } 8 \ \text{ with } \eta = -1 \, .
\end{aligned}
\tag{3.25}
$$

Exercise: check that this can be summarized as: the charge conjugation is symmetric in an Euclidean space with $s - t$ spacelike dimensions. ∎
In these cases, we can thus define a reality condition as in (3.22), and such spinors will be denoted as 'Majorana spinors'. Another way of expressing the Majorana condition is that the 'Majorana conjugate' equals the 'Dirac conjugate'. They are respectively

$$
\bar{\lambda} \equiv \lambda^T \mathcal{C} \, , \qquad \bar{\lambda}^C \equiv \lambda^\dagger A \alpha^{-1} \, .
\tag{3.26}
$$

Majorana spinors can thus be thought as spinors $\lambda_1 + i\lambda_2$, where λ_1 and λ_2 have real components, but these are related by the above condition. The exact value α, if it is of modulus 1, is irrelevant for our purpose here. However, it determines whether e.g. $\bar{\psi}\psi$ or $\int d^d x \bar{\psi} \, \partial\!\!\!/ \psi$ is hermitian. We will come back to this issue in section 3.4.

Consider now that the right had side of (3.20) is not 1, thus -1. Then there is still another possibility if we have extended supersymmetry. Indeed, then one can define a 'symplectic Majorana condition'

$$
\lambda_i^* \equiv \left(\lambda^i \right)^* = B\Omega_{ij}\lambda^j \, ,
\tag{3.27}
$$

where Ω is some antisymmetric matrix, with $\Omega\Omega^* = -1$. This is e.g. the way to have real spinors in $d = 6$ Minkowski space.

Having two projections, to chiral spinors and to Majorana spinors, one may ask whether they can be *combined, i.e. can we define a reality condition respecting the chiral projection*. As $(\Gamma_*)^* = (-)^{d/2+t} B\Gamma_* B^{-1}$ the condition is that $d/2 + t = 0$ mod 2, i.e.

$$
\text{MW spinor}: s - t = 0 \text{ mod } 4 \, .
\tag{3.28}
$$

Table 2 gives the resulting possibilities, and its caption gives a summary.

d \ t	0		1		2		3	
1	M	1	M	1				
2	M$^-$	2	MW	1	M$^+$	2		
3		4	M	2	M	2		4
4	SMW	4	M$^+$	4	MW	2	M$^-$	4
5		8		8	M	4	M	4
6	M$^+$	8	SMW	8	M$^-$	8	MW	4
7	M	8		16		16	M	8
8	MW	8	M$^-$	16	SMW	16	M$^+$	16
9	M	16	M	16		32		32
10	M$^-$	32	MW	16	M$^+$	32	SMW	32
11		64	M	32	M	32		64
12	SMW	64	M$^+$	64	MW	32	M$^-$	64

Table 2: *Possible spinors in various dimensions, and for various number of time directions (modulo 4). M stands for Majorana spinors. For even dimensions, $M^\pm$ indicates which sign of η should be used. MW indicates the possibility of Majorana–Weyl spinors. For even dimensions one can always have Weyl spinors. Symplectic Majorana spinors are always possible when the Majorana condition is not possible. SMW indicates the possibility of symplectic Majorana–Weyl spinors. The number indicates the real dimension of the minimal spinor.*

3.3 4 dimensions

As an example take the Minkowski space in 4 dimensions. In order to have Majorana spinors we choose $\eta = 1$, and hence $\epsilon = 1$. There exists a Majorana representation where the matrices γ_μ are real:

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}. \tag{3.29}$$

The charge conjugation matrix is proportional to γ_0, say $\mathcal{C} = z\gamma_0$, with $|z| = 1$. As $A = \gamma_0$, the Majorana condition is $\lambda^* = z\alpha\lambda$. Thus with an appropriate choice of signs, $z\alpha = 1$, Majorana spinors are just pure real spinors in this basis.

In another representation that I present now, γ_5 is diagonal. That basis is useful to have manifestly chiral spinors. These are then just spinors that have only two upper components λ^A, while antichiral have two lower components, denoted as $\lambda_{\dot{A}}$, (where A and $\dot{A}$ can be 1 or 2). The explicit realization is

$$\gamma_0 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ i\mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \quad (3.30)$$

and with this choice

$$\mathcal{C} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \qquad \text{with} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.31)$$

This basis makes the connection to the '2-component formalism' easy. In that formulation, indices are raised or lowered with ε in a NW-SE convention

$$\chi^A = \varepsilon^{AB}\chi_B, \qquad \chi_A = \chi^B\varepsilon_{BA}, \quad (3.32)$$

which implies $\varepsilon^{AB}\varepsilon_{BC} = -\delta^A{}_C$, and the Majorana condition for a spinor $(\zeta^A, \zeta_{\dot{A}})$ is then

$$\zeta_A = (\zeta_{\dot{A}})^*. \quad (3.33)$$

I do not use this convention, but use chiral spinors, avoiding the indices.

A third representation that is often used, has diagonal γ_0.

$$\gamma_0 = \begin{pmatrix} i\mathbb{1}_2 & 0 \\ 0 & -i\mathbb{1}_2 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}. \quad (3.34)$$

The charge conjugation is $\mathcal{C} = \gamma_0\gamma_2$ and the Majorana condition amounts to

$$\lambda^* = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\lambda. \quad (3.35)$$

3.4 Technical tips

3.4.1 Complex conjugation

To consider complex conjugation in practice, it is useful to consider the operation C, defined by

$$\lambda^C \equiv \alpha^{-1}B^{-1}\lambda^*. \quad (3.36)$$

This is thus chosen such that the Majorana spinors are those for which $\lambda^C = \lambda$. Note that the bar operation, defined by the first equation of (3.26), on

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λ^C gives $\bar\lambda^C$ as defined in the second equation of (3.26). The square of this operation gives the identity $(\lambda^C)^C = \lambda$ if the right hand side of (3.20) is 1. In case that this is -1, we should modify the definition of the C operation similar to (3.27). One can check, using (3.21), that

$$\bar\lambda^* = \alpha\eta^t\bar\lambda^C B^{-1}\,. \qquad (3.37)$$

We are now equipped to discuss complex conjugation of bi-spinors. First of all, we still have to determine whether complex conjugation changes the order of fermions or not. In fact, both types of complex conjugation are possible. I introduce a new sign factor to reflect this choice, say β. Thus for two fermionic quantities,

$$(\lambda\chi)^* = \beta\lambda^*\chi^* = -\beta\chi^*\lambda^*\,, \qquad \beta = \pm 1\,. \qquad (3.38)$$

Consider now the complex conjugation of $\bar\chi M\lambda$, where M is some matrix in spinor space. With the previous equations one easily finds

$$(\bar\chi M\lambda)^* = \beta\alpha^2\eta^t\bar\chi^C B^{-1}M^* B\lambda^C\,. \qquad (3.39)$$

We can thus choose in any case α appropriate in order that e.g. $\bar\chi\lambda$ is real. Then we define $M^C \equiv B^{-1}M^* B$, thus e.g.

$$\Gamma_a^C = -\eta(-)^t\Gamma_a \qquad \Gamma_*^C = (-)^{d/2+t}\Gamma_*\,, \qquad (3.40)$$

the latter for even dimensions.
Rule: to perform complex conjugation of a bispinor, first add a factor $\beta\alpha^2\eta^t$ and then replace fields and matrices by their C-conjugates, for which (3.40) gives the important relations.

