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Hamiltonian origins of a special collection of $D = 8$ topological BF models

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Abstract

Here, we address the Hamiltonian setting underlying a special $D = 8$ collection of topological BF models with the field spectrum consisting in a system forms of degree zero, one, three, and four. All the necessary ingredients of a standard Hamiltonian approach to degenerate systems are introduced in order to reveal the structure and properties of the generating set of gauge symmetry specific to the Lagrangian framework.

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1 Introduction

Topological field theories [1] are best known for their relationship to some interacting, non-Abelian versions of the Poisson algebra [2] specific to a large variety of Poisson sigma models [3]–[9], which, in turn, are very important for a consistent formulation of $D = 2$ gravity [10]–[20]. Meanwhile, pure $D = 3$ gravity is nothing but a topological BF theory, while, concerning the higher dimensional case, General Relativity and supergravity in Ashtekar formalism can be related to some topological BF models in the presence of certain extra constraints [21]–[24]. In this context, BF theories on higher dimensional spacetimes and their properties may be helpful at clarifying some key aspects of corresponding gravity/supergravity theories.

This paper is devoted to the delicate problem of explaining the Hamiltonian origins of the generating set of gauge symmetries used in the literature in connection to the Lagrangian formulation of a finite collection of free, topological BF models with a non-standard field spectrum, consisting in four sets of form fields with the form degree equal to 0, 1, 3, and 4. This is done using the Hamiltonian approach to degenerate field theories [25] in a complex framework, where the first- and second-class constraints are not separated from the start and some of the first-class ones are reducible. The results exposed here add to the previous ones obtained by the authors and related to various aspects of single or several topological BF models emerging from a Lagrangian or Hamiltonian approach [26]–[37] and, in addition, enlightens the Hamiltonian roots of our previous findings [38] regarding the construction of consistent Lagrangian interactions that can be added precisely to the above mentioned collection of free BF models.

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Our paper is divided into introduction, nine main sections, and conclusions. Section 2 briefly exposes the Lagrangian formulation for the considered, finite collection of free topological BF models evolving on a Minkowski $D = 8$ space-time of 'mostly positive' signature and sets the problem to be investigated, while Section 3 introduces the main conventions and notations related to the Hamiltonian approach. Next, the primary constraints and canonical Hamiltonian are constructed in Section 4 and then the Dirac–Bergmann algorithm is applied in order to determine the secondary constraints of this model. It is argued that the entire constraint set is not yet separated into its first- and second-class constraint subsets and the separation procedure is effectively solved in Section 6, after which the main properties of the equivalent formulation of the overall constraint hypersurface are discussed in Section 7. The second-class constraint subset is eliminated via passing to the strictly first-class hypersurface by means of the Dirac bracket in Section 8, so a purely first-class model, equivalent to the initial one, is revealed. Then, the extended and total Hamiltonian actions are constructed together with their specific sets of gauge transformations in Sections 9 and 10, finally leading to the searched for generating set of gauge symmetries of the initial Lagrangian action.

2 Setting the problem

The Lagrangian setting of a free model describing a non-standard collection of BF models in $D = 8$ relies on the action

$$\begin{aligned} S^{\mathrm{L}}_{\mathrm{BF}}\left[\Phi^A\right] &= \int_{\mathbb{R}^{7,1}} \left(H^a_{\mu_1}\partial^{\mu_1}\phi_a + H^a_{\mu_1\mu_2\mu_3\mu_4}\partial^{[\mu_1}V^{\mu_2\mu_3\mu_4]}_a\right) d^8x \\ &\equiv \int_{\mathbb{R}^{7,1}} \mathcal{L}_{\mathrm{BF}}\left(\Phi^A, \partial^\mu\Phi^A\right) d^8x, \end{aligned} \tag{1}$$

with the space-time the 8-dimensional Minkowski manifold $\mathbb{R}^{7,1}$ (with a 'mostly positive' signature metric, $\mathrm{diag}(- + \cdots +)$). The 'non-standard' designation refers to the presence of a reduced field spectrum compared to the maximally allowed one, consisting in a finite collection of (purely real) scalar–vector and three-form–four-form pairs $\left\{\phi_a, H^a_{\mu_1}\right\}$ and $\left\{V^{\mu_1\mu_2\mu_3}_a, H^a_{\mu_1\mu_2\mu_3\mu_4}\right\}$ respectively

$$\Phi^A \equiv \left\{\phi_a, V^{\mu_1\mu_2\mu_3}_a, H^a_{\mu_1\mu_2\mu_3\mu_4}, H^a_{\mu_1}\right\}_{a=\overline{1,A}}. \tag{2}$$

The collection indices are symbolized by letters from the beginning of Latin alphabet, $a, b, c, \ldots = \overline{1, A}$ ($A \geq 2$). In (1) the notation $[\mu_1 \ldots \mu_m]$ signifies full antisymmetry with respect to the indices between brackets, where we consider only independent terms, without normalization factors. It can be checked that a generating set of gauge symmetries of action (1) can be taken as

$$\begin{aligned} \delta_{\Omega^{\alpha_1}}\Phi^A \ : \ &\left\{\delta_{\Omega^{\alpha_1}}\phi_a = 0, \quad \delta_{\Omega^{\alpha_1}}V^{\mu_1\mu_2\mu_3}_a = \partial^{[\mu_1}\epsilon^{\mu_2\mu_3]}_a, \right. \\ &\left. \delta_{\Omega^{\alpha_1}}H^a_{\mu_1\ldots\mu_4} = -5\partial^\lambda\xi^a_{\lambda\mu_1\ldots\mu_4}, \quad \delta_{\Omega^{\alpha_1}}H^a_{\mu_1} = -2\partial^\lambda\xi^a_{\lambda\mu_1}\right\}. \end{aligned} \tag{3}$$

with the gauge parameters (consisting of fully antisymmetric arbitrary fields on the chosen $D = 8$ space-time) compactly denoted by

$$\Omega^{\alpha_1} \equiv \left\{\epsilon^{\mu_1\mu_2}_a, \xi^a_{\mu_1\ldots\mu_5}, \xi^a_{\mu_1\mu_2}\right\}. \tag{4}$$

The main features of the above generating set of gauge symmetries is that it is purely Abelian and off-shell reducible of order 6. Meanwhile, each sector of non-trivial gauge symmetries from (3) is independently reducible, with the partial reducibility order 2, 3, and respectively 6.

In a previous paper, [38], we constructed all consistent interactions that can be added to (1) via a deformation method based on a cohomological approach within the antifield-BRST framework. In this process, we also deformed the generating set (3), without changing the field spectrum number or the underlying number of independent gauge symmetries, into one specific to the interacting BF model. Consequently, we arrived at the deformed Lagrangian action

$$\bar{S}^{\mathrm{L}}_{\mathrm{BF}}\left[\Phi^{A}\right]=\int_{\mathbb{R}^{7,1}}\left[H^{a}_{\mu_1}\partial^{\mu_1}\phi_a+H^{a}_{\mu_1\mu_2\mu_3\mu_4}\left(\partial^{[\mu_1}V^{\mu_2\mu_3\mu_4]}_{a}\right.\right.$$
$$\left.\left.+\frac{\lambda}{2}\varepsilon^{\mu_1\cdots\mu_8}Z_{ab}\left(\phi\right)H^{b}_{\mu_5\mu_6\mu_7\mu_8}\right)\right]d^8x\equiv\int_{\mathbb{R}^{7,1}}\overline{\mathcal{L}}_{\mathrm{BF}}\left(\Phi^{A},\partial^{\mu}\Phi^{A}\right)d^8x \tag{5}$$

and the associated generating set of gauge symmetries

$$\begin{aligned}\bar{\delta}_{\Omega^{\alpha_1}}\Phi^{A}\quad:\quad&\left\{\bar{\delta}_{\Omega^{\alpha_1}}\phi_a=0,\quad\bar{\delta}_{\Omega^{\alpha_1}}V^{\mu_1\mu_2\mu_3}_{a}=\partial^{[\mu_1}\epsilon^{\mu_2\mu_3]}_{a}-\lambda\varepsilon^{\mu_1\cdots\mu_8}Z_{ab}\left(\phi\right)\xi^{b}_{\mu_4\cdots\mu_8},\right.\\&\bar{\delta}_{\Omega^{\alpha_1}}H^{a}_{\mu_1\mu_2\mu_3\mu_4}=-5\partial^{\rho}\xi^{a}_{\rho\mu_1\mu_2\mu_3\mu_4},\\&\left.\bar{\delta}_{\Omega^{\alpha_1}}H^{a}_{\mu_1}=-2\partial^{\rho}\xi^{a}_{\rho\mu_1}-5\lambda\varepsilon^{\rho_1\cdots\rho_8}\frac{\partial Z_{bc}\left(\phi\right)}{\partial\phi_a}H^{b}_{\rho_1\cdots\rho_4}\xi^{c}_{\mu_1\rho_5\cdots\rho_8}\right\}.\end{aligned} \tag{6}$$

In the above, λ stands for the coupling constant, $Z_{ab}\left(\phi\right)$ represent a set of symmetric functions depending on the undifferentiated BF scalar fields that parameterize the self-interactions among the BF fields and are otherwise arbitrary

$$Z_{ab}\left(\phi\right)=Z_{ba}\left(\phi\right), \tag{7}$$

and $\varepsilon^{\mu_1\cdots\mu_8}$ is the 'inverse' of the standard $D=8$ Levi–Civita symbol $\varepsilon_{\mu_1\cdots\mu_8}$, defined via $\varepsilon_{01\ldots7}=1$ plus full antisymmetry. The deformed generating set is endowed with a more complex structure compared to the free version: its gauge algebra is open, some of the reducibility functions are modified with respect to the free limit, the reducibility relations partly hold only on-shell, and the partial reducibility order of the gauge transformations of the 3-forms is raised by one unit (the overall reducibility order of the deformed set of gauge symmetries is of course preserved with respect to the free limit).

