The Space Time Geometry inside a Proton

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Abstract

This work is an effort in reconciliation of the theory of General Relativity with the laws of Quantum phenomena. A proton is a composite form of most fundamental particles called quarks and gluons. Confinement of those most fundamental particles at the core of a proton is the cause of stability of the particle. Some publications point out that proton is one kind of Gaussian Wave Packet and with the help of quantum chromo dynamics (QCD) pressure and energy density distribution inside a proton can be depicted. However, inside a static proton what should be the nature of space time geometry or gravity is tried to be found out in this work.

Keywords: Gaussian Wave Packet, Schwarzschild's solution, form factor, space time geometry.

1. Introduction

A set of Gaussian One Particle Wave Packet [1-3]at initial time $t = 0$ is considered as a perfect fluid droplet which shows spherical symmetry. Total energy of a particle is distributed throughout the wave packet. Physical properties of a particle like GWP are characterized by quantum mechanical treatment. But general relativity is a satisfactory theory of gravitation for a massive object. The static spherically symmetric solution of Schwarzschild representing the gravitational field surrounding a spherically symmetric object of Mass M has an important role in understanding the nature of gravity. A massive object is a collective form of a huge number of atomic nuclei (i.e. proton and neutron) and a proton may be considered as a Gaussian wave packet [4]. Therefore, properties of gravity of an extended object or massive object are rooted in GWP. But, reconciliation of general relativity with the laws of quantum physics or with QCD remains a problem. The recent publications on pressure distribution and energy density distribution inside a proton [5-8] may open up a new area of research on the fundamental gravitational properties of protons and neutrons.

In this work, a trial has been made to show how, general relativistic view of gravity may be introverted inside a proton.

2. Energy Density and Pressure Distribution Inside a Proton

A single fundamental particle is considered as a kind of Gaussian Wave Packet whereas proton is a composite fundamental particle which may be presented by a Gaussian Wave Function [4]:

$$\psi(r) = (2\pi R_P^2/3)^{-3/4} e^{-(3/4R_P^2)\, r^2} \text{or, } \ \psi(r) = Ae^{-B\, r^2}$$

where, R_P is rms radius of the localized proton probability distribution, $A = (2\pi R_P^2/3)^{-3/4}$ and $B = (3/4R_P^2)$. Now, position probability density ($\psi^*\psi = A^2 e^{-2B\, r^2}$) for a proton obtained from the above equation implies that at $r = 0$, position probability density of a Gaussian Wave Packet is maximum and that is zero at $r = \infty$. This picture gives a description that particle is not confined in a fixed width rather, widespread from $r = 0$ to $r = \infty$ followed by the position probability density distribution. This means that Energy of a particle is distributed from $r = 0$ to $r = \infty$ followed by the position probability density distribution. Now, the possible definition of energy density is [9]

$$\rho^E(r) = \frac{1}{2}(\psi^*\hat{H}\psi + \psi\hat{H}\psi^*) \quad \text{or,} \quad \rho^E(r) = \frac{\hbar^2}{2\mu}(\nabla\psi^*).(\nabla\psi) + \psi^*V(r)\psi$$

It is expressly clear that $\rho^E(r)$ in the above equations have two factors for Gaussian Wave Packet, one is non exponential part and other is exponential part. For Gaussian wave form of proton the exponential factor is e^{-2Br^2}. So, one may read

$$\rho^E(r) = \rho_0^E e^{-2Br^2} \tag{1}$$

where, at $r = 0$, $\rho^E(r) = \rho_0^E$ which is non exponential part as said above. However, works related to pressure and energy density distribution inside a proton [5-8] and Gaussian wave form of proton [4] are very relevant to this work and we are interested to introduce general relativity to depict the gravity inside a linearly static proton.

With the help of QCD, a picture of total energy density inside a proton is given by [6]

$$T_{00}(r) = m \int \frac{d^3\Delta}{(2\pi)^3} \; e^{-ir\Delta} [A(t) - \frac{t}{4m^2}\{A(t) - 2J(t) + D(t)\}]$$

where, in Breit frame, t represents total momentum transferred as discussed in [5-8] and it is linked with a term Δ as $-\Delta^2 = t$ and $\Delta = (0, \; \Delta)$. $A(t)$, $J(t)$ and $D(t)$ are three form factors contributed by quark and gluons [6]. The energy density is the function of r; it is independent of polarization and normalized as $\int d^3r \; T_{00}(r) = m$. The shear forces $s(r)$ and pressure$p(r)$ can be computed from $D(t)$ as follows [6]:

$$s(r) = -\frac{1}{4m} r \frac{d}{dr}\frac{1}{r}\frac{r}{dr} \; D(r), \; p(r) = \frac{1}{6m}\frac{1}{r^2}\frac{d}{dr} \; r^2 \frac{r}{dr} \; D(r)$$

where, $D(r) = \int \frac{d^3\Delta}{(2\pi)^3} \; e^{-i\Delta \boldsymbol{r}} \; D(-\Delta^2)$.