As an example consider Minkowski spaces where $\eta = \epsilon = 1$, thus in dimensions $d = 2, 3, 4 \mod 8$, containing 4, 10 and 11 dimensions. We choose α according to the convention on complex conjugation of bi-spinors as $\beta\alpha^2 = 1$. Then (3.39) implies that we can perform complex conjugation by using the C operation. Majorana spinors are thus considered as just real, and the same applies for the ordinary gamma matrices. However γ_5, the Γ_* in 4 dimensions, behaves as a pure imaginary matrix. In 10 dimensions this matrix can also be considered as real, which reflects the possibility of Majorana–Weyl spinors in that dimension.
Exercise:

See how this should be set up in the case of symplectic Majorana spinors. ∎

In extended supergravity in 4 dimensions, the chirality is indicated by the position of the i, j index (index running over $1, \ldots N$ for N-extended supergravity). See e.g. (4.6). The choice of chirality for the spinor with an upper (lower) index can change for each spinor. It is chosen conveniently on the first occurrence of the spinor. With the rules of the C operation, we see that h.c. effectively interchanges upper and lower indices. The chiral spinor Q^i is not a Majorana spinor. Check that $(Q^i)^C = Q_i$.

3.4.2 Gamma matrix manipulations

I still give some useful identities for calculations in arbitrary dimensions [6]. For a product of two antisymmetrized gamma matrices, one can use

$$\Gamma_{a_1 \ldots a_i} \Gamma^{b_1 \ldots b_j} = \sum_{k=|i-j|}^{i+j} \frac{i! j!}{s! t! u!} \delta^{[b_1}_{[a_i} \cdots \delta^{b_s}_{a_{t+1}} \Gamma_{a_1 \ldots a_t]}{}^{b_{s+1} \ldots b_j]} \tag{3.41}$$

$$s = \tfrac{1}{2}(i + j - k), \qquad t = \tfrac{1}{2}(i - j + k), \qquad u = \tfrac{1}{2}(-i + j + k).$$

In [6] a few extra rules are given and a diagrammatic technique is explained that is based on the work of Kennedy [7].

For contractions of repeated gamma matrices, one has the formula

$$\Gamma_{b_1 \ldots b_k} \Gamma_{a_1 \ldots a_\ell} \Gamma^{b_1 \ldots b_k} = c_{k,\ell} \Gamma_{a_1 \ldots a_\ell}$$

$$c_{k,\ell} = (-)^{k(k-1)/2} k! (-)^{k\ell} \sum_i^{\min(k,\ell)} \binom{\ell}{i} \binom{D-\ell}{k-i} (-)^i, \tag{3.42}$$

for which tables were given in [6] in dimensions 4, 10, 11 and 12, and which can be easily obtained from a computer programme.

Further, there is the Fierz relation. We know that the gamma matrices are matrices in dimension $\Delta = 2^{\mathrm{Int}\ d/2}$, and that a basis of $\Delta \times \Delta$ matrices is given by the set

$$\{\mathbb{1}, \Gamma_a, \Gamma_{a_1 a_2}, \ldots, \Gamma^{a_1 \ldots a_{[D]}}\} \qquad \text{where} \qquad \begin{cases} [D] = D & \text{for even } D \\ [D] = (D-1)/2 & \text{for odd } D, \end{cases} \tag{3.43}$$

of which only the first has nonzero trace. This is the basis of the general Fierz formula for an arbitrary matrix M in spinor space:

$$M_\alpha{}^\beta \Delta = \sum_{k=0}^{[D]} (-)^{k(k-1)/2} k! \, (\Gamma_{a_1 \dots a_k})_\alpha{}^\beta \, \mathrm{Tr} \left(\Gamma^{a_1 \dots a_k} M \right) . \qquad (3.44)$$

More useful Fierz identities can be found in [7].

3.4.3 Spinor indices

In these notes, I mostly omit spinor indices. However, in some applications, this is difficult to accomplish or one prefers spinor indices. I will give here a way to include spinor indices in a practical way, independent of the dimension of spacetime. First of all, spinors get a lower spinor index. So for a spinor λ, I write λ_α. Gamma matrices act on these spinors, and therefore the expression $\Gamma_a \lambda$ becomes $(\Gamma_a)_\alpha{}^\beta \lambda_\beta$. Note the position of the spinor indices on a usual Gamma matrix. This is then the same for products of Gamma matrices.

Then I want to be able to raise and lower indices. Indeed, e.g. to discuss symmetries of Gamma matrices, the $(\Gamma_a)_\alpha{}^\beta$ can not be used, as we can not interchange upper and lower indices. To that purpose, I introduce matrices $\mathcal{C}^{\alpha\beta}$ and $\mathcal{C}_{\alpha\beta}$, which will be related to the charge conjugation matrix below. The convention that I adopt, is that I raise and lower indices always in the NorthWest–SouthEast (NW-SE) convention. That means that the contraction indices should appear in that relative position, i.e.

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta} \lambda_\beta , \qquad \lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha} . \qquad (3.45)$$

In order for these two equations to be consistent, we should have

$$\mathcal{C}^{\alpha\beta} \mathcal{C}_{\gamma\beta} = \delta_\gamma^\alpha , \qquad \mathcal{C}_{\beta\alpha} \mathcal{C}^{\beta\gamma} = \delta_\alpha^\gamma . \qquad (3.46)$$

I choose the identifications such that the Majorana conjugate $\bar\lambda$ is written as λ^α. Comparing (3.45) and (3.26), we conclude that $\mathcal{C}^{\alpha\beta}$ is $\mathcal{C}^T$, and $\mathcal{C}_{\alpha\beta}$ is then $\mathcal{C}^{-1}$. One may check that the symmetries of $\mathcal{C}\Gamma^{(n)}$ or $\Gamma^{(n)}\mathcal{C}^{-1}$, which were discussed before, is now the symmetry of $(\Gamma_a)^{\alpha\beta}$ or $(\Gamma_a)_{\alpha\beta}$.
Exercise : Check that $\bar\chi\lambda = \chi^\alpha \lambda_\alpha = \pm \chi_\alpha \lambda^\alpha$ where the sign is $-\epsilon$, i.e. $+(-)$ if the charge conjugation matrix is (anti)symmetric. The same is true for spinor indices at any place: $(\Gamma_a)_\alpha{}^\beta \lambda_\beta = \pm (\Gamma_a)_{\alpha\beta} \lambda^\beta$.

4 Supersymmetry algebras

4.1 Haag–Łopuszański–Sohnius result as introduction

After the introduction with bosonic symmetries, and the recapitulation of spinor properties, I can finally consider fermionic symmetries. This means that the parameters of some transformations can be anticommuting numbers. For the generators we then have anticommutators rather than commutators. Indeed, consider

$$\delta(\epsilon) = \epsilon^A Q_A \,, \tag{4.1}$$

for anticommuting parameters ϵ^A and hence operators Q_A, which I will denote further as 'odd', distinguishing them from the bosonic 'even' operators. We have then

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \epsilon_2^B \epsilon_1^A (Q_A Q_B + Q_B Q_A) \equiv \epsilon_2^B \epsilon_1^A \{Q_A, Q_B\} \,. \tag{4.2}$$

Here, I introduced a notation for the anticommutator, and I also introduce a general notation

$$[T_A, T_B\} = T_A T_B - (-)^{AB} T_B T_A \,. \tag{4.3}$$

The notation $[,\}$ is a commutator except when both operators are odd, when it is an anticommutator. In the notation $(-)^A$ one should thus understand A as an even number, e.g. 0, if T_A is even, and A as an odd number, e.g. 1, if T_A is odd.