Our main aim is to investigate the Hamiltonian formalism adapted to constrained field theories in order to prove that (3) indeed represents an appropriate generating set of gauge symmetries of (1) and, in this context, also reveal the origins of its properties. Indeed, the deformation method employed in [38] guarantees that (6) is a deformed generating set of gauge symmetries of (5) as long as (3) plays the same role at the level of the free action (1), so one still needs to determine a generating set of gauge symmetries of the initial, uncoupled model.

3 Preliminaries

We adopt the following notations for the canonical momentum densities corresponding to the (real) field spectrum in (2)

$$\Pi_A\equiv\left\{p^{a},p^{a}_{\mu_1\mu_2\mu_3},P^{\mu_1\mu_2\mu_3\mu_4}_{a},P^{\mu_1}_{a}\right\}_{a=\overline{1,A}} \tag{8}$$

and work only with the independent time versus space components of both fields and momenta taking into account their antisymmetry (where appropriate), chosen as

$$V_a^{\mu_1\mu_2\mu_3} \quad \rightarrow \quad \left\{V_a^{0ij}, V_a^{ijk}\right\}_{i,j,k=\overline{1,7}}, \quad p^a_{\mu_1\mu_2\mu_3} \rightarrow \left\{p^a_{0ij}, p^a_{ijk}\right\}_{i,j,k=\overline{1,7}}, \tag{9}$$

$$H^a_{\mu_1\mu_2\mu_3\mu_4} \quad \rightarrow \quad \left\{H^a_{0ijk}, H^a_{ijkl}\right\}_{i,j,k,l=\overline{1,7}}, \quad P_a^{\mu_1\mu_2\mu_3\mu_4} \rightarrow \left\{P_a^{0ijk}, P_a^{ijkl}\right\}_{i,j,k,l=\overline{1,7}}. \tag{10}$$

The canonical momenta are defined in the standard fashion as the partial derivatives of the Lagrangian density with respect to the time derivatives of the corresponding fields

$$\Pi_B = \frac{\partial \mathcal{L}_{\mathrm{BF}}}{\partial \dot{\Phi}^B}\left(\Phi^A, \partial^\mu \Phi^A\right). \tag{11}$$

Since we work on the Minkowski space-time $\mathbb{R}^{7,1}$, where the covariant and contravariant time derivatives differ by a global (-1) factor, we need to set an additional convention in (11)

$$\dot{\Phi}^B \equiv \partial^0 \Phi^B, \tag{12}$$

such that (11) reads explicitly

$$\Pi_B = \frac{\partial \mathcal{L}_{\mathrm{BF}}}{\partial\left(\partial^0 \Phi^B\right)}\left(\Phi^A, \partial^\mu \Phi^A\right). \tag{13}$$

Next, we need to introduce one more key element, namely, define either the kinetic terms in the Hamiltonian density or the non-vanishing fundamental Poisson brackets among the fields and canonical momenta since they are correlated via a matrix inversion operation. We favor the kinetic term route and choose to take into account only the independent field/momenta components (9)–(10) with no normalization factors

$$\begin{aligned} \forall x \in \mathbb{R}^{7,1} \quad : \quad \left(\dot{\Phi}^A \Pi_A\right)(x) = {} & \left[\left(\partial^0 \phi_a\right) p^a + \left(\partial^0 V_a^{0ij}\right) p^a_{0ij} + \left(\partial^0 V_a^{ijk}\right) p^a_{ijk} \right. \\ & \left. + \left(\partial^0 H^a_{0ijk}\right) P_a^{0ijk} + \left(\partial^0 H^a_{ijkl}\right) P_a^{ijkl} + \left(\partial^0 H^a_\mu\right) P_a^\mu\right](x). \end{aligned} \tag{14}$$

The previous choice fully determines the expression of the non-vanishing (equal-time) fundamental Poisson brackets among the independent components of the fields and their conjugated momenta in condensed DeWitt notations like

$$\left[\Phi^A, \Pi_B\right]_t = \delta^A_B, \tag{15}$$

where each discrete index now automatically includes a continuous one and δ^A_B designates the true projector on the subspace of tensor indices A and B

$$\left[\Phi^A(x), \Pi_B(y)\right]_t = \delta^A_B \delta^7(\mathbf{x}-\mathbf{y}), \quad x = \{t, \mathbf{x}\}, \quad y = \{t, \mathbf{y}\}. \tag{16}$$

According to our choices and conventions, (15) take the concrete form

$$\left[\phi_a, p^b\right]_t = \delta^b_a, \quad \left[V_a^{0ij}, p^b_{0kl}\right]_t = \frac{1}{2}\delta^b_a \delta^{[i}_k \delta^{j]}_l, \quad \left[V_a^{ijk}, p^b_{lmn}\right]_t = \frac{1}{3!}\delta^b_a \delta^{[i}_l \delta^j_m \delta^{k]}_n, \tag{17}$$

$$\left[H^a_{0ijk}, P_b^{0lmn}\right]_t = \frac{1}{3!}\delta^a_b \delta^l_{[i} \delta^m_j \delta^n_{k]}, \quad \left[H^a_{ijkl}, P_b^{mnpq}\right]_t = \frac{1}{4!}\delta^a_b \delta^m_{[i} \delta^n_j \delta^p_k \delta^q_{l]}, \quad \left[H^a_\mu, P_b^\nu\right]_t = \delta^a_b \delta^\nu_\mu. \tag{18}$$

Now, we have at hand all major elements to develop the Hamiltonian analysis of the free BF model.

4 Primary constraints and canonical Hamiltonian

The canonical approach to the free BF model under investigation begins with the computation of momentum densities in (8) via the former set of relations in (11) based on the Lagrangian density $\mathcal{L}_{\mathrm{BF}}\left(\Phi^{A},\partial^{\mu}\Phi^{A}\right)$ in (1), which provides only primary constraints (no time derivatives of the fields may be expressed from (11) in terms of momentum densities and possibly their spatial derivatives since the rank of the Hessian density associated with $\mathcal{L}_{\mathrm{BF}}\left(\Phi^{A},\partial^{\mu}\Phi^{A}\right)$ is equal to zero, so the free model under study is degenerate), whose surface is defined by

$$\Sigma_{\mathrm{prim}}:\left\{\chi_{a}\equiv P_{a}^{0},\quad \tilde{\chi}^{a}\equiv p^{a}-H_{0}^{a},\quad \chi_{a}^{ijk}\equiv P_{a}^{0ijk},\quad \tilde{\chi}_{ijk}^{a}\equiv p_{ijk}^{a}-4H_{0ijk}^{a},\right. \tag{19}$$

$$\left.G_{a}^{i}\equiv P_{a}^{i},\quad G_{ij}^{a}\equiv p_{0ij}^{a},\quad G_{a}^{ijkl}\equiv P_{a}^{ijkl}\right\}\approx 0, \tag{20}$$

where '$\approx$' is the weak equality symbol, signifying equalities that hold on Σ_{prim}, by contrast to '$=$', which we keep for strong equalities (that take place everywhere on the phase-space). The primary constraint surface is regular, being defined in this case only by independent functions. The canonical Hamiltonian of this model is defined on Σ_{prim} in a standard manner via the Legendre transform of the initial Lagrangian function

$$H_{t}=\int_{\mathbb{R}^{7}}\left[\left(\dot{\Phi}^{A}\Pi_{A}\right)(t,\mathbf{x})-\mathcal{L}_{\mathrm{BF}}\left(\Phi^{A}\left(t,\mathbf{x}\right),\partial^{\mu}\Phi^{A}\left(t,\mathbf{x}\right)\right)\right]d\mathbf{x}\Big|_{\Sigma_{\mathrm{prim}}}. \tag{21}$$

Based on our convention (14) and the expression of $\mathcal{L}_{\mathrm{BF}}$ from (1), we infer

$$H_{t}\approx-\int_{\mathbb{R}^{7}}\left(H_{i}^{a}\partial^{i}\phi_{a}+12V_{a}^{0ij}\partial^{k}H_{0kij}^{a}+H_{ijkl}^{a}\partial^{[i}V_{a}^{jkl]}\right)d\mathbf{x}, \tag{22}$$

being understood that each term from the density Hamiltonian in the right-hand side of (22) is taken at $x=(t,\mathbf{x})\in\mathbb{R}^{1,7}$.