Now, computation of energy density and pressure physically from this picture and direct use of its in Gaussian wave packet to solve Einstein's field equation is very difficulties. So, we are interested in finding an alternating way to derive energy density and pressure distribution function inside a proton.
If, the form factors $A(t)$ and $D(t)$ are known, then the energy density and pressure in the nucleon centre can be computed directly as [6]

$$T_{00}(0) = \quad \rho_0 = \frac{m}{4\pi^2}\int_{-\infty}^{0} dt\sqrt{-t} \; [A(t) - \frac{t}{4m^2}D(t)] \tag{2A}$$

$$p(0) = \; p_0 = \frac{1}{24\pi^2 \, m}\int_{-\infty}^{0} dt\sqrt{-t} \;\; tD(t) \tag{2B}$$

Now, with the help of (1) and (2A) we obtain energy density (in gravitational unit)

$$\rho(r) = \rho_0 e^{-2B r^2} = \left(\frac{m}{4\pi^2}\int_{-\infty}^{0} dt\sqrt{-t}\ [A(t) - \frac{t}{4m^2}D(t)]\right) e^{-2B r^2} \quad (3)$$

Again, we have to find out pressure distribution function inside proton which is a GWP. This distribution must be useful in the solution of Einstein's static field equation where (2B) represents the central pressure inside a proton. Experimental results on proton [5], very relevant to our work, state,

"Here we report a measurement of the pressure distribution experienced by the quarks in the proton. We find a strong repulsive pressure near the center of the proton (up to 0.6 femtometers) and a binding pressure at greater distances. The average peak pressure near the Centre is about 10^{35} ***pascals".***

"The distribution has a positive core and a negative tail of the $r^2 p(r)$ distribution as a function of r, with a zero crossing near $r = 0.6\, fm$. The regions where repulsive and binding pressures dominate are separated in radial space, with the repulsive distribution peaking near $r = 0.25\, fm$, and the maximum of the negative pressure that is responsible for the binding occurring near $r = 0.8\, fm$."

Now, after a careful observation on the above statement in double invited coma and the related diagram given in [5], we choose a supporting function for pressure distribution

$$p(r) = p_0\left(1 - \alpha r^{\beta}\right)e^{-\eta r^2} = \left(\frac{1}{24\pi^2 m}\int_{-\infty}^{0} dt\sqrt{-t}\ \ tD(t)\right)\left(1 - \alpha r^{\beta}\right)e^{-\eta r^2} \quad (4)$$

where, α , β , η are three constants, explicitly linked with essential conditions as given below

a) $p(r)$ is zero at $r = \infty$ and it is maximum (p_0) at $r = 0$

b) The pressure distribution function $\mathrm{p}(r)$ must satisfies the von Laue stability condition [5,10]: $\int_0^{\infty} r^2 \mathrm{p}(r) dr = 0$. This implies that $\int_0^{r_0} r^2 p(r) dr = -\int_{r_0}^{\infty} r^2 \mathrm{p}(r) dr$ where, $0 < r_0 < \infty$ and r_0 is the nodal point. According to the statement given above in double invited coma, $r_0 \approx 0.6\, fm$.

c) $\frac{d}{dr}\left(r^2 p(r)\right) = 0$ at the position r where $r^2 \mathrm{p}(r)$ is extremum. According to the statement given in double invited coma, $r^2 \mathrm{p}(r)$ is extremum at $r \approx 0.25\, fm$ and at $r \approx 0.8\, fm$

The known value of p_0 and values of α, β and η as a result of the solution of three equations obtained from **b)** and **c)** can show the exact form of pressure distribution function. However, with the help of (3) and (4), we have to find out space time geometry inside a proton.

3. Space Time Geometry Inside Proton

A linearly static proton is considered an incompressible perfect fluid droplet widespread from $r = 0$ to $r = \infty$. So, one can solve Einstein's static field equation in side proton. This section may be initiated by representing the Einstein's static field equation

$$R_{\mu\nu} - g_{\mu\nu}R/2 + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}\ .$$

Now, the general form of line element in a GWP at any position r may be presented by the equation

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 + e^{\nu}d\tau^2 \tag{5}$$

where, λ and ν are function of r only and those will be zero at $r = \infty$. This leads to get the concerned fundamental tensor

$$g_{kl} = \begin{pmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & e^{\nu} \end{pmatrix} \tag{6}$$

It is well known that the energy momentum tensor in mixed tensor form is

$$T_k^l = (p+\rho) g_{\alpha}^{k} \frac{dx^{\alpha}}{ds}\frac{dx^{l}}{ds} - g_k^l p \tag{7}$$