These algebras should also satisfy a Jacobi identity. Keeping the parameters in place, such a Jacobi identity takes the same form as the pure bosonic one:

$$\left[\epsilon_1^A T_A, \left[\epsilon_2^B T_B, \epsilon_3^C T_C\right]\right] + \text{ cyclic in 1, 2, 3 } = 0 \,. \tag{4.4}$$

Removing the parameters, signs appear, but an easy form to remember is

$$[[T_A, T_B\}, T_C\} = [T_A, [T_B, T_C\}\} - (-)^{AB} [T_B, [T_A, T_C\}\} \,. \tag{4.5}$$

The last sign factor just reflects the symmetry of the commutator in the left-hand side.

These definitions lead to a lot of extensions of algebras. We will come back to the mathematical part in section 4.6. First we consider the physical input. As a generalisation of the result of [1], reviewed in section 2.1, Haag, Łopuszański and Sohnius [8] considered the same question allowing

for fermionic symmetries. As for [1], their result contained two parts: one general possibility (which we will label as the super-Poincaré algebra), and one for massless fields only, the superconformal algebras. An important part of their result is that super-Poincaré algebras may contain 'central charges', which we will discuss in section 4.4. They did suppose a flat space, but we are also interested in the possibility of a curved space, i.e. the super-(anti) de Sitter algebras.

First I will give some qualitative remarks on supersymmetry in general. Then I will consider the super-Poincaréalgebras, first without central charges in section 4.3, then with central charges in section 4.4. I will turn to super-anti-de Sitter algebras in section 4.5, but will have to interrupt that analysis to give an overview of simple superalgebras in general (section 4.6). Finally I will turn to the superconformal algebras in section 4.7.

4.2   Basic properties of Poincaré supersymmetry

In general the algebra contains respectively the Poincaré, conformal or (anti) de Sitter algebra, and a number of (Lorentz) scalar generators on the even side, and spin-1/2 odd generators. For the Poincaré case these are generators Q_α^i, where α is a spinor index, and i is a representation index for some part of the bosonic scalar generators outside of the Poincaré algebra. Remember that the whole discussion here is in 4 dimensions, where the spinors can be split in a left and a right chirality, which we indicate by the position of the i index:

$$Q_\alpha^i = \tfrac{1}{2}(1 - \gamma_5)_\alpha{}^\beta Q_\beta^i \,, \qquad Q_{i\alpha} = \tfrac{1}{2}(1 + \gamma_5)_\alpha{}^\beta Q_{i\beta} \,. \tag{4.6}$$

Their anticommutator takes the form

$$\{Q_\alpha^i, Q_{\beta j}\} = \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu \delta_j^i \,, \tag{4.7}$$

and I will return to the meaning of the charge conjugation matrix, C, in section 3. This is the defining relation of supersymmetries: they square to the translations. One of the consequences is that it has a dimension 1/2, where the dimension of translations is conventionally taken to be 1, i.e. as a mass dimension. So e.g. if a scalar ϕ of dimension 1 transforms in a fermion[6]

$$\delta_Q \phi = \bar{\epsilon}\lambda \,, \tag{4.8}$$

---

[6]Spinor indices are deleted, and details of the notation will follow later.

then, as ϵ has dimension $-1/2$, the dimension of λ is $1/2$ higher than that of ϕ, and its supersymmetry transformation contains ϕ always with a derivative or a mass. Of course, in minimal actions of scalars and spinors, the former have dimension 1, and the latter $3/2$, so we are not unhappy with this conclusion.

Theorem: *There are an equal number of bosonic and fermionic degrees of freedom in any realization of the supersymmetry algebra when translations are an invertible operation.*

Consider the commutator (4.7) on the space of all bosons. The first Q transforms the space to a space of fermions, while the second one brings us back to bosons, translated by P_μ.

$$\text{bosons} \xrightarrow{\ Q\ } \text{fermions} \xrightarrow{\ Q\ } \text{bosons translated by } P_\mu$$

The latter is an invertible operator, which proves that the last space is as large as the first one, and thus also the middle one, the space of the fermions has the same dimension. This proves the well-known fact that there are an equal number of bosonic and of fermionic states. It should hold when the algebra (4.7) holds. So this equality of bosonic and fermionic states should hold e.g. for on-shell states, but also for off-shell states if the algebra is also realized off-shell. On the other hand, it is not a general fact of a superalgebra. One needs the invertibility of the square of fermionic generators. This does not hold for massless states in an Euclidean spacetime, or in 1 dimension. In these cases the $P^2 = 0$ condition for a massless state implies $P = 0$, and there is thus no invertibility.

4.3 Super-Poincaré algebras

Now we can study the superalgebras which are important in supersymmetry. I start with the result of [8] in 4 dimensions. Apart from the bosonic Poincaré algebra (2.1), and the defining relation of supersymmetry (4.7), there is of course the statement that the supersymmetries are spinors under the Lorentz group[7]:

$$[M_{\mu\nu}, Q_\alpha^i\,] = -\tfrac{1}{4}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta^i\,. \tag{4.9}$$

The supersymmetries may also rotate under an automorphism group T_A:

$$[T_A, Q_\alpha^i\,] = (U_A)^i{}_j Q_\alpha^j\,, \qquad [T_A, Q_{\alpha i}\,] = (U_A)_i{}^j Q_{\alpha j}\,. \tag{4.10}$$

---

[7]As the reader probably already noticed, we denote Γ matrices in 4 dimensions as γ_μ

The latter equation is obtained from the former by complex conjugation, thus the complex conjugate of $(U_A)^i{}_j$ is denoted as $(U_A)_i{}^j$, consistent with the rule mentioned at the end of section 3.4.1. The Jacobi identity $[TTQ]$ implies that these matrices form a representation of the algebra of the T_A. And as T_A has to commute with P_μ, as follows already from the Coleman–Mandula result, it is easy to check that the $[TQ_iQ^j]$ Jacobi identity implies that

$$(U_A)_i{}^j = -(U_A)^j{}_i \equiv -\left((U_A)_j{}^i\right)^* . \tag{4.11}$$

Thus these are unitary matrices and the automorphism group is $U(N)$, for $i = 1, \ldots, N$.