We associate Lagrange multiplier densities with all the primary constraint functions (that define Σ_{prim})

$$\chi_{a}\rightarrow u^{a},\quad \tilde{\chi}^{a}\rightarrow\tilde{u}_{a},\quad \chi_{a}^{ijk}\rightarrow u_{ijk}^{a},\quad \tilde{\chi}_{ijk}^{a}\rightarrow\tilde{u}_{a}^{ijk}, \tag{23}$$

$$G_{a}^{i}\rightarrow v_{i}^{a},\quad G_{ij}^{a}\rightarrow v_{a}^{ij},\quad G_{a}^{ijkl}\rightarrow v_{ijkl}^{a}. \tag{24}$$

For further convenience, we split the BF field/canonical momenta spectra (2) and (8) into two complementary, independent subsets

$$\Phi^{A}=\left\{\Phi^{A_{0}},\Phi^{\tilde{A}_{0}}\right\},\Phi^{A_{0}}=\left\{V_{a}^{0ij},H_{ijkl}^{a},H_{i}^{a}\right\}_{a=\overline{1,A}},\Phi^{\tilde{A}_{0}}=\left\{\phi_{a},V_{a}^{ijk},H_{0ijk}^{a},H_{0}^{a}\right\}_{a=\overline{1,A}}, \tag{25}$$

$$\Pi_{A}=\left\{\Pi_{A_{0}},\Pi_{\tilde{A}_{0}}\right\},\Pi_{A_{0}}=\left\{p_{0ij}^{a},P_{a}^{ijkl},P_{a}^{i}\right\}_{a=\overline{1,A}},\Pi_{\tilde{A}_{0}}=\left\{p^{a},p_{ijk}^{a},P_{a}^{0ijk},P_{a}^{0}\right\}_{a=\overline{1,A}}, \tag{26}$$

such that the non-vanishing fundamental Poisson brackets are fully decoupled between them, meaning that

$$\left[\Phi^{A_{0}},\Pi_{B_{0}}\right]_{t}=\delta_{B_{0}}^{A_{0}},\quad \left[\Phi^{\tilde{A}_{0}},\Pi_{\tilde{B}_{0}}\right]_{t}=\delta_{\tilde{B}_{0}}^{\tilde{A}_{0}}, \tag{27}$$

while

$$\left[\Phi^{A_{0}},\Pi_{\tilde{B}_{0}}\right]_{t}=0=\left[\Phi^{\tilde{A}_{0}},\Pi_{B_{0}}\right]_{t}. \tag{28}$$

Consequently, we can express Σ_{prim} and H_{t} as

$$\Sigma_{\mathrm{prim}}\quad:\quad\left\{\chi_{\tilde{A}_{0}}\equiv\Pi_{\tilde{A}_{0}}-f_{\tilde{A}_{0}}^{\mathrm{lin}}\left(\Phi^{\tilde{B}_{0}}\right),\quad G_{A_{0}}\equiv\Pi_{A_{0}}\right\}\approx 0, \tag{29}$$

$$H_t \approx -\int_{\mathbb{R}^7} \Phi^{A_0} \tilde{G}_{A_0} \left(\left[\Phi^{\tilde{B}_0} \right]_1 \right) d\mathbf{x} \equiv \int_{\mathbb{R}^7} \mathcal{H}(t, \mathbf{x})\, d\mathbf{x} \tag{30}$$

and rewrite the Lagrange multipliers (23) and (24) in a compact manner like

$$\chi_{\tilde{A}_0} \to u^{\tilde{A}_0}, \quad G_{A_0} \to v^{A_0}. \tag{31}$$

The notation $f^{\text{lin}}_{\tilde{A}_0}\left(\Phi^{\tilde{B}_0}\right)$ signify some functions that are linear in the fields $\Phi^{\tilde{B}_0}$, while $\left[\Phi^{\tilde{B}_0}\right]_1$ means a (local) dependence of $\Phi^{\tilde{B}_0}$ and their spatial first-order derivatives $\partial^i \Phi^{\tilde{B}_0}$.

All Lagrange multipliers are at present independent of the fields and canonical momentum densities (so their Poisson brackets with all these variables strongly vanish), being viewed as some extra fields defined on the space-time manifold $\mathbb{R}^{1,7}$, all real-valued and inheriting the antisymmetry properties of the corresponding primary constraint functions wherever applicable. Their role here is to (locally) instate the invertibility of the Legendre transformation between the space $\left(\Phi^A(x), \partial^0 \Phi^A(x)\right)$ and the surface Σ_{prim} embedded now in $\left(\Phi^A(x), \Pi_A(x), u^{\tilde{A}_0}(x), v^{A_0}(x)\right)$.

The Hamiltonian dynamics (equations of motion) of this model follows from the variational principle

$$\delta \left[\int_{\mathbb{R}^{7,1}} \left(\dot{\Phi}^A \Pi_A - \mathcal{H} - u^{\tilde{B}_0} \chi_{\tilde{B}_0} - v^{B_0} G_{B_0} \right) d^8 x \right] = 0$$

and can be written as (again in DeWitt condensed notations)

$$\dot{F} = [F, H]_t + u^{\tilde{B}_0} \left[F, \chi_{\tilde{B}_0}\right]_t + v^{B_0} \left[F, G_{B_0}\right]_t, \tag{32}$$

with F an arbitrary function of the canonical fields/momenta. Explicitly, (32) is nothing but

$$\begin{aligned} \partial^0 F(t, \mathbf{x}) = \int_{\mathbb{R}^7} \Big(& [F(t, \mathbf{x}), \mathcal{H}(t, \mathbf{y})] + u^{\tilde{B}_0}(t, \mathbf{y}) \left[F(t, \mathbf{x}), \chi_{\tilde{B}_0}(t, \mathbf{y})\right] \\ & + v^{B_0}(t, \mathbf{y}) \left[F(t, \mathbf{x}), G_{B_0}(t, \mathbf{y})\right] \Big) d\mathbf{y}. \end{aligned} \tag{33}$$

5 Dirac–Bergmann algorithm

Next, we apply the Dirac–Bergmann algorithm in order to generate all the other constraints to which the free model under investigation might be subject.

In the first step of this algorithm we ask that the primary constraints are preserved in time, so we use (32) and successively replace F with every primary constraint function in (29), asking that

$$\dot{\chi}_{\tilde{A}_0} \approx 0, \quad \dot{G}_{A_0} \approx 0, \tag{34}$$

which gives a set of equations known as the consistency conditions related to the primary constraints. Based on (33), after some simple computation we find that the previous consistency conditions lead to two different outcomes. Thus, the former subset in (34) fixes all the Lagrange multiplier densities $u^{\tilde{A}_0}$ to

$$u^{\tilde{A}_0} : \left\{ u^a \approx -\partial^m H^a_m, \quad \tilde{u}_a \approx 0, \quad u^a_{ijk} \approx -\partial^m H^a_{mijk}, \quad \tilde{u}^{ijk}_a \approx V^{0[ij,k]}_a \right\}, \tag{35}$$

where we used the notation $f^{,k}(x) = \partial f(x)/\partial x_k$ (in our case $f^{,k}(x) = f_{,k}(x)$ due to the chosen positively defined metric in the spatial index sector). By contrast, the latter set of consistency conditions from (34) generates a new set of constraints, independent of (29), known as the secondary constraints of the analyzed model

$$\Sigma_{\text{sec}} : \tilde{G}_{A_0}\left(\left[\Phi^{\tilde{B}_0}\right]_1\right) = \left\{\tilde{G}^i_a \equiv \partial^i \phi_a, \quad \tilde{G}^a_{ij} \equiv 12\partial^m H^a_{0mij}, \quad \tilde{G}^{ijkl}_a \equiv \partial^{[i} V^{jkl]}_a\right\} \approx 0, \quad (36)$$

where the functions $\tilde{G}_{A_0}$ are precisely those present in the canonical Hamiltonian (30). Although we use the same generic weak equality symbol in both (29) and (36), it is obvious that $\tilde{G}_{A_0}$ are not vanishing on Σ_{prim}. This unified notation emphasizes that at each $x \in \mathbb{R}^{7,1}$ the model under consideration exhibits now a constraint hypersurface $\Sigma_{\text{total}} = \Sigma_{\text{prim}} \cap \Sigma_{\text{sec}}$, defined by (29) and (36).

These different outcomes related to the two subsets of consistency conditions are due to the complementary properties of the primary constraint subsets that define (29): while $\chi_{\tilde{A}_0}$ are second-class

$$C_{\tilde{A}_0\tilde{B}_0} \equiv \left[\chi_{\tilde{A}_0}, \chi_{\tilde{B}_0}\right]_t = \begin{pmatrix} \mathbf{0} & \delta^b_a & \mathbf{0} & \mathbf{0} \\ -\delta^a_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{4}{3!}\delta^b_a \delta^{[i}_{i'} \delta^j_{j'} \delta^{k]}_{k'} \\ \mathbf{0} & \mathbf{0} & -\frac{4}{3!}\delta^a_b \delta^{[i'}_i \delta^{j'}_j \delta^{k']}_k & \mathbf{0} \end{pmatrix}, \quad (37)$$

G_{A_0} are first-class and Abelian

$$\left[G_{A_0}, G_{B_0}\right]_t = \left[G_{A_0}, \chi_{\tilde{B}_0}\right]_t = 0. \quad (38)$$

In (37) and (38) we kept the original order of constraint functions, provided in (19) and (19). The presence of the global factor $\delta^7(\mathbf{x} - \mathbf{y})$ in the right-hand side (37) is understood in view of condensed DeWitt notations.

Due to the appearance of secondary constraints, the Dirac–Bergmann algorithm unfolds a second step, where we require the consistency of (36)

$$\dot{\tilde{G}}_{A_0} \approx 0. \quad (39)$$

We cannot stress enough that $\dot{\tilde{G}}_{A_0}$ in (39) is still of the form (32) in the sense that no supplementary Lagrange multipliers are added with respect to (36), but, subsequently to the necessary computation, we replace the Lagrange multipliers corresponding to $\chi_{\tilde{B}_0}$ with their expressions fixed in the first step (namely, (35) with $\tilde{A}_0 \longrightarrow \tilde{B}_0$). In this manner, we arrive at

$$\dot{\tilde{G}}^{\,i}_a \approx 0 \Leftrightarrow \partial^i \tilde{u}_a \approx 0, \quad \dot{\tilde{G}}^{\,a}_{ij} \approx 0 \Leftrightarrow 12\partial^n u^a_{nij} \approx 0, \quad \dot{\tilde{G}}^{\,ijkl}_a \approx 0 \Leftrightarrow \partial^{[i} \tilde{u}^{jkl]}_a \approx 0 \quad (40)$$

and observe that all these conditions are identically satisfied on the solutions (35). As a consequence, the Dirac–Bergmann stops after its second step and brings no new information compared to the first step (no new constraints emerged and no additional Lagrange multipliers were fixed).

