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On viscous space-time, geodesic curves and a possible relation to MOND

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Abstract

Embarking on a theoretical exploration, the author delves into the prospect of integrating Modified Newtonian Dynamics (MOND) and the incorporation of viscosity into curved spacetime within the framework of general relativity. MOND, a proposition that modifies the laws of gravity at low accelerations, has been put forward as a potential explanation for galactic dynamics without invoking dark matter, although this aspect is not in the center of interest of this text. Conversely, the incorporation of viscosity through the viscosity tensor $\Pi\mu\nu$ aims to capture the effects of viscosity on particle motion in curved spacetime. While these two ideas are independent, there is an intrigue in exploring their potential connection and how they might interact. Employing a mathematical and perturbative approach, the author investigates how MOND and viscosity might influence particle trajectories under specific conditions.

Keywords: Modified Newtonian Dynamics Theory (MOND), viscosity in space-time, general relativity, modified gravity, particle trajectories, low accelerations, mathematical approximation, perturbations, metric connections, Christoffel symbols, interpolation, viscosity as a force, geometric viscosity, galactic dynamics.

Discussion

The path of this article progresses from less to more. From a small question to broader conclusions, following the process the author followed in writing it. A sphere is a particular case of an ellipse in the gravitational field? A planet or a star forms a sphere when gravitationally settled, but the objects orbiting them, or they themselves, trace ellipses (and sometimes circles) Why? Yes, a sphere can be considered as a particular case of an ellipse in the context of gravity. Both spheres formed by planets or stars and ellipses traced by orbiting objects are related to the gravitational forces acting between celestial bodies.

When a massive body, such as a planet or a star, forms or settles due to gravity, it tends to take on a spherical shape. This is because the gravitational force acts uniformly from all directions towards the centre of mass of the body, it has central symmetry. In that case we have the particular case of an ellipse whose foci occupy the same point (the semi-major axis is zero): the centre of a sphere. This balances the internal pressures and, as the body collapses, all parts are compressed towards the centre, forming the sphere: this is the way in which a massive body can have the lowest gravitational potential energy.

In contrast, when a smaller object, such as a moon or a satellite, orbits a massive body (such as a planet or a star), it follows an elliptical trajectory also due to gravity. This is because the gravitational force varies as the object moves in its orbit, as there is another force at work: the centripetal force that causes the object to move. Gravity is stronger when the object is closer to the massive body and weaker when it is further away. In the extragalactic context, an elliptical galaxy can be modelled as several concentric ellipses sharing a flat area of space.

Kepler's laws describe the elliptical orbits of moving objects under the influence of a central force, such as gravity (Russell, J. L., 1964). These laws were formulated by Johannes Kepler in the 17th century and are:

1. The orbit of an object around a massive body is an ellipse, with the massive body located at one of the foci of the ellipse.
2. The line connecting the massive body to the object will sweep equal areas at equal time intervals.
3. The square of the orbital period (the time it takes for an object to make one complete revolution around the massive body) is proportional to the cube of the length of the semi-major axis of the ellipse.

The formation of spheres and the elliptical orbits (and other trajectories) of celestial bodies can be deduced from the Einstein field equations, which are the core of the theory of general relativity. The Einstein field equations are a set of partial differential equations that describe how matter and energy affect the curvature of space-time. These equations are given by:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

where $G\mu\nu$ is the Einstein tensor, which describes the curvature of space-time, G is the gravitational constant, and $T\mu\nu$ is the energy-momentum tensor, which describes the distribution of matter and energy.

For a massive body, such as a planet or a star, the energy-momentum tensor $T\mu\nu$ is not zero in the region occupied by the body. Solving the Einstein field equations with the appropriate conditions for a massive body yields a solution that describes the geometry of space-time around the object. For a spherically symmetric body, the solution to the Einstein field equations describes a curvature of space-time that resembles a sphere around the object. This is known as the Schwarzschild metric, which describes the gravitational field of a non-rotating spherical object (Kruskal, M. D., 1960). On the other hand, for an object in orbit around a massive body, the Einstein field equations also allow one to derive the trajectories of the orbits. The general solution for an orbiting object is known as the Kerr metric, which describes the gravitational field of a rotating spherical object. With these solutions it can be shown that a smaller orbiting object will follow an elliptical trajectory around a massive object, as predicted by the theory of general relativity and Kepler's laws.