We can make a general statement about the isometry groups in view of table 2. The 4-dimensional $t = 1$ example is general for even dimensions where no Majorana–Weyl spinor is possible, thus indicated by M in the table. Then the automorphism group is $U(N)$. In fact, in this formulation we have not used the Majorana property, but rather the Weyl-spinor formulation. In all even dimensions we can use Weyl spinors, and this leads to spinors of the same dimension as the Majorana ones (they are the same in another notation). If we would have the possibility of Majorana–Weyl, then there is an extra reality condition, and we have two factors as automorphism groups of the left and right spinors, i.e. we obtain $SO(N_L) \times SO(N_R)$. For odd dimensions, we can not use the chiral formulation, and if there are Majorana spinors, the result is $SO(N)$. If there are symplectic Majorana conditions, then the same argument of the Jacobi identity $[TQQ]$ leads to the preservation of the symplectic metric, and the automorphism algebra is thus reduced to $USp(N)$ (even N)[8]. If there is a SMW spinor, then there are two such factors. In summary, the automorphism groups are obtained from the entries of table 2:

$$
\begin{aligned}
M \text{ and } d \text{ odd} &\quad : \quad SO(N) \\
M \text{ and } d \text{ even} &\quad : \quad U(N) \\
MW &\quad : \quad SO(N_L) \times SO(N_R) \\
S \text{ (empty in table)} &\quad : \quad USp(N) \\
SMW &\quad : \quad USp(N_L) \times USp(N_R) .
\end{aligned}
\tag{4.12}
$$

We can now write a list of supersymmetry theories in Minkowski spaces of

_____

[8]There is often confusion about the notation of Sp groups. We use notation such that $Sp(2)$ is the smallest symplectic group, see section 4.6.

various dimensions and with various extensions. Table 3, which I essentially

d \ Q	32	16	8	4
11	X			
10	$\overset{IIB}{(2,0)}\ \overset{IIA}{(1,1)}$	$\overset{I}{(1,0)}$		
9	X	X		
8	X	X		
7	X	X		
6	$(2,2)$	$(2,0)\ (1,1)$	$(1,0)$	
5	X	X	X	
4	$N=8$	$N=4$	$N=2$	$N=1$

Table 3: *Pure supergravity theories in dimensions $4 \leq d \leq 11$ with the number of independent supercharges equal to $Q = 32, 16, 8$ and 4. In 3 spacetime dimensions, pure supergravity does not describe propagating degrees of freedom and is a topological theory.*

copied from [9], gives an overview of possible supersymmetry models[9]. This follows mainly from table 2. The limit of 32 supercharges comes from the requirement that one can not construct field theories with helicities larger than 2. I have not explained representations of the algebras, and therefore can not really prove this. However, one can understand that in 4 dimensions any supersymmetry raises or lowers the helicity by $1/2$. Therefore with $N = 8$ we already fill the whole range from 2 to -2. Similarly rigid supersymmetry models are only possible for $N \leq 4$, i.e. $Q \leq 16$. We still remark that other values of Q, i.e. $Q = 12, 20, 24$, are also possible in $d = 4$, but are rarely used.

Let me summarize the result for the super Poincaré algebras in 4 dimensions, omitting the automorphisms (as can consistently be done for the super-Poincaré case):

$$\begin{aligned}
[M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta^{[\rho}_{[\mu}M_{\nu]}{}^{\sigma]}\,, \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu}P_{\rho]}\,,
\end{aligned}$$

---

[9]Note that the theories are not always unique. Certainly if one allows arbitrary signs in kinetic terms for some fields.

$$
\begin{aligned}
[P_\mu, P_\nu \,] &= 0 \\
\{Q_\alpha^i, Q_{\beta j}\} &= \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu \delta_j^i \\
[M_{\mu\nu}, Q_\alpha^i \,] &= -\frac{1}{4}(\gamma_{\mu\nu} Q)_\alpha^i \\
[P_\mu, Q_\alpha^i \,] &= 0 \,.
\end{aligned}
\tag{4.13}
$$

Remark that we may also consistently remove the Lorentz algebra generators M. Thus the algebra is a semi direct sum of the part with P and Q only, with the Lorentz algebra and with the automorphism algebra.

Let us check the reality property of the anticommutator of 2 supersymmetries, and generalize to other dimensions and signatures. Complex conjugation gives

$$
\beta\alpha^2 B \left\{(Q^i)^C, (Q_j)^C\right\} B^T = -\eta(-)^t B\Gamma_\mu \mathcal{C}^{-1} B^T \eta^t P^\mu \,,
\tag{4.14}
$$

where spinor indices are implicit. The factor β results from the fact that complex conjugation may changes the order of fermions if $\beta = -1$. Note that it does not change the order in which the generators act. Thus there is only that factor, but the operators keep their order. The β spinor index, which was explicit in (4.7), is at the right side of the equation, and that is the reason of B^T at the end of the left hand side, as $Q_\beta^* = \alpha Q_\gamma^C (B^T)^\gamma{}_\beta$. For the right hand side of (4.14), I used

$$
\mathcal{C}^{-1,*} = \mathcal{C}^T = \eta^t B \mathcal{C}^{-1} B^T \,,
\tag{4.15}
$$

and the bosonic operators are real. Remark that if we move the η^t to the left hand side, we find here again the factors of (3.39) and (3.40). Such commutators in general behave as bi-spinors, as one can imagine multiplying them with spinor parameters at the left and right. Thus they follow the rule mentioned at page 24.

The symmetry of the commutator (and of $\Gamma_\mu \mathcal{C}^{-1}$) implies that in the right hand side in 4 dimensions we can use

$$
\left\{(Q^i)^C, (Q_j)^C\right\} = \left\{Q_i, Q^j\right\} = \left\{Q^j, Q_i\right\} \,.
\tag{4.16}
$$

Then the reality consistency is proven if we take (for Minkowski spaces with $\eta = 1$) conventions with

$$
\beta\alpha^2 = 1 \,.
\tag{4.17}
$$

With these methods it is easy to check how this can be generalised to other dimensions, with other types of spinors. E.g. in $d = 11$ (Minkowski, $t = 1$) there are no chiral spinors, and we just have

$$\{Q_\alpha, Q_\beta\} = \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu. \tag{4.18}$$

The reality goes as above, one can just neglect the i indices.

In $d = 10$ one can have Majorana–Weyl spinors. For having a similar $\{Q,Q\} = P$ anticommutator, the $\Gamma_\mu P^\mu$ should thus appear in the anticommutator between two supersymmetries of the same chirality. One can ask in general the question whether one can have the translations P_μ in the right hand side of the anticommutator between two chiral supersymmetries $\{Q_L, Q_L\}$. If that is the case, then

$$
\begin{aligned}
\{Q_L, Q_L\} &= \tfrac{1}{2}\left(1 + \Gamma_*\right)\gamma^\mu \mathcal{C}^{-1} P_\mu \tfrac{1}{2}\left(1 + \Gamma_*\right)^T \\
&= \tfrac{1}{4}\left(1 + \Gamma_*\right)\left(1 - (-)^{d/2}\Gamma_*\right)\gamma^\mu P_\mu \mathcal{C}^{-1} \\
&= \tfrac{1}{4}\left(1 - (-)^{d/2}\right)\left(1 + \Gamma_*\right)\gamma^\mu \mathcal{C}^{-1} P_\mu,
\end{aligned}
$$

where I used that

$$
\begin{aligned}
\Gamma_*^T &= (-\mathrm{i})^{d/2+t}\Gamma_d^T \ldots \Gamma_1^T = (-\mathrm{i})^{d/2+t}\mathcal{C}\Gamma_d \ldots \Gamma_1 \mathcal{C}^{-1} \\
&= (-)^{d/2}\mathcal{C}\Gamma_*\mathcal{C}^{-1}. \tag{4.19}
\end{aligned}
$$

Thus $d/2$ should be odd, as it is the case in 10 dimensions. If there is one such chiral supersymmetry, we denote this as $(1,0)$ supersymmetry. If there are two, then this is $(2,0)$ (which is usually called IIB). There can be two supersymmetries of opposite chirality which is denoted as $(1,1)$, or IIA supersymmetry. Remark that the result (4.19) also checks that in 4 dimensions the P_μ does not appear in the anticommutator of 2 generators of the same chirality, but, as immediately shown in (4.7), in the anticommutator of a chiral and an antichiral supersymmetry.