This conclusion seems counterintuitive at the first sight since (40) signalize that there are some non-vanishing Poisson brackets among the secondary constraint functions $\tilde{G}_{A_0}$ and the primary constraint functions $\chi_{\tilde{B}_0}$, namely,

$$\left[\tilde{G}_{A_0}, \chi_{\tilde{B}_0}\right]_t = \begin{pmatrix} \mathbf{0} & \delta^b_a \partial^i_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2\delta^a_b \partial^{[k}_y \delta^l_i \delta^{m]}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{3!}\delta^b_a \partial^{[i}_x \delta^j_m \delta^k_n \delta^{l]}_p \end{pmatrix}, \quad (41)$$

so we expect that $\tilde{G}_{A_0}$ include some second-class constraints that extend the primary ones, $\chi_{\tilde{A}_0}$. We omitted again the global factor $\delta^7(\mathbf{x}-\mathbf{y})$ on which the right-hand side of (41) acts, but it must be taken into consideration when various computations are performed. On the other hand, no other Lagrange multipliers (i.e., among v^{B_0}) are being fixed since the Poisson brackets between $\tilde{G}_{A_0}$ and G_{B_0} vanish strongly (they involve only variables that are not canonically conjugated ones to the others)

$$\left[\tilde{G}_{A_0}, G_{B_0}\right]_t \equiv \left[\tilde{G}_{A_0}\left(\left[\Phi^{\tilde{C}_0}\right]_1\right), \Pi_{A_0}\right]_t = \mathbf{0}. \tag{42}$$

Moreover, (40) are identically satisfied on the initial solutions (35), but $\tilde{G}_{A_0}$ and $\chi_{\tilde{B}_0}$ are not cross-reducible ($\chi_{\tilde{B}_0}$ are independent among themselves and with respect to $\tilde{G}_{A_0}$, while $\tilde{G}_{A_0}$ are indeed reducible, but only among themselves).

6 Separation of constraints into first- and second-class subsets

The eccentric situation exposed in the final part of the previous section is explained by a less common aspect of degenerate theories, namely, that the entire set of constraints (primary and secondary) is not yet fully separated into its first- and second-class counterparts.

More precisely, due to (38) and (42) (their Poisson brackets with all the constraint functions from $\Sigma_{\text{prim}} \cap \Sigma_{\text{sec}}$ vanish strongly), it is obvious that the primary constraints G_{A_0} are already separated, being purely first-class with respect to all the constraints from the theory, namely (29) and (36). Therefore, we only need to perform the separation procedure with respect to the constraint subset $\left\{\chi_{\tilde{A}_0}, \tilde{G}_{A_0}\right\} \approx 0$. On behalf of results (37) and (41) completed with the obvious ones

$$\left[\tilde{G}_{A_0}, \tilde{G}_{B_0}\right]_t \equiv \left[\tilde{G}_{A_0}\left(\left[\Phi^{\tilde{C}_0}\right]_1\right), \tilde{G}_{B_0}\left(\left[\Phi^{\tilde{D}_0}\right]_1\right)\right]_t = \mathbf{0}, \tag{43}$$

we find that the matrix of the Poisson brackets among the constraint functions $\left\{\chi_{\tilde{A}_0}, \tilde{G}_{A_0}\right\}$ reads

$$\begin{pmatrix} \left[\chi_{\tilde{A}_0}, \chi_{\tilde{B}_0}\right]_t & \left[\chi_{\tilde{A}_0}, \tilde{G}_{B_0}\right]_t \\ \left[\tilde{G}_{A_0}, \chi_{\tilde{B}_0}\right]_t & \left[\tilde{G}_{A_0}, \tilde{G}_{B_0}\right]_t \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{0} & \delta_a^b & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\delta_b^a & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\delta_b^a \partial_y^m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{4}{3!}\delta_a^b \delta_m^{[i} \delta_n^j \delta_p^{k]} & \mathbf{0} & 2\delta_a^b \partial_x^{[i} \delta_m^j \delta_n^{k]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{4}{3!}\delta_b^a \delta_i^{[m} \delta_j^n \delta_k^{p]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{3!}\delta_b^a \partial_y^{[m} \delta_i^n \delta_j^p \delta_k^{q]} \\ \mathbf{0} & \delta_a^b \partial_x^i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2\delta_b^a \partial_y^{[m} \delta_i^n \delta_j^{p]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{3!}\delta_a^b \partial_x^{[i} \delta_m^j \delta_n^k \delta_p^{l]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{44}$$

where the global factor $\delta^7(\mathbf{x}-\mathbf{y})$ in the right-hand side of the above formula is understood. The previous matrix (in general restricted to the entire constraint surface, but this is irrelevant here due to the functional linearity of all constraint functions in the canonical variables, which implies that all its elements are independent of the canonical variables) possesses a set of non-trivial null vectors (its rank is locally equal to the number of independent components of $\chi_{\tilde{A}_0}$)

$$\begin{pmatrix} \left[\chi_{\tilde{A}_0}, \chi_{\tilde{B}_0}\right]_t & \left[\chi_{\tilde{A}_0}, \tilde{G}_{B_0}\right]_t \\ \left[\tilde{G}_{A_0}, \chi_{\tilde{B}_0}\right]_t & \left[\tilde{G}_{A_0}, \tilde{G}_{B_0}\right]_t \end{pmatrix} \begin{pmatrix} \lambda^{\tilde{B}_0}_{C_0} \\ \lambda^{B_0}_{C_0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{45}$$

(again in DeWitt notations), provided by

$$\begin{pmatrix} \lambda^{\tilde{B}_0}_{C_0} \\ \lambda^{B_0}_{C_0} \end{pmatrix} = \begin{pmatrix} -\delta^b_c \partial^{m'}_z & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{4!}\delta^b_c \partial^{[m'}_z \delta^{n'}_m \delta^{p'}_n \delta^{q']}_p \\ \mathbf{0} & -\frac{1}{2}\delta^c_b \partial^{[m}_y \delta^n_{m'} \delta^{p]}_{n'} & \mathbf{0} \\ \delta^b_c \delta^{m'}_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\delta^c_b \delta^{[m}_{m'} \delta^{n]}_{n'} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{4!}\delta^b_c \delta^{[m'}_m \delta^{n'}_n \delta^{p'}_p \delta^{q']}_q \end{pmatrix}. \tag{46}$$

In turn, this set of null vectors generates the true first-class constraint subset of the model under study among $\left\{\chi_{\tilde{B}_0}, \tilde{G}_{B_0}\right\}$, of course without affecting the overall surface (defined by (19), (20), and (36)), via

$$\tilde{G}_{A_0} \to \tilde{G}'_{A_0} \equiv \begin{pmatrix} \chi_{\tilde{B}_0} & \tilde{G}_{B_0} \end{pmatrix} \begin{pmatrix} \lambda^{\tilde{B}_0}_{A_0} \\ \lambda^{B_0}_{A_0} \end{pmatrix}, \tag{47}$$

such that only the secondary constraints (40) are modified. Making the remaining computations in (47) via (46) (where we make the replacement $C_0, z \to A_0, x$), we finally determine the searched for first-class constraint subset

$$\tilde{G}_{A_0} \quad \to \quad \tilde{G}'_{A_0} : \left\{ \tilde{G}'^i_a = \tilde{G}^i_a - \partial^i \chi_a \equiv \partial^i \left(\phi_a - P^0_a\right), \; \tilde{G}'^a_{ij} = \tilde{G}^a_{ij} + 3\partial^m \tilde{\chi}^a_{mij} \right. \tag{48}$$

$$\left. \equiv 3\partial^m p^a_{mij}, \; \tilde{G}'^{ijkl}_a = \tilde{G}^{ijkl}_a - \frac{1}{4}\partial^{[i}\chi^{jkl]}_a \equiv -\left(V^{[ijk,l]}_a - \frac{1}{4}P^{0[ijk,l]}_a\right) \right\} \approx 0. \tag{49}$$

Naturally, the primary ones keep their initial expressions, namely, (19) and (20).