One might now ask whether MOND (Modified Newtonian Dynamics) theory could be related to the difference between spherical and elliptical trajectories in certain low acceleration situations. MOND is a modified theory of gravity that proposes a modification of Newton's laws at extremely low acceleration scales (Milgrom, M., 2014).

In the MOND framework, it is suggested that gravity behaves differently than it does according to Newton's laws (and Einstein's general relativity) in these low-acceleration situations. At very low acceleration scales, MOND introduces a modification of the gravitational forces, which can have effects on the orbits of celestial bodies. When gravitational accelerations are really very low, the differences between the gravity predicted by MOND and Newtonian or relativistic gravity can become more apparent. In the context of a sphere and an ellipse, MOND could influence the trajectories of objects in orbit around a massive body when the accelerations go to low values.

The scenario where the differences between MOND and standard gravitational theories (such as general relativity and Newton's law of gravitation) are best reflected is in a Friedmann-Robertson-Walker (FRW) universe. The FRW model describes a large-scale homogeneous and isotropic universe, in which the expansion of space-time is governed by the field equations of general relativity (Akbar, M. & Cai, R. G., 2006).

In a FRW universe, the space-time metric is described by the following form:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

where ds^2 is the line element (space-time interval), a(t) is the scaling factor that determines the expansion of the universe as a function of time t, r is the radial coordinate, θ is the latitude coordinate and ϕ is the longitude coordinate.

The FRW model of the universe is based on the observation that it appears to be homogeneous and isotropic on a large scale, which would indicate that it has the same appearance from any position and direction in space. There is another model of the universe, the De Sitter model, which is a particular case of the FRW model, but with one important feature: in the De Sitter model, the expansion of the universe is purely exponential and contains no matter (there is no matter density in that universe) (Polyakov, A. M., 2008). In this model, the curvature of space-time is dominated by the cosmological constant (which acts as a form of dark energy), and there is no dynamical evolution of matter density over time.

Since MOND proposes a modification of gravity in low-acceleration situations and affects the dynamics of bodies as a function of the matter distribution, it is in a FRW universe that the differences between MOND and standard gravitational theories would be best reflected. In a De Sitter universe, the matter density does not vary with time, so differences in gravitational dynamics related to MOND might be less evident or have no significant impact compared to the FRW model, where the evolution of the matter density is an important feature.

That said, let us add more complication to the matter. In an Einstein field equation with viscosity tensor added to contemplate a turbulent option $G\mu\nu = 8\pi(T\mu\nu + \Pi\mu\nu)$, already proposed by the author (Quiroga, E., 2023), can the above mentioned circular trajectories also be deduced as a particular case of elliptic trajectories in the gravitational field? The addition of the viscosity tensor $\Pi\mu\nu$ to the Einstein field equations introduces a modification that considers the effects of viscosity in the gravitational field. This modification may have implications for the dynamics of the trajectories of celestial bodies in the gravitational field, but its specific effect will depend on the form and properties of the viscosity tensor. In the standard Einstein field equations (without the addition of the viscosity tensor), the trajectories of celestial bodies in a gravitational field follow Kepler's laws, which include elliptical, circular and parabolic trajectories as particular cases, depending on the energy and angular momentum of the body in question. Circular trajectories are simply a particular form of elliptical orbits, where the semi-major axis is zero. When the viscosity tensor is introduced into the Einstein field equations, the gravitational dynamics can be modified and affect the trajectories of celestial bodies in the gravitational field. The viscosity tensor can represent effects such as resistance to motion or energy dissipation, which could influence how objects move in the gravitational field.

The viscosity tensor $\Pi\mu\nu$ added as a proposal by the author in the Einstein field equations represents the presence of a viscous medium in the very structure of space-time. The introduction of this tensor modifies the field equations of general relativity, which may have interesting consequences for the dynamics of celestial bodies and hence their trajectories. The factors that imply how viscosity would influence the trajectories of celestial bodies are diverse and complex. Some of them are:

A. Viscosity intensity: The magnitude of the viscosity tensor $\Pi\mu\nu$ will determine the drag force experienced by moving bodies. As the viscosity increases, the resistance to flow will also increase, which could lead to more significant deviations in the trajectories.