Exercise. Check that for $d/2$ even, consistency is obtained if P_μ appears in the anticommutator of a chiral with an antichiral supersymmetry. Further check that for $d/2$ odd, the anticommutator of two supersymmetries of the same chirality can only have $\Gamma^{(n)}\mathcal{C}^{-1}$ for n odd, while for opposite chirality n should be even. This is reversed for $d/2$ even. ∎

An important consequence of supersymmetry, which we did not mention yet, is contained in the *positivity statements* of the energy. To consider these, we

want the anticommutator between a supersymmetry and its complex conjugate:

$$\{Q_i, (Q_j)^*\} = \alpha \left\{ Q_i, (Q_j)^C \right\} B^T = \alpha \delta_i^j \Gamma_\mu A^{-1} P^\mu \,, \qquad (4.20)$$

where the i, j indices are put as for $d = 4$, but one can forget them for e.g. $d = 11$. The left hand side is real and positive definite if we do not interchange spinors in complex conjugation ($\beta = 1$). Then we can choose $\alpha = 1$, consistent with (4.17). For $\beta = -1$ the left hand side is imaginary, but α is then an imaginary unit. We can therefore conclude that we can choose α, consistent with (4.17), such that

$$\alpha^{-1} \{Q_i, (Q_j)^*\} \geq 0 \,. \qquad (4.21)$$

As $A = \Gamma_0$ in Minkowski spaces, this exhibits the positive energy statements in supersymmetry. For a state at rest, i.e. $P^\mu = \delta_0^\mu M$, this implies $M \geq 0$.

4.4 Poincaré algebras with central charges

Finally we come to an extra possibility, which are the central charges. They have been found in the classification of Haag–Lopuszański–Sohnius [8], and were realized in the massive (and short) hypermultiplets in [10]. This was in 4-dimensional Minkowski space, to which we will restrict first, before discussing the generalization to higher dimensions.

I did not yet give the commutator of two supersymmetries of equal chirality. As I do not want new symmetries with Lorentz indices, the general possibility is

$$\{Q_{\alpha i}, Q_{\beta j}\} = \mathcal{C}_{\alpha\beta}^{-1} \omega_{ij}^M Z_M \,, \qquad (4.22)$$

where M labels different central charges Z^M, characterized by independent antisymmetric matrices ω_{ij}^M. Indeed, the properties of spinors imply that ω_{ij}^M should be antisymmetric. Now one has to check all Jacobi identities, and at the end one obtains that these Z^M should commute with all other generators including the supersymmetries. Therefore, these are called 'central charges'. Considering finally $[TQQ]$, we find a restriction on the automorphism group. Indeed, one obtains

$$U_i{}^k \omega_{kj} - \omega_{ik} U_j{}^k = 0 \,. \qquad (4.23)$$

Let us consider as an example $N = 2$. Then for ω_{ij}, there is only ε_{ij}, as this is the only antisymmetric tensor. We thus have 1 (complex) central charge:

$$\{Q_{\alpha i}, Q_{\beta j}\} = \mathcal{C}_{\alpha\beta}^{-1} \varepsilon_{ij} Z \,. \qquad (4.24)$$

If there is such a central charge, then the automorphism group is not $U(2)$, but rather the $U(1) \times U(1)$ which commutes with ε_{ij}.

Central charges are essential in various application due to the modified positivity statements. Indeed, the modified algebra allows us to obtain another positivity condition.

I will show the argument for $N = 2$, but it can easily be generalized. The complex conjugate of (4.24) is[10]

$$\beta \left\{ Q^{*\alpha i}, Q^{*\beta j} \right\} = -\varepsilon^{ij} \mathcal{C}^{\alpha\beta} Z^* . \tag{4.25}$$

To diagonalize the set of anticommutators for the supersymmetries, I define

$$A_{\alpha i} \equiv Q_{\alpha i} + \alpha^{-1} e^{i\theta} \varepsilon_{ij} \mathcal{C}_{\alpha\beta}^{-1} Q^{*\beta j} , \tag{4.26}$$

where $e^{i\theta}$ is an arbitrary phase factor. The complex conjugate is

$$A^{*\alpha i} \equiv (A_{\alpha i})^* = Q^{*\alpha i} + \alpha e^{-i\theta} \varepsilon^{ij} Q_{j\beta} \mathcal{C}^{\beta\alpha} = \alpha e^{-i\theta} \varepsilon^{ij} A_{j\beta} \mathcal{C}^{\beta\alpha} . \tag{4.27}$$

Consider now their anticommutators on a state for which $P^\mu = \delta_0^\mu M$:

$$\alpha^{-1} \left\{ A_{\alpha i}, A^{*\beta j} \right\} = \delta_i^j \delta_\alpha^\beta (2M + Z e^{-i\theta} + Z^* e^{i\theta}) . \tag{4.28}$$

As the left hand side is a positive definite operator, the right hand side should be positive for all θ, which shows that

$$M \geq |Z| . \tag{4.29}$$

This is an important result, giving a lower limit to the mass, dependent on the central charge. This is called the *BPS bound*. One can now also see that if the equality is satisfied for a state (a 'BPS state'), then that state is invariant under part of the supersymmetry generators. That state preserves thus part of the supersymmetry. Another role of the central charges, is that one can have supersymmetry multiplets with central charges such that the bound is saturated. Such multiplets have less components than others, and are therefore called short multiplets. These are the characteristics of algebras with central charges.

Knowing of this possibility of central charges in 4 dimensions, one may try to generalize it to *higher dimensions*. One looks again for generators

---

[10]I use here $Q^{*\alpha i} \equiv (Q_{i\alpha})^*$. The tensor ε^{ij} is the same as ε_{ij}, namely $\varepsilon^{12} = \varepsilon_{12} = 1$, but I write one or the other according to the index structure of the equation.

in the $\{Q, Q\}$ anticommutator. This was first considered in the rheonomic approach to $d = 11$ supergravity [11], for solving $d = 10$ super-Yang–Mills superspace constraints [12], and while looking for superconformal algebras in higher dimensions [6]. The presence of these operators will lead to positivity bounds, similar to the one that we found in $N = 2$, $d = 4$.

Consider as an example $d = 11$, with one supersymmetry. In the $\{Q, Q\}$ anticommutator one can think of all sorts of objects

$$\{Q_\alpha, Q_\beta\} = \sum_n \left(\Gamma^{(n)} \mathcal{C}^{-1} \right)_{\alpha\beta} Z_{(n)}. \tag{4.30}$$

But of course, the right hand side should also be symmetric. Considering table 1, we see that this restricts us already to $n = 1, 2$ mod 4, and for an odd dimension, the independent ones are those up to $n = (d-1)/2$. This implies that we have at most

$$\{Q_\alpha, Q_\beta\} = \left(\Gamma^\mu \mathcal{C}^{-1} \right)_{\alpha\beta} P_\mu + \left(\Gamma^{\mu\nu} \mathcal{C}^{-1} \right)_{\alpha\beta} Z^2_{\mu\nu} + \left(\Gamma^{\mu\nu\rho\sigma\tau} \mathcal{C}^{-1} \right)_{\alpha\beta} Z^5_{\mu\nu\rho\sigma\tau}. \tag{4.31}$$

The Z^2 and Z^5 are thus new generators in this anticommutator. In fact, they are not 'central' in the algebra, i.e. they do not commute with all the other generators. As they have spacetime indices, they do not commute with the Lorentz rotations. The terminology which is in use, refers to the fact that these generators have the same physical consequences as the real central charges in 4 dimensions, i.e. they lead to BPS bounds similar to (4.29), with the limit obtained by states which preserve part of the supersymmetry.