Thus, we managed to construct an equivalent description of the full constraint hypersurface Σ_{total} in terms of some (now) fully separated first- and respectively second-class sectors

$$\Sigma_{\text{total}} : \left\{ \chi_a \equiv P^0_a, \quad \tilde{\chi}^a \equiv p^a - H^a_0, \quad \chi^{ijk}_a \equiv P^{0ijk}_a, \quad \tilde{\chi}^a_{ijk} \equiv p^a_{ijk} - 4H^a_{0ijk}, \right. \tag{50}$$

$$G^i_a \equiv P^i_a, \quad G^a_{ij} \equiv p^a_{0ij}, \quad G^{ijkl}_a \equiv P^{ijkl}_a, \tag{51}$$

$$\left. \tilde{G}'^i_a \equiv \partial^i \left(\phi_a - P^0_a\right), \; \tilde{G}'^a_{ij} \equiv 3\partial^m p^a_{mij}, \; \tilde{G}'^{ijkl}_a \equiv -\left(V^{[ijk,l]}_a - \frac{1}{4}P^{0[ijk,l]}_a\right) \right\} \approx 0. \tag{52}$$

We notice that (48) and (49) depend now, via the former relations in (29) and also (36), on some linear combinations of the spatial derivatives of both fields $\Phi^{\tilde{C}_0}$ and their conjugated momentum densities $\Pi_{\tilde{C}_0}$, by contrast to the initial constraint functions $\tilde{G}_{A_0}$, which involved only the fields $\Phi^{\tilde{C}_0}$. Written alternatively, in terms of the complementary field/momentum spectra (25) and (26), the overall constraint hypersurface equations are expressed like

$$\Sigma_{\text{total}} : \left\{ \chi_{\tilde{A}_0} \equiv \Pi_{\tilde{A}_0} - f^{\text{lin}}_{\tilde{A}_0}\left(\Phi^{\tilde{C}_0}\right), G_{A_0} \equiv \Pi_{A_0}, \tilde{G}'_{A_0} \equiv g^{\text{lin}}_{A_0}\left(\partial^m \Phi^{\tilde{C}_0}, \partial^m \Pi_{\tilde{C}_0}\right) \right\} \approx 0. \tag{53}$$

7 Properties of constraints. First-class Hamiltonian

Let us take a closer look at the properties of constraints (53) that characterize the $D = 8$ free BF model under study. On the one hand, the subset defined by $\chi_{\tilde{A}_0} \approx 0$ is second-class and irreducible since the matrix of their Poisson brackets, given in (37), is invertible, with the elements of the inverse provided by

$$C^{\tilde{B}_0 \tilde{C}_0} = \begin{pmatrix} \mathbf{0} & -\delta_c^b & \mathbf{0} & \mathbf{0} \\ \delta_b^c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{4!}\delta_c^b \delta_{i'}^{[i''} \delta_{j'}^{j''} \delta_{k'}^{k'']} \\ \mathbf{0} & \mathbf{0} & \frac{1}{4!}\delta_b^c \delta_{i''}^{[i'} \delta_{j''}^{j'} \delta_{k''}^{k']} & \mathbf{0} \end{pmatrix}, \tag{54}$$

where the global factor $\delta^7 (\mathbf{y} - \mathbf{z})$ in the right-hand side of the previous formula is understood. On the other hand, the subset defined by $G_{A_0} \approx 0$ and $\tilde{G}'_{A_0} \approx 0$ is first-class and Abelian since

$$[G_{A_0}, G_{B_0}]_t = \left[G_{A_0}, \tilde{G}'_{B_0}\right]_t = \left[\tilde{G}'_{A_0}, \tilde{G}'_{B_0}\right]_t = \mathbf{0}, \tag{55}$$

$$\left[G_{A_0}, \chi_{\tilde{B}_0}\right]_t = \left[\tilde{G}'_{A_0}, \chi_{\tilde{B}_0}\right] = \mathbf{0}. \tag{56}$$

We stress that in general the Poisson bracket between an arbitrary first- and any second-class constraint function does not need to vanish strongly, but weakly, on the entire constraint hypersurface (of both first- and second-class constraints), while the Poisson bracket between any two first-class constraint functions also vanishes weakly, but strictly on the first-class constraint hypersurface.

It is also important to mention that $G_{A_0} \approx 0$ are irreducible, while $\tilde{G}'_{A_0} \approx 0$ are off-shell reducible of order 6. The reducibility relations are completely separated among the three first-class constraint function sets in (52), being expressed in terms of three independent relation towers, whose right-hand sides vanish strongly (hold off-shell) at each order strictly higher than one (the reducibility relations of order one involve the first-class constraint functions themselves). In more detail, the subset $\tilde{G}'^{i}_{a}$ is reducible of maximum order, equal to 6, $\tilde{G}'^{a}_{ij}$ displays the lowest partial reducibility order, equal to 2, while $\tilde{G}'^{ijkl}_{a}$ exhibits a partial reducibility order equal to 3. The off-shell feature of the reducibility relations associated with the first-class constraint functions is again a mark of the absence of self-interactions.

In view of the subsequent analysis, we need the first-order reducibility functions $\tilde{Z}'^{A_0}{}_{A_1}$ corresponding to $\tilde{G}'_{A_0}$ (written also in DeWitt notations), involved in the first-order reducibility relations $\tilde{G}'_{A_0} \tilde{Z}'^{A_0}{}_{A_1} = \mathbf{0}$

$$\tilde{Z}'^{A_0}{}_{A_1} = \begin{pmatrix} \delta_b^a \partial_{[y}^{[m} \delta_{i]}^{n]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\delta_a^b \partial_x^{[i} \delta_m^{j]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{4!}\delta_b^a \partial_y^{[m} \delta_i^n \delta_j^p \delta_k^q \delta_{l]}^{r]} \end{pmatrix}. \tag{57}$$

Another key ingredient of the Hamiltonian approach to degenerate first-class theories is the construction of a first-class Hamiltonian (a first-class function with respect to all the constraint functions of the theory, whose initial piece is precisely the canonical Hamiltonian), H'. It is interesting to notice that we already have at hand all the elements required for its structure. Indeed, during the first step of the Dirac–Bergmann algorithm we fixed all the Lagrange multiplier densities $u^{\tilde{A}_0}$ corresponding to the second-class constraints

$\chi_{\tilde{A}_0} \approx 0$ (see (35)). Since there are no further second-class constraints in the theory, it is easy to see that a possible choice is

$$H'_t = H_t + \int_{\mathbb{R}^7} u^{\tilde{A}_0} \chi_{\tilde{A}_0} \left(\Pi_{\tilde{C}_0}, \Phi^{\tilde{C}_0} \right). \tag{58}$$

Indeed, using (30) and (35), we find that (58) can be expressed like

$$H'_t = -\int_{\mathbb{R}^7} \left[H^a_i \left(\tilde{G}^i_a - \partial^i \chi_a \right) + V^{0ij}_a \left(\tilde{G}^a_{ij} + 3\partial^k \tilde{\chi}^a_{kij} \right) + H^a_{ijkl} \left(\tilde{G}^{ijkl}_a - \frac{1}{4} \partial^{[i} \chi^{jkl]}_a \right) \right] d\mathbf{x}. \tag{59}$$

By means of (48) and (49) it follows that

$$H'_t = -\int_{\mathbb{R}^7} \left(H^a_i \tilde{G}'^i_a + V^{0ij}_a \tilde{G}'^a_{ij} + H^a_{ijkl} \tilde{G}'^{ijkl}_a \right) d\mathbf{x}, \tag{60}$$

which can be finally rendered in a compact manner in terms of $\tilde{G}'_{A_0}$ and the various field/momenta subsets from (25) and (26) as

$$H'_t = -\int_{\mathbb{R}^7} \Phi^{A_0} \tilde{G}'_{A_0} \left(\partial^m \Pi_{\tilde{C}_0}, \partial^m \Phi^{\tilde{C}_0} \right) d\mathbf{x}. \tag{61}$$

Now, taking into account (53), we directly infer that

$$\left[\Phi^{A_0}, \chi_{\tilde{B}_0} \left(\Pi_{\tilde{C}_0}, \Phi^{\tilde{C}_0} \right) \right]_t = \mathbf{0}, \quad \left[\Phi^{A_0}, G_{B_0} \right]_t \equiv \left[\Phi^{A_0}, \Pi_{B_0} \right]_t = \delta^{A_0}_{B_0}, \tag{62}$$

$$\left[\Phi^{A_0}, \tilde{G}'_{B_0} \left(\partial^m \Pi_{\tilde{C}_0}, \partial^m \Phi^{\tilde{C}_0} \right) \right]_t = \mathbf{0}, \tag{63}$$

which further combined with the relations from (55) and (55) that involve $\tilde{G}'_{A_0}$ allow us to write that

$$\left[H', \chi_{\tilde{B}_0} \right]_t = 0, \quad \left[H', G_{B_0} \right]_t = -\tilde{G}'_{B_0}, \quad \left[H', \tilde{G}'_{B_0} \right]_t = 0, \tag{64}$$

so H'_t may indeed be taken as a first-class Hamiltonian with respect to the full constraint set (53).

8 Elimination of second-class constraints. Dirac bracket

Once we separated the entire constraint set of the free BF model into its true first- and second-class subsets, we can eliminate the second-class constraints from the theory via substituting the weak equality with the strong one in (50)

$$\chi_{\tilde{A}_0} \equiv \left\{ \chi_a, \ \tilde{\chi}^a, \ \chi^{ijk}_a, \ \tilde{\chi}^a_{ijk} \right\} = 0, \tag{65}$$

which is equivalent to

$$P^0_a = 0, \quad p^a = H^a_0, \quad P^{0ijk}_a = 0, \quad p^a_{ijk} = 4H^a_{0ijk}. \tag{66}$$

In this way, we pass from the initial phase-space to the reduced one, (locally) parameterized, according to our choice (66), by

$$\left\{ \phi_a, V^{0ij}_a, V^{ijk}_a, H^a_{0ijk}, H^a_{ijkl}, H^a_0, H^a_i, p^a_{0ij}, P^{ijkl}_a, P^i_a \right\}. \tag{67}$$