Where, p and ρ are pressure and mass density respectively in gravitational unit. Now, equations (6) and (7) are linked with each other by Einstein's field equation as given below

$$R_k^l - g_k^l R/2 + \Lambda g_k^l = -8\pi T_k^l \tag{8}$$

Equation (7) and (8) leads to the relation as given below

$$R_k^l - g_k^l R/2 + \Lambda g_k^l = -8\pi (p+\rho) g_{\alpha}^{k} \frac{dx^{\alpha}}{ds}\frac{dx^{l}}{ds} - g_k^l p \tag{9}$$

Proton, a Gaussian Wave Packet, is like a incompressible perfect fluid droplet and distribution of equivalent mass inside it is considered as linearly static. So, one may read $\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0$. Therefore, we obtain from (5) $\frac{d\tau}{ds} = e^{-\nu/2}$ and consequently from (7)

$$T_1^1 = T_2^2 = T_3^3 = -p \text{ and } T_4^4 = \rho \tag{10}$$

Now, using the above results in equation (9) eventually, we reach to the three effective equations [11] as given below.

$$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \tag{11}$$

$$8\pi p = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right) + \Lambda \tag{12}$$

$$8\pi \rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \tag{13}$$

where, $\nu' = \frac{d\nu}{dr}$, $\lambda' = \frac{d\lambda}{dr}$. Differentiation of (11) yields

$$8\pi \frac{dp}{dr} = e^{-\lambda}\left(\frac{\nu''}{r} - \frac{\nu'}{r^2} - \frac{2}{r^3}\right) - \lambda' e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{2}{r^3} \tag{14}$$

Again, equating (11) and (12) and rearranging, we get

$$e^{-\lambda}\left(\frac{\nu"}{r}-\frac{\nu'}{r^2}-\frac{2}{r^3}\right)-\lambda' e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{2}{r^3}=-e^{-\lambda}\,(\frac{\lambda'}{r}+\frac{\nu'}{r})\frac{\nu'}{2} \tag{15}$$

where, $8\pi(p+\rho)=-e^{-\lambda}\,(\frac{\lambda'}{r}+\frac{\nu'}{r})$. With the help of (15), we obtain from (14)

$$\frac{dp}{p+\rho}=-\frac{d\nu}{2} \tag{16}$$

However, with the help of (3) and (4), we obtain from (16)

$$\int_{\infty}^{r}\frac{p_0 e^{-\eta r^2}(2\eta r+\alpha\beta r^{\beta-1}-2\alpha\eta r^{\beta+1})}{p_0(1-\alpha r^{\beta})e^{-\eta r^2}+\rho_0 e^{-2Br^2}}dr=\int_{0}^{\nu}\frac{d\nu}{2} \tag{17}$$

$$\text{Therefore, } e^{\nu}=\exp\,(\int_{\infty}^{r}\frac{2p_0 e^{-\eta r^2}(2\eta r+\alpha\beta r^{\beta-1}-2\alpha\eta r^{\beta+1})}{p_0(1-\alpha r^{\beta})e^{-\eta r^2}+\rho_0 e^{-2Br^2}}dr) \tag{18}$$

Again, from (13) we obtain the relation as given below

$$d(re^{-\lambda})=[1-(8\pi\rho+\Lambda)r^2]dr \Rightarrow \int_0^r d(re^{-\lambda})=\int_0^r [1-(8\pi\rho+\Lambda)r^2]dr$$

$$\Rightarrow e^{\lambda}=\left(1-\frac{8\pi\rho_0}{r}\int_0^r r^2 e^{-2Br^2}dr-\frac{\Lambda r^2}{3}\right)^{-1} \tag{19}$$

where, ρ is as given in (3). Eventually, with the help of (18) and (19) we obtain from (5)

$$ds^2=-\left(1-\frac{8\pi\rho_0}{r}\int_0^r r^2\exp\,(-2Br^2)dr-\frac{\Lambda r^2}{3}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2$$

$$+\exp\,(\int_{\infty}^{r}\frac{2p_0 e^{-\eta r^2}(2\eta r+\alpha\beta r^{\beta-1}-2\alpha\eta r^{\beta+1})}{p_0(1-\alpha r^{\beta})e^{-\eta r^2}+\rho_0 e^{-2Br^2}}dr)d\tau^2 \tag{20}$$

This is the form of space time geometry inside a proton where, ρ_0 and p_0 are given in (2A) and (2B) respectively.

4. Conclusion

Eventually, it is concluded that proton is a kind of Gaussian Wave Packet, inside this wave packet the solution of Einstein's field equation is as given in (20). With the help of QCD and assuming that a proton is a GWP, we get a picture on pressure and energy density distribution inside proton. Then, the process of the solution of Einstein's field equation as shown in the text leads to the equation (20). In the same way, space time geometry inside any single particle Gaussian wave packet may be depicted if, one can find out pressure and energy density distribution inside it. Therefore, this effort may be a step for reconciliation of the theory of General Relativity with the laws of Quantum phenomena.

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