In 4 dimensions the scalar central charges are gauged by a vector, and the charges are produced by a particle. In 11 dimensions various solutions produce such charges, e.g. M2 or M5 branes, as has be reviewed by J. Gomis in this school, and I refer the reader to [13, 14].

4.5 Super-anti-de Sitter algebras

Remember that in the bosonic case, the Poincaré algebra is not a simple algebra, but a contraction $R \to \infty$ of the simple (anti) de Sitter algebra (2.2). The same will apply for the corresponding superalgebras. But in this section we will first go step by step to construct a super anti-de Sitter algebra.

Due to the last commutator of (2.2), we find that we need a modification with respect to the super-Poincaré algebra to satisfy the Jacobi identity $[P, P, Q]$, see (4.5). Indeed, it can not be satisfied anymore with $[P, Q] = 0$.

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For covariance (or consistency with Lorentz transformations, i.e. $[M, P, Q]$ Jacobi), we should have (at least for $N = 1$)

$$[P_\mu, Q] = x\Gamma_\mu Q \,, \tag{4.32}$$

where x is an arbitrary number. But taking the complex conjugate and using (3.22) and (3.19) we obtain

$$x = -\eta(-)^t x^* \,. \tag{4.33}$$

Continuing with the $[P, P, Q]$ Jacobi identity, we get (using (2.2))

$$x^2 = \frac{1}{16\,R^2} \,. \tag{4.34}$$

Specifying to $t = 1$, we thus find that for $\eta = 1$, which is e.g. the case in $d = 11$ and $d = 4$ if there are Majorana spinors, x should be real. Thus R^2 is positive, and this implies, see the discussion following (2.2), that we can only have the anti-de Sitter algebra, rather than de Sitter. Indeed, the de Sitter algebra would require $R^2 < 0$, which is not consistent with (4.34) and real x. For $N = 1$ in $d = 4$ or $d = 11$ this thus proves that one should consider anti-de Sitter, rather than de Sitter algebras. For extended supersymmetry, one could have introduced an antisymmetric matrix in the ij indices in (4.32), and my arguments for anti-de Sitter are thus not complete. For $d = 10$ with $N = 1$ chiral supersymmetry one can not have either de Sitter nor anti-de Sitter, as (4.32) is not consistent with the chirality projection.

Continuing in this way with Jacobi identities [6], we obtain the following result for $d = 4$, $N = 1$:

$$
\begin{aligned}
\{Q_{L\alpha}, Q_{R\beta}\} &= \left(\Gamma_\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu + 2x \left(\Gamma_{\mu\nu} \mathcal{C}^{-1}\right)_{\alpha\beta} M_{\mu\nu} \\
[P_\mu, Q] &= x\Gamma_\mu Q \\
[M_{\mu\nu}, Q] &= -\tfrac{1}{4}\Gamma_{\mu\nu} Q \\
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu[\rho} M_{\sigma]\nu} - \eta_{\nu[\rho} M_{\sigma]\mu} \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]} \\
[P_\mu, P_\nu] &= 8\,x^2\, M_{\mu\nu} \,,
\end{aligned}
\tag{4.35}
$$

while in $d = 11$ we obtain

$$\{Q_\alpha, Q_\beta\} = \left(\Gamma_\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu + 2x \left(\Gamma_{\mu\nu} \mathcal{C}^{-1}\right)_{\alpha\beta} M_{\mu\nu}$$

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$$+\frac{1}{5!}\left(\Gamma^{\mu\nu\rho\sigma\tau}\mathcal{C}^{-1}\right)_{\alpha\beta}Z^5_{\mu\nu\rho\sigma\tau}$$

$$\left[Z^5_{\mu\nu\rho\sigma\tau},Q\right] = \frac{x}{2}\Gamma_{\mu\nu\rho\sigma\tau}Q$$

$$\left[Z^5_{\mu_1\ldots\mu_5},M^{\nu\rho}\right] = 5\delta^{[\nu}_{[\mu_1}Z^{5\,\rho]}{}_{\mu_2\ldots\mu_5]}, \tag{4.36}$$

with the other rules of (4.35) unchanged. Remark that we need now also the 5-index 'central charge' generator in order to be able to close the algebra. However, it is even less 'central' than before, because this generator now also has a non-zero commutator with Q. The Z^2 of (4.31) has been identified here with the Lorentz transformations.

In fact, the $d=11$ super adS algebra is not of the 'HLS' form. Its bosonic algebra is not $adS \times G$. Here it is $Sp(32)$. Indeed the generators of P_μ, $M_{\mu\nu}$ and Z^5 constitute

$$\binom{11}{1}+\binom{11}{2}+\binom{11}{5}=11+55+462=528=\frac{32.33}{2} \tag{4.37}$$

generators. We can combine them in the 32×32 matrix form

$$M=\frac{1}{x}\Gamma^\mu P_\mu + \Gamma^{\mu\nu}M_{\mu\nu}+\frac{1}{x}\Gamma^{\mu\nu\rho\sigma\tau}Z^5_{\mu\nu\rho\sigma\tau} \tag{4.38}$$

to see this algebra explicitly. The full superalgebra is $OSp(1|32)$, but for that we should first give an introduction to superalgebras. Then we will be able to classify also the super adS algebras in other dimensions.

4.6 Superalgebras

Lie superalgebras have been classified in [15]. I do not have the time to go through the full classification mechanism of course, but will consider the most important superalgebras, the 'simple Lie superalgebras', which have no non-trivial invariant subalgebra. However, one should know that in superalgebras there are more subtle issues, as e.g. not any semi-simple superalgebra is the direct sum of simple superalgebras. A good review is [16]. The fermionic generators of such superalgebras are in representations of the bosonic part. If that 'defining representation' of the bosonic algebra in the fermionic generators is completely reducible, the algebra is said to be 'of classical type'. The others are 'Cartan type superalgebras' $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$, which we will further neglect. For further reference, I give the list of the real forms

of superalgebras 'of classical type'. But, for fixing my notations on algebras, it might be useful first to recapitulate a similar list for the bosonic algebras. That is done in table 4. The conventions which I use for groups is that

Table 4: Real forms of simple bosonic Lie algebras. For the exceptional algebras, I do not repeat the obvious line that the compact form itself is one of the real forms. The second number in the notation for these real forms is the number of non-compact $-$ number of compact generators.