The induced bracket on the reduced phase-space is nothing but the bracket induced on the second-class constraint surface, which is precisely the Dirac bracket associated with the second-class constraints (also in DeWitt notations)

$$\left.[F,G]_t\right|_{\chi_{\tilde{A}_0}=0} = [F,G]_t^* \equiv [F,G]_t - \left[F,\chi_{\tilde{A}_0}\right]_t C^{\tilde{A}_0\tilde{B}_0}\left[\chi_{\tilde{B}_0},G\right]_t, \tag{68}$$

with $C^{\tilde{A}_0\tilde{B}_0}$ the inverse of the matrix of the Poisson brackets among the second-class constraints, (37), provided in (54). Systematically applying (68), we generate the non-vanishing Dirac brackets among the reduced phase-space variables (67) in the form

$$\left[\phi_a, H_0^b\right]_t^* = \delta_a^b, \quad \left[V_a^{0ij}, p_{0mn}^b\right]_t^* = \left[V_a^{0ij}, p_{0mn}^b\right]_t = \frac{1}{2}\delta_a^b\delta_m^{[i}\delta_n^{j]}, \tag{69}$$

$$\left[V_a^{ijk}, H_{0mnp}^b\right]_t^* = \frac{1}{4!}\delta_a^b\delta_m^{[i}\delta_n^j\delta_p^{k]}, \quad \left[H_{ijkl}^a, P_b^{mnpq}\right]_t^* = \left[H_{ijkl}^a, P_b^{mnpq}\right]_t \tag{70}$$

$$= \frac{1}{4!}\delta_b^a\delta_{[i}^m\delta_j^n\delta_k^p\delta_{l]}^q, \quad \left[H_i^a, P_b^m\right]_t^* = \left[H_i^a, P_b^m\right]_t = \delta_b^a\delta_i^m. \tag{71}$$

The elimination of second-class constraints leaves us with a purely first-class BF model (in the Dirac bracket) obtained from (51) and (52) where we set (65)

$$\Sigma_{\text{first}} \quad : \quad \left\{G_{A_0}, \tilde{G}_{A_0}\right\} \approx 0, \tag{72}$$

$$G_{A_0} \quad : \quad \left\{G_a^i \equiv P_a^i, \quad G_{ij}^a \equiv p_{0ij}^a, \quad G_a^{ijkl} \equiv P_a^{ijkl}\right\}, \tag{73}$$

$$\tilde{G}'_{A_0} \quad \to \quad \tilde{G}_{A_0} : \left\{\tilde{G}_a^i \equiv \partial^i\phi_a, \quad \tilde{G}_{ij}^a \equiv 12\partial^k H_{0kij}^a, \quad \tilde{G}_a^{ijkl} \equiv \partial^{[i}V_a^{jkl]}\right\}. \tag{74}$$

There is no surprise that (73) and (74) are reverted to the initial primary and secondary constraints of the free model under consideration, (20) and (36), respectively. Indeed, the functions G_{A_0} do not depend on any of the fields/momentum densities involved in the $\chi_{\tilde{A}_0}$'s, while (52) reduce to (36) via setting (65) in (48) and (49). Along the same line, the first-class Hamiltonian (61) is also reverted to the original canonical Hamiltonian (30) due to its alternative expression (59) including the second-class constraint functions $\chi_{\tilde{A}_0}$

$$H'_t \to H_t = -\int_{\mathbb{R}^7}\Phi^{A_0}\tilde{G}_{A_0}d\mathbf{x} \equiv -\int_{\mathbb{R}^7}\left(H_i^a\partial^i\phi_a + 12V_a^{0ij}\partial^k H_{0kij}^a + H_{ijkl}^a\partial^{[i}V_a^{jkl]}\right)d\mathbf{x}. \tag{75}$$

Nevertheless, the main difference between the initial setting, with the second-class constraints unsolved, and the present one is that (72) and (75) are all true first-class functions with respect to the Dirac bracket (68). This is automatically enforced by the well-known property of the Dirac bracket according to which any first-class function with respect to a given constraint set in the Poisson bracket remains so after passing to the Dirac one. More precisely, the first-class constraint set (72) remains Abelian with respect to the Dirac bracket

$$\left[G_{A_0}, G_{B_0}\right]_t^* = \mathbf{0}, \quad \left[G_{A_0}, \tilde{G}_{B_0}\right]_t^* = \mathbf{0}, \quad \left[\tilde{G}_{A_0}, \tilde{G}_{B_0}\right]_t^* = \mathbf{0}, \tag{76}$$

with G_{A_0} irreducible and $\tilde{G}_{A_0}$ off-shell reducible of order 6. The first-order reducibility functions corresponding to the latter constraint set in (72) are still given by (57)

$$\tilde{G}_{A_0}\tilde{Z}^{A_0}{}_{A_1} = \mathbf{0}, \quad \tilde{Z}^{A_0}{}_{A_1} = \tilde{Z}'^{A_0}{}_{A_1} \equiv \begin{pmatrix} \delta_b^a\partial_{[y}^{[m}\delta_{i]}^{n]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\delta_a^b\partial_x^{[i}\delta_m^{j]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{4!}\delta_b^a\partial_y^{[m}\delta_i^n\delta_j^p\delta_k^q\delta_{l]}^{r]} \end{pmatrix}. \tag{77}$$

The first-order behavior of (75) with respect to the overall constraint set (72) is stipulated within the next formulas

$$[H, G_{A_0}]_t^* = -\tilde{G}_{A_0}, \quad \left[H, \tilde{G}_{A_0}\right]_t^* = \mathbf{0}. \tag{78}$$

Although in general the distinction between primary and secondary constraints may be lost in the process of eliminating the second-class subset, especially when a separation process into first- and second-class constraint subsets is previously required and implies a mixing among primary and secondary ones, this is not the case here. Indeed, on the one hand $G_{A_0} \approx 0$ were not affected by the separation procedure and were deduced using only the definitions of canonical momentum densities (see the beginning of Section 4) and not the equations of motion, so we can correctly still view them as the primary (first-class) constraints of the free BF model under study on the reduced phase-space. On the other hand $\tilde{G}_{A_0} \approx 0$ were initially obtained precisely in the first step of the Dirac–Bergmann algorithm, which is entirely based on the Hamiltonian equations of motion for constrained dynamical systems, so they were by construction secondary constraints. Nevertheless, they passed through a separation procedure and were replaced by some combinations $\tilde{G}'_{A_0}$ among $\tilde{G}_{A_0}$ and the primary second-class constraint functions $\chi_{\tilde{A}_0}$ (see Section 6). The key point is that on the reduced phase-space they gain the initial expressions $\tilde{G}_{A_0}$ that used the Hamiltonian equations of motion since the separation process did not require the addition of terms depending on the primary first-class constraints G_{A_0}. In this light, we are entitled to regard $\tilde{G}_{A_0} \approx 0$ as the secondary (first-class) constraints of the uncoupled BF model on the reduced phase-space. This discussion is important in view of the next two sections, where we tackle the gauge symmetries of the BF model by passing from the Hamiltonian to the Lagrangian setting.

9 Extended action and its gauge symmetries

We are now in the position to determine all the relevant gauge symmetries of the free BF model under consideration. Indeed, it is precisely the presence of first-class constraints at the Hamiltonian level that generates the true gauge symmetries of any Lagrangian action. Strictly speaking, it is the Hamiltonian formalism based on the primary first-class constraint set (known as the total formalism) that is equivalent to the Lagrangian one. Still, this would deny the gauge symmetries due to the presence of secondary first-class constraints since, according to the Dirac conjecture, every first-class constraint (primary or secondary) induces gauge symmetries. In view of this, one needs in a first step a larger setting, known as the extended Hamiltonian formalism, where all the gauge transformations of canonical variables are taken into account, irrespective of the primary or secondary character of the generating first-class constraint function. Still, the equivalence between the Hamiltonian and the Lagrangian formulations is not lost, being restored in the second step, where we pass from the extended to the total Hamiltonian formalism, which further enables the correct passing to the Lagrangian one. The key point is that in this process all the gauge symmetries are unfolded at the Lagrangian level, including those that originate in the presence of secondary first-class constraints.

Special attention should be paid in our case to the adaptation of standard extended and total formalisms since we eliminated the second-class constraints and passed to the reduced phase-space endowed with the induced bracket (Dirac) instead of the original (Poisson) one. Actually, we are now in the position to develop an extended formalism with fewer variables, where the second-class constraints were solved inside the extended action. As a consequence, the kinetic terms in the extended action are no longer given by (14), but must be replaced with the ones specific to the reduced phase-space, that may be read from (69)–(71)

$$\text{Kinred} \equiv \dot{\phi}_a H_0^a + \dot{V}_a^{0ij} p_{0ij}^a + 4\dot{V}_a^{ijk} H_{0ijk}^a + \dot{H}_{ijkl}^a P_a^{ijkl} + \dot{H}_i^a P_a^i, \tag{79}$$

where we maintain our initial convention $\dot{F}(x) \equiv \partial^0 F(x)$.