Compact	Real Form	Maximal compact subalg.
$SU(n)$	$SU(p, n-p)$	$SU(p) \times SU(n-p) \times U(1)$
$SU(n)$	$S\ell(n)$	$SO(n)$
$SU(2n)$	$SU^*(2n)$	$USp(2n)$
$SO(n)$	$SO(p, n-p)$	$SO(p) \times SO(q)$
$SO(2n)$	$SO^*(2n)$	$U(n)$
$USp(2n)$	$Sp(2n)$	$U(n)$
$USp(2n)$	$USp(2p, 2n-2p)$	$USp(2p) \times USp(2n-2p)$
$G_{2,-14}$	$G_{2,2}$	$SU(2) \times SU(2)$
$F_{4,-52}$	$F_{4,-20}$	$SO(9)$
$F_{4,-52}$	$F_{4,4}$	$USp(6) \times SU(2)$
$E_{6,-78}$	$E_{6,-26}$	$F_{4,-52}$
$E_{6,-78}$	$E_{6,-14}$	$SO(10) \times SO(2)$
$E_{6,-78}$	$E_{6,2}$	$SU(6) \times SU(2)$
$E_{6,-78}$	$E_{6,6}$	$USp(8)$
$E_{7,-133}$	$E_{7,-25}$	$E_{6,-78} \times SO(2)$
$E_{7,-133}$	$E_{7,-5}$	$SO(12) \times SU(2)$
$E_{7,-133}$	$E_{7,7}$	$SU(8)$
$E_{8,-248}$	$E_{8,-24}$	$E_{7,-133} \times SU(2)$
$E_{8,-248}$	$E_{8,8}$	$SO(16)$

$Sp(2n) = Sp(2n, \mathbb{R})$ (always even entry), and $USp(2m, 2n) = U(m, n, \mathbb{H})$. $S\ell(n)$ is $S\ell(n, \mathbb{R})$. Further, $SU^*(2n) = S\ell(n, \mathbb{H})$ and $SO^*(2n) = O(n, \mathbb{H})$. Note that in these bosonic algebras there are the following isomorphisms,

some of which will be important later[11]

$$
\begin{aligned}
SO(2,1) &= S\ell(2) = SU(1,1) = Sp(2) \\
SO(4) &= SU(2) \times SU(2)\,, \qquad SO(2,2) = S\ell(2) \times S\ell(2) \\
SO^*(4) &= SU(1,1) \times SU(2) \\
SO(5) &= USp(4)\,, \qquad SO(4,1) = USp(2,2)\,, \qquad SO(3,2) = Sp(4) \\
SO(6) &= SU(4)\,, \qquad SO(5,1) = SU^*(4) \\
&\qquad SO(4,2) = SU(2,2)\,, \qquad SO(3,3) = S\ell(4) \\
SO^*(6) &= SU(3,1) \\
SO^*(8) &= SO(6,2)\,.
\end{aligned}
\qquad (4.39)
$$

The similar table for Lie superalgebras 'of classical type', and their real forms [17] is given in table 5. In this table, 'defining representation' gives the fermionic generators as a representation of the bosonic subalgebra. The 'number of generators' gives the numbers of (bosonic,fermionic) generators in the superalgebra. I mention first the algebra as an algebra over $\mathbb{C}$, and then give different real forms of these algebras. With this information, you can reconstruct all properties of these algebras, up to a few exceptions. The names which I use for the real forms is for some algebras different from those in the mathematical literature [17], and chosen such that it is most suggestive of its bosonic content. There are isomorphisms as $SU(2|1) = OSp(2^*|2,0)$, and $SU(1,1|1) = S\ell(2|1) = OSp(2|2)$. In the algebra $D(2,1,\alpha)$ the three $S\ell(2)$ factors of the bosonic group in the anticommutator of the fermionic generators appear with relative weights 1, α and $-1-\alpha$. The real forms contain respectively $SO(4) = SU(2) \times SU(2)$, $SO(3,1) = S\ell(2,\mathbb{C})$ and $SO(2,2) = S\ell(2) \times S\ell(2)$. In the first and last case α should be real, while $\alpha = 1 + ia$ with real a for $p = 1$. In the limit $\alpha = 1$ one has the isomorphisms $D^p(2,1,1) = OSp(4-p,p|2)$.

4.7 Superconformal algebras

In this section I review the classification of superconformal algebras of a standard type, to be explained soon. At the same time, however, also the adS

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[11]Note that the equality sign is not correct for the groups. For the algebras there are these isomorphism, but it are rather the covering groups of the orthogonal groups which are mentioned at the right hand sides.

superalgebras of the similar type will be classified, due to the isomorphism (2.17).

In supersymmetric theories, the conformal symmetry implies the presence of a second supersymmetry S, usually denoted as 'special supersymmetry'. Indeed, the commutator of the special conformal transformations, and the ordinary supersymmetry Q implies this S due to $[K_\mu, Q] = \Gamma_\mu S$.

The anticommutator $\{Q, S\}$ generates also an extra bosonic algebra (sometimes called R symmetry). The whole superalgebra can be represented in a supermatrix, as e.g. (symbolically)

$$\begin{pmatrix} SO(d,2) & Q+S \\ Q-S & R \end{pmatrix} . \tag{4.40}$$

We can consider such superalgebras in general. That is what Nahm [18] did in his classification. The requirements for a superconformal algebra in d or a super-adS algebra in $d+1$ are:

1. $SO(d,2)$ should appear as a factored subgroup of the bosonic part of the superalgebra. For Nahm, this requirement was motivated by the Coleman–Mandula theorem, but one can demand it also in order that the bosonic algebra is the isometry algebra of a space which has the adS space as a factor.

2. fermionic generators should sit in a spinorial representation of that group.

To find the list of superconformal algebras, one has to consider the isomorphisms of groups $SO(d,2)$ which are in (4.39). Then the analysis is straightforward, and the result are algebras with maximal $d = 6$[12]. The result[13] is given in Table 6, except for $d = 2$. For $d = 2$ the finite bosonic adS or conformal algebra is $SO(2,2) \approx SO(2,1) \oplus SO(2,1)$, i.e. the sum of two $d = 1$ algebras. The super-adS or superconformal algebra is then the sum of two $d = 1$ algebras of Table 6. Notice that these are the finite

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[12]Note the particular case of $d = 6$, where we use the notation $OSp(8^*|N)$ for the superconformal algebra. Often, including previous articles of ourselves, it was written as $OSp(6,2|4)$, not paying attention to the existing real forms. In fact, in the series $OSp(m - p, p|2n)$ the algebra $Sp(2n)$ is non-compact. Thus, a compact R-symmetry group, $USp(2n)$, is not possible in that series. The possibility of a compact R-symmetry exists due to the isomorphism $SO^*(8) = SO(6,2)$, such that one can use the next line of table 5. This thus works only for the signature (6,2).

[13]We mention the superalgebra with compact R-symmetry group.

part of infinite dimensional superconformal algebras in 2 dimensions. For a classification of the infinite superconformal algebras, see [19]. One may also relax the condition that the bosonic algebra contains the algebra $SO(d,2)$ as a factored subgroup of the whole bosonic algebra, and suffice with having it as some subgroup. Then the other bosonic symmetries are not necessarily scalars and the Coleman–Mandula theorem is violated. However, this may still be relevant where branes are present and has been used e.g. in [6] to propose the $OSp(1|32)$ as super adS_{11} or $conf_{10}$. In that case one has (4.31) This algebra is now known as the M-theory algebra [14].