Due to the fact that in the second step we need to eliminate the reference to the secondary first-class constraints, we recall the discussion from the end of the previous section and add an extra index (1) to all quantities related to the primary first-class constraints and an additional index (2) in relation to all ingredients connected to the secondary ones, also redenoting the constraints in this process:

$$\left\{G_{A_0}, \tilde{G}_{A_0}\right\} \approx 0 \to \left\{G_{(1)A_0}, G_{(2)A_0}\right\} \approx 0, \tag{80}$$

$$G_{(1)A_0} \equiv \left\{G_{(1)a}^i \equiv P_a^i, \quad G_{(1)ij}^a \equiv p_{0ij}^a, \quad G_{(1)a}^{ijkl} \equiv P_a^{ijkl}\right\}, \tag{81}$$

$$G_{(2)A_0} \equiv \left\{G_{(2)a}^i \equiv \partial^i \phi_a, \quad G_{(2)ij}^a \equiv 12\partial^k H_{0kij}^a, \quad G_{(2)a}^{ijkl} \equiv \partial^{[i} V_a^{jkl]}\right\}. \tag{82}$$

A major ingredient of the extended formalism is the introduction of Lagrange multiplier (densities) with respect to all the first-class constraints

$$G_{(1)A_0} \approx 0 \to v^{(1)A_0} \equiv \begin{pmatrix} v^{(1)a}{}_i \\ v^{(1)ij}{}_a \\ v^{(1)a}{}_{ijkl} \end{pmatrix}, \quad G_{(2)A_0} \approx 0 \to v^{(2)A_0} \equiv \begin{pmatrix} v^{(2)a}{}_i \\ v^{(2)ij}{}_a \\ v^{(2)a}{}_{ijkl} \end{pmatrix}. \tag{83}$$

They are some arbitrary extra fields (real in our case and fully antisymmetric in their spatial Lorentz indices, where appropriate) defined on $\mathbb{R}^{7,1}$ that extend the reduced phase-space and bring no contribution to the kinetic term (their Dirac brackets with all the reduced phase-space variables are strongly vanishing). The extended Hamiltonian density is nothing but the sum between the density of the first-class Hamiltonian (75) and the linear combination of first-class constraint functions with the Lagrange multipliers as coefficients

$$\mathcal{H}_\mathrm{E} = \mathcal{H} + v^{(1)A_0} G_{(1)A_0} + v^{(2)A_0} G_{(2)A_0}. \tag{84}$$

Combining (79) with (84) and taking into account formulas (81), (82), and (75), we arrive at the extended action of our free BF model as

$$\begin{aligned} &S_\mathrm{BF}^\mathrm{E}\left[\phi_a, V_a^{0ij}, V_a^{ijk}, H_{0ijk}^a, H_{ijkl}^a, H_0^a, H_i^a, p_{0ij}^a, P_a^{ijkl}, P_a^i, v^{(1)A_0}, v^{(2)A_0}\right] = \\ &\int_{\mathbb{R}^{7,1}} (\text{Kinred} - \mathcal{H}_\mathrm{E})\, d^8x \equiv \int_{\mathbb{R}^{7,1}} \Big(\dot{\phi}_a H_0^a + \dot{V}_a^{0ij} p_{0ij}^a + 4\dot{V}_a^{ijk} H_{0ijk}^a + \\ &\dot{H}_{ijkl}^a P_a^{ijkl} + \dot{H}_i^a P_a^i + H_i^a \partial^i \phi_a + 12 V_a^{0ij} \partial^k H_{0kij}^a + H_{ijkl}^a \partial^{[i} V_a^{jkl]} \\ &- v^{(1)a}{}_i P_a^i - v^{(1)ij}{}_a p_{0ij}^a - v^{(1)a}{}_{ijkl} P_a^{ijkl} - v^{(2)a}{}_i \partial^i \phi_a \\ &- 12 v^{(2)ij}{}_a \partial^k H_{0kij}^a - v^{(2)a}{}_{ijkl} \partial^{[i} V_a^{jkl]}\Big)\, d^8x. \end{aligned} \tag{85}$$

Regarding its gauge symmetries, we need to construct on the one hand the gauge transformations of the reduced phase-space variables and on the other hand those of the Lagrange multiplier densities. Related to the former, these are some local transformations generated by every first-class constraint functions (primary and secondary) via the Dirac bracket in terms of some arbitrary parameters (known as the gauge parameters)

$$\delta_\epsilon F = \left[F, G_{(1)A_0}\right]_t^* \epsilon^{(1)A_0} + \left[F, G_{(2)A_0}\right]_t^* \epsilon^{(2)A_0}, \tag{86}$$

with F a function defined on the reduced phase-space. In our case, the gauge parameters sets $\epsilon^{(1)A_0}$ and $\epsilon^{(2)A_0}$ respectively corresponding to the primary and secondary first-class constraints (80) exhibit a multi-component structure, similar to (81) and (82)

$$\epsilon^{(1)A_0} \equiv \begin{pmatrix} \epsilon^{(1)a}{}_{i} \\ \epsilon^{(1)ij}{}_{a} \\ \epsilon^{(1)a}{}_{ijkl} \end{pmatrix}, \quad \epsilon^{(2)A_0} \equiv \begin{pmatrix} \epsilon^{(2)a}{}_{i} \\ \epsilon^{(2)ij}{}_{a} \\ \epsilon^{(2)a}{}_{ijkl} \end{pmatrix}, \tag{87}$$

being some arbitrary real fields on $\mathbb{R}^{7,1}$ (antisymmetric where appropriate). After some computation, (86) provides the searched for gauge transformations of the extended action in the reduced phase-space sector under the form

$$\delta_\epsilon \phi_a = 0,\ \delta_\epsilon V_a^{0ij} = \epsilon^{(1)ij}{}_{a},\ \delta_\epsilon V_a^{ijk} = -\partial^{[i}\epsilon^{(2)jk]}{}_{a},\ \delta_\epsilon H^a_{0ijk} = \partial^l \epsilon^{(2)a}{}_{lijk}, \tag{88}$$

$$\delta_\epsilon H^a_{ijkl} = \epsilon^{(1)a}{}_{ijkl},\ \delta_\epsilon H^a_0 = \partial^i \epsilon^{(2)a}{}_{i},\ \delta_\epsilon H^a_i = \epsilon^{(1)a}{}_{i},\ \delta_\epsilon P_a^{ijkl} = 0,\ \delta_\epsilon p^a_{0ij} = 0,\ \delta_\epsilon P^i_a = 0. \tag{89}$$

Concerning the gauge transformations of the Lagrange multiplier densities, we first need to recall the properties of the first-class constraint functions and of the first-class Hamiltonian and, in this context, add some new elements:

- all the constraints are purely Abelian (see (76))

$$\left[G_{(1)A_0}, G_{(1)B_0}\right]_t^* = \mathbf{0}, \quad \left[G_{(1)A_0}, G_{(2)B_0}\right]_t^* = \mathbf{0}, \quad \left[G_{(2)A_0}, G_{(2)B_0}\right]_t^* = \mathbf{0}; \tag{90}$$

- the concrete relations expressing the first-class property of H_t, (78) take a matrix form in terms of the new notations (80)

$$\left(\left[H, G_{(1)B_0}\right]_t^* \quad \left[H, G_{(2)B_0}\right]_t^*\right) = \left(G_{(1)A_0} \quad G_{(2)A_0}\right) \begin{pmatrix} V^{(1)A_0}_{(1)B_0} & V^{(1)A_0}_{(2)B_0} \\ V^{(2)A_0}_{(1)B_0} & V^{(2)A_0}_{(2)B_0} \end{pmatrix}, \tag{91}$$

$$V^{(1)A_0}_{(1)B_0} = \mathbf{0}, \quad V^{(1)A_0}_{(2)B_0} = \mathbf{0}, \quad V^{(2)A_0}_{(2)B_0} = \mathbf{0}, \tag{92}$$

$$\begin{aligned} V^{(2)A_0}_{(1)B_0} &\equiv \begin{pmatrix} V^{(2)am}_{(1)ib} & V^{(2)ab}_{(1)imn} & V^{(2)amnpq}_{(1)ib} \\ V^{(2)ijm}_{(1)ab} & V^{(2)ijb}_{(1)amn} & V^{(2)ijmnpq}_{(1)ab} \\ V^{(2)am}_{(1)ijklb} & V^{(2)ab}_{(1)ijklmn} & V^{(2)amnpq}_{(1)ijklb} \end{pmatrix} \\ &= \begin{pmatrix} -\delta^a_b \delta^m_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{2}\delta^b_a \delta^i_{[m}\delta^{j]}_{n]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{4!}\delta^b_a \delta^{[m}_i \delta^n_j \delta^p_k \delta^{q]}_l \end{pmatrix}; \end{aligned} \tag{93}$$

- the first-class constraints are reducible (see (77)), with the first-order reducibility relations and respectively functions written in a matrix form in terms of notations (80) like

$$\left(G_{(1)A_0} \quad G_{(2)A_0}\right) \begin{pmatrix} Z^{(1)A_0}_{A_1} \\ Z^{(2)A_0}_{A_1} \end{pmatrix} = \mathbf{0}, \tag{94}$$

$$Z^{(1)A_0}_{A_1} = \mathbf{0}, \quad Z^{(2)A_0}_{A_1} \equiv \tilde{Z}^{A_0}{}_{A_1} = \begin{pmatrix} \delta^a_b \partial^{[m}_{[y} \delta^{n]}_{i]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\delta^b_a \partial^{[i}_x \delta^{j]}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{4!} \delta^a_b \partial^{[m}_y \delta^n_i \delta^p_j \delta^q_k \delta^{r]}_{l]} \end{pmatrix}; \quad (95)$$

- for every reducibility relation in (94) we add an extra gauge parameter, arriving thus to an entire new set (again defined on $\mathbb{R}^{7,1}$ and antisymmetric in their spatial Lorentz indices wherever appropriate) of gauge parameters, denoted by

$$\tilde{\epsilon}^{A_1} \equiv \begin{pmatrix} \tilde{\epsilon}^b_{mn} \\ \tilde{\epsilon}^m_b \\ \tilde{\epsilon}^b_{mnpqr} \end{pmatrix}, \quad (96)$$

  whose presence is essential in establishing later both the spacetime locality and the Lorentz covariance of the Lagrangian formulation of the analyzed model.