To recognize it as $OSp(1|32)$, one can write it as

$$\{Q_\alpha, Q_\beta\} = M_{\alpha\beta}, \qquad [M_{\alpha\beta}, Q_\gamma] = Q_{(\alpha}\mathcal{C}_{\beta)\gamma}$$
$$[M_{\alpha\beta}, M_{\gamma\delta}] = \mathcal{C}_{\alpha(\gamma}M_{\delta)\beta} + \mathcal{C}_{\beta(\gamma}M_{\delta)\alpha}. \tag{4.41}$$

Exercise: Obtain the commutators in (4.36) and the Lorentz algebra from this general rule, defining first $M_{\alpha\beta}$ as the right hand side of the anticommutator of the supersymmetries. ∎

In this way it is clear how all the bosonic generators form $Sp(32)$ where the antisymmetric metric in that algebra is $\mathcal{C}_{\alpha\beta}$.

These superconformal algebras are used in adS/CFT correspondence, or other applications in M-theory. I have been first interested in them for the construction of general matter couplings in super-Poincaré theories. This goes along the lines explained at the end of section 2.5. For reviews on this method in 4 dimensions, see [20, 2]. For 6 dimensions it has be applied for (1,0) supersymmetry in [21, 3] and for (2,0) supersymmetry in [22].

Acknowledgments.

A Conventions

I use the metric signature $(-+\ldots+)$. If you prefer the opposite, insert a minus sign for every upper index which you see, or for an explicit metric η_{ab} or $g_{\mu\nu}$. The Gamma matrices Γ_a should then be multiplied by an i to have this change of signature.

The curved indices are denoted by $\mu, \nu, \ldots = 0, \ldots, d-1$ and the flat ones by $a, b, \ldots$. (Anti)symmetrization is done with weight one: $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$ and $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$.

The anticommuting Levi–Civita tensor is taken to be

$$\varepsilon_{12\ldots d} = 1, \qquad \varepsilon^{12\ldots d} = (-)^t, \tag{A.1}$$

where t is the number of timelike directions. If we use the index value 0 for Minkowski space, then $\varepsilon_{01\ldots(d-1)} = 1$. The contraction identity for these tensors is $(p + n = d)$

$$\varepsilon_{a_1\ldots a_n b_1\ldots b_p}\varepsilon^{a_1\ldots a_n c_1\ldots c_p} = (-)^t\, p!\, n!\, \delta^{[c_1}_{[b_1}\ldots \delta^{c_p]}_{b_p]}. \tag{A.2}$$

For the local case, we can still define constant tensors

$$\varepsilon_{\mu_1\ldots\mu_d} = e^{-1} e^{a_1}_{\mu_1}\ldots e^{a_d}_{\mu_d}\varepsilon_{a_1\ldots a_d}, \qquad \varepsilon^{\mu_1\ldots\mu_d} = e\, e^{\mu_1}_{a_1}\ldots e^{\mu_d}_{a_d}\varepsilon^{a_1\ldots a_d}. \tag{A.3}$$

They are thus *not* obtained from each other by raising or lowering indices with the metric.

Exercises.

1. Show that the tensors in (A.3) are indeed constants, i.e. that arbitrary variations of the vierbein cancel in the full expression. You will have to use the so-called 'Schouten identities', which means that antisymmetrizing in more indices than the range of the indices, gives zero.

2. If one defines in even dimensions $d = 2n$ the dual of an n-tensor as

$$\tilde{F}_{a_1\ldots a_n} = (\mathrm{i})^{d/2+t}\frac{1}{n!}\varepsilon_{a_1\ldots a_d}F^{a_d\ldots a_{n+1}}, \tag{A.4}$$

check that

$$\tilde{\tilde{F}}_{a_1\ldots a_n} = F_{a_1\ldots a_n}, \qquad \Gamma_{a_1\ldots a_n} = \Gamma_*\tilde{\Gamma}_{a_1\ldots a_n}. \tag{A.5}$$

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Table 5: Lie superalgebras of classical type.

Name	Range	Bosonic algebra	Defining repres.	Number of generators
$SU(m\|n)$	$m \geq 2$ $m \neq n$ $m = n$	$SU(m) \oplus SU(n)$ $\oplus U(1)$ no $U(1)$	$(m,\bar{n}) \oplus$ $(\bar{m},n)$	$m^2 + n^2 - 1,$ $2mn$ $2(m^2-1), 2m^2$
$S\ell(m\|n)$ $SU(m-p,p\|n-q,q)$ $SU^*(2m\|2n)$		$S\ell(m) \oplus S\ell(n)$ $SU(m-p,p) \oplus SU(n-q,q)$ $SU^*(2m) \oplus SU^*(2n)$	$\oplus SO(1,1)$ $\oplus U(1)$ $\oplus SO(1,1)$ $\left.\right\}$ if $\left.\right\} m \neq n$	
$OSp(m\|n)$	$m \geq 1$ $n = 2,4,..$	$SO(m) \oplus Sp(n)$	(m,n)	$\frac{1}{2}(m^2 - m + n^2 + n), mn$
$OSp(m-p,p\|n)$ $OSp(m^*\|n-q,q)$		$SO(m-p,p) \oplus Sp(n)$ $SO^*(m) \oplus USp(n-q,q)$	n even m,n,q even	
$D(2,1,\alpha)$	$0 < \alpha \leq 1$	$SO(4) \oplus S\ell(2)$	$(2,2,2)$	$9, 8$
$D^p(2,1,\alpha)$		$SO(4-p,p) \oplus S\ell(2)$		$p = 0,1,2$
$F(4)$		$\overline{SO(7)} \oplus S\ell(2)$	$(8,2)$	$21, 16$
$F^p(4)$		$SO(7-p,p) \oplus S\ell(2)$		$p = 0,1,2,3$
$G(3)$		$G_2 \oplus S\ell(2)$	$(7,2)$	$14, 14$
$G_p(3)$		$G_{2,p} \oplus S\ell(2)$		$p = -14, 2$
$P(m-1)$	$m \geq 3$	$S\ell(m)$	$(m \otimes m)$	$m^2 - 1, m^2$
$Q(m-1)$	$m \geq 3$	$SU(m)$	Adjoint	$m^2 - 1, m^2 - 1$
$Q(m-1)$ $Q((m-1)^*)$ $UQ(p,m-1-p)$		$S\ell(m)$ $SU^*(m)$ $SU(p,m-p)$		

Table 6: Super adS_{d+1} or $conf_d$ algebras.

d	superalgebra	R	number of fermionic
1	$OSp(N|2)$	$O(N)$	$2N$
	$SU(N|1,1)$	$SU(N) \times U(1)$ for $N \neq 2$	$4N$
	$SU(2|1,1)$	$SU(2)$	8
	$OSp(4^*|2N)$	$SU(2) \times USp(2N)$	$8N$
	$G(3)$	G_2	14
	$F^0(4)$	$SO(7)$	16
	$D^0(2,1,\alpha)$	$SU(2) \times SU(2)$	8
3	$OSp(N|4)$	$SO(N)$	$4N$
4	$SU(2,2|N)$	$SU(N) \times U(1)$ for $N \neq 4$	$8N$
	$SU(2,2|4)$	$SU(4)$	32
5	$F^2(4)$	$SU(2)$	16
6	$OSp(8^*|N)$	$USp(N)$ (N even)	$8N$