We have now at hand all the necessary ingredients necessary at expressing the gauge transformations of the Lagrange multiplier densities (83) in a compact, matrix form (in DeWitt condensed notations)

$$\begin{pmatrix} \delta_\epsilon v^{(1)A_0} \\ \delta_\epsilon v^{(2)A_0} \end{pmatrix} = \begin{pmatrix} \dot{\epsilon}^{(1)A_0} \\ \dot{\epsilon}^{(2)A_0} \end{pmatrix} - \begin{pmatrix} V^{(1)A_0}_{(1)B_0} & V^{(1)A_0}_{(2)B_0} \\ V^{(2)A_0}_{(1)B_0} & V^{(2)A_0}_{(2)B_0} \end{pmatrix} \begin{pmatrix} \epsilon^{(1)B_0} \\ \epsilon^{(2)B_0} \end{pmatrix} - \begin{pmatrix} Z^{(1)A_0}_{A_1} \\ Z^{(2)A_0}_{A_1} \end{pmatrix} \tilde{\epsilon}^{A_1}. \quad (97)$$

Using the previous relations (91)–(96), performing the necessary computations, and reverting to the initial notation from convention (12), we finally infer

$$\delta_\epsilon v^{(1)a}{}_i = \partial^0 \epsilon^{(1)a}{}_i, \quad \delta_\epsilon v^{(1)ij}{}_a = \partial^0 \epsilon^{(1)ij}{}_a, \quad \delta_\epsilon v^{(1)a}{}_{ijkl} = \partial^0 \epsilon^{(1)a}{}_{ijkl}, \quad (98)$$

$$\delta_\epsilon v^{(2)a}{}_i = \partial^0 \epsilon^{(2)a}{}_i + \epsilon^{(1)a}{}_i + 2\partial^j \tilde{\epsilon}^a_{ji}, \quad \delta_\epsilon v^{(2)ij}{}_a = \partial^0 \epsilon^{(2)ij}{}_a + \epsilon^{(1)ij}{}_a + \partial^{[i} \tilde{\epsilon}^{j]}_a, \quad (99)$$

$$\delta_\epsilon v^{(2)a}{}_{ijkl} = \partial^0 \epsilon^{(2)a}{}_{ijkl} + \epsilon^{(1)a}{}_{ijkl} + 5\partial^m \tilde{\epsilon}^a_{mijkl}. \quad (100)$$

In conclusion, for a special $D = 8$ free BF model we generated the extended Hamiltonian action (85) and computed its gauge symmetries, (88), (89), and (98)–(100). Although not yet equivalent with the starting Lagrangian formulation, this setting reveals the complete set of Hamiltonian gauge symmetries, including those induced by the secondary first-class constraints and their associated reducibility.

10 Total and Lagrangian actions and their gauge symmetries

The next step is focused on the total action, which is obtained from the extended one via setting zero all the Lagrange multipliers associated with the secondary first-class constraints

$$v^{(2)A_0} = 0 \quad (101)$$

and whose gauge symmetries follow from those of the extended action where we implement (101) together with their direct consequences

$$\delta_\epsilon v^{(2)A_0} = 0 \quad \Leftrightarrow \quad \Big\{ \epsilon^{(1)a}{}_i = -\partial^0 \epsilon^{(2)a}{}_i - 2\partial^j \tilde{\epsilon}^a_{ji}, \quad \epsilon^{(1)ij}{}_a = -\partial^0 \epsilon^{(2)ij}{}_a - \partial^{[i} \tilde{\epsilon}^{j]}_a,$$

$$\epsilon^{(1)a}_{\quad ijkl} = -\partial^0 \epsilon^{(2)a}_{\quad ijkl} - 5\partial^m \tilde{\epsilon}^a_{mijkl} \Big\} . \tag{102}$$

Enforcing the previous relations in (85) and (88), (89) together with (98), we generate the total action

$$\begin{aligned}
&S^{\mathrm{T}}_{\mathrm{BF}}\left[\phi_a, V^{0ij}_a, V^{ijk}_a, H^a_{0ijk}, H^a_{ijkl}, H^a_0, H^a_i, p^a_{0ij}, P^{ijkl}_a, P^i_a, v^{(1)A_0}\right] = \\
&\int_{\mathbb{R}^{7,1}} \left(\dot{\phi}_a H^a_0 + \dot{V}^{0ij}_a p^a_{0ij} + 4\dot{V}^{ijk}_a H^a_{0ijk} + \right. \\
&\dot{H}^a_{ijkl} P^{ijkl}_a + \dot{H}^a_i P^i_a + H^a_i \partial^i \phi_a + 12 V^{0ij}_a \partial^k H^a_{0kij} + H^a_{ijkl} \partial^{[i} V^{jkl]}_a \\
&\left. - v^{(1)a}_{\quad i} P^i_a - v^{(1)ij}_{\quad a} p^a_{0ij} - v^{(1)a}_{\quad ijkl} P^{ijkl}_a \right) d^8x
\end{aligned} \tag{103}$$

as well as its gauge symmetries

$$\delta_\epsilon \phi_a = 0,\ \delta_\epsilon V^{0ij}_a = -\partial^0 \epsilon^{(2)ij}_{\quad a} - \partial^{[i} \tilde{\epsilon}^{j]}_a,\ \delta_\epsilon V^{ijk}_a = -\partial^{[i} \epsilon^{(2)jk]}_{\quad a},\ \delta_\epsilon H^a_{0ijk} = \partial^l \epsilon^{(2)a}_{\quad lijk}, \tag{104}$$

$$\delta_\epsilon H^a_{ijkl} = -\partial^0 \epsilon^{(2)a}_{\quad ijkl} - 5\partial^m \tilde{\epsilon}^a_{mijkl},\ \delta_\epsilon H^a_0 = \partial^i \epsilon^{(2)a}_{\quad i},\ \delta_\epsilon H^a_i = -\partial^0 \epsilon^{(2)a}_{\quad i} - 2\partial^j \tilde{\epsilon}^a_{ji}, \tag{105}$$

$$\delta_\epsilon P^{ijkl}_a = 0,\ \delta_\epsilon p^a_{0ij} = 0,\ \delta_\epsilon P^i_a = 0,\ \delta_\epsilon v^{(1)a}_{\quad i} = -\partial^0 \left(\partial^0 \epsilon^{(2)a}_{\quad i} + 2\partial^j \tilde{\epsilon}^a_{ji}\right), \tag{106}$$

$$\delta_\epsilon v^{(1)ij}_{\quad a} = -\partial^0 \left(\partial^0 \epsilon^{(2)ij}_{\quad a} + \partial^{[i} \tilde{\epsilon}^{j]}_a\right),\ \delta_\epsilon v^{(1)a}_{\quad ijkl} = -\partial^0 \left(\partial^0 \epsilon^{(2)a}_{\quad ijkl} + 5\partial^m \tilde{\epsilon}^a_{mijkl}\right). \tag{107}$$

The total action is completely equivalent to the Lagrangian one, (1), since the remaining momentum and Lagrange multiplier densities in (103) are auxiliary variables, that may be eliminated (algebraically) on their own equations of motion

$$v^{(1)a}_{\quad i} = \partial^0 H^a_i, \quad v^{(1)ij}_{\quad a} = \partial^0 V^{0ij}_a, \quad v^{(1)a}_{\quad ijkl} = \partial^0 H^a_{ijkl}, \tag{108}$$

$$P^i_a = 0, \quad p^a_{0ij} = 0, \quad P^{ijkl}_a = 0 \tag{109}$$

and, consequently, (103) where we replace (108) and (109) reduces to (1). Accordingly, the gauge symmetries of the Lagrangian action itself follow from (104) and (105) where in principle one uses (108) and (109), but this is not necessary here since the gauge transformations of the fields do not depend on either the momentum or the Lagrange multiplier densities.

This set of gauge transformations of the Lagrangian action of the special $D = 8$ collection of free BF models, coming from the Hamiltonian analysis, is by construction a generating set of gauge symmetries of $S^{\mathrm{L}}_{\mathrm{BF}}\left[\Phi^A\right]$. It can be easily brought to the manifestly covariant form (3) via some simple rearrangements of the various gauge parameters present in (104) and (105) into the independent components of some arbitrary real tensors as in (4)

$$\epsilon^{\mu\nu}_a : \left\{\epsilon^{0i}_a \equiv \tilde{\epsilon}^i_a,\ \epsilon^{ij}_a \equiv -\epsilon^{(2)ij}_{\quad a}\right\}, \quad \xi^a_{\mu\nu} : \left\{\xi^a_{0i} \equiv \frac{1}{2}\epsilon^{(2)a}_{\quad i}\ \xi^a_{ij} \equiv \tilde{\epsilon}^a_{ij}\right\}, \tag{110}$$

$$\xi^a_{\lambda\mu\nu\rho\sigma} : \left\{\xi^a_{0ijkl} \equiv \frac{1}{5}\epsilon^{(2)a}_{\quad ijkl},\ \xi^a_{ijklm} = \tilde{\epsilon}^a_{ijklm}\right\}. \tag{111}$$

We now observe that, if we had ignored the gauge transformations induced by the first-order reducibility relations, independent of those resulting from the presence of the first-class constraints, we would not have been able to obtain a covariant form of the gauge transformations of the Lagrangian action. Also, if we directly considered the total action within the Hamiltonian formulation, without taking into account the gauge transformations generated by the secondary first-class constraints of the analyzed model, we would have destroyed both the Lorentz covariance and the spacetime locality at the Lagrangian level.

11 Conclusions

The main conclusion of this work is that the construction of consistent Lagrangian interactions in gauge field theories, in particular in a special class of $D = 8$ topological BF models via, for instance, the deformation method based on the antifield-BRST method, still needs a careful investigation of the Hamiltonian roots of the initial Lagrangian formulation. Indeed, it is only in this fundamental context that the main geometric properties of the meaningful Lagrangian quantities, such as the allowed generating sets of gauge symmetries of the Lagrangian action, are revealed.

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