B. Structure of the viscosity tensor: The specific structure of the viscosity tensor can have an impact on how the drag force is distributed along the trajectory of a body. Depending on the symmetry and anisotropy of the tensor, trajectory deviations could be asymmetric or depend on the angle of motion.

C. Velocity and mass of the celestial body: The influence of viscosity can depend on the velocity and mass of the celestial body. More massive bodies with higher velocities may experience more significant resistance to motion, leading to larger deviations in their trajectories.

D. Characteristics of the viscous medium: The nature of the viscous medium causing the resistance to movement will also play an important role. The density and viscosity of the medium (the space-time itself) will affect the magnitude of the drag force.

We can attempt to characterise the influence of the viscosity tensor on the trajectories of celestial bodies mathematically using the Einstein field equations with the addition of the viscosity tensor. However, we should note that this can be a rather technical analysis and would require specific solutions of the modified equations, which can be complex and is not always available analytically.

In the following, I will give a general description of how the viscosity tensor might affect the equations of motion of celestial bodies, but I will not provide a complete mathematical solution:

1. Einstein field equations with viscosity: The modified Einstein field equation with the addition of the viscosity tensor is (Quiroga, E., 2023):

   $G\mu\nu=8\pi(T\mu\nu+\Pi\mu\nu)$

   Where:

- $G\mu\nu$ is the Einstein tensor describing the curvature of space-time.
- $T\mu\nu$ is the energy-momentum tensor representing the distribution of mass and energy in space-time.
- $\Pi\mu\nu$ is the viscosity tensor representing the viscous effects in space-time.

2. Equations of motion of celestial bodies: To study the trajectories of celestial bodies in space-time with viscosity, we need to solve the equations of motion for particles under

the influence of gravity and the drag force due to viscosity. This requires solving the modified geodesic equations in the presence of the viscosity tensor.

The geodesic equations describe how particles follow trajectories in curved space-time under the influence of gravity and are expressed by the following equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{1}{2}g^{\mu\nu}\frac{dg_{\alpha\nu}}{d\tau}\frac{dx^\alpha}{d\tau}$$

Where:

- x^μ represents space-time coordinates.
- τ is the affine parameter (a parameter that parameterises the trajectory).
- ${}^{\mu}{}_{\alpha\beta}\,\Gamma$ are the Christoffel symbols defining the metric connections.
- $g^{\mu\nu}$ is the metric tensor describing the geometry of space-time.

3. Specific solutions: To fully characterise the effect of the viscosity tensor on the trajectories of celestial bodies, we need to solve the Einstein field equations with viscosity and the modified geodesic equations for specific systems involving the presence of viscosity. This may require the specification of a particular metric, a viscosity tensor and appropriate initial conditions.

Without wishing to over-complicate this article, the author would like to add some additional considerations on specific solutions:

1. Metric and Viscosity Tensor: To study specific systems, a metric describing the space-time geometry and a viscosity tensor representing the resistance to flow in that context must be specified. The choices of metric and viscosity tensor will depend on the problem under study and the characteristics of the viscous medium involved.
2. Initial Conditions: To solve the equations of motion (geodesic equations) and obtain specific trajectories for celestial bodies, appropriate initial conditions must be established. These initial conditions will determine the initial position and velocity of the bodies at a given time, from which the equations can be integrated to obtain the trajectories over time.
3. Approximations and Numerical Methods: Since the Einstein field equations with viscosity are highly non-linear and complex, in most cases, it is not possible to find exact analytical solutions. Instead, approximation techniques or numerical methods are used to obtain approximate solutions that describe the behaviour of the system.
4. Special Cases: In certain cases, exact analytical solutions can be found for simplified systems with special symmetries and specific properties of the viscosity tensor. For example, homogeneous and isotropic systems could be studied, where symmetries simplify the equations and allow more tractable solutions to be obtained.

The geodesic equations are fundamental to the theory of general relativity, as they describe how objects follow trajectories in curved space-time. These equations are essential for understanding how gravity affects the motion of celestial bodies, and play a crucial role in predicting planetary orbits, particle trajectories and other gravitationally related phenomena (Pasterski, S. et al., 2016).

In extending the conclusions related to the geodesic equations, we can highlight some important points:

1. Geodesics and Trajectories of Celestial Bodies: The geodesic equations describe the natural trajectories of celestial bodies in curved space-time. When a body moves under the influence of gravity alone (with no other significant external forces), it follows a geodesic in space-time. For a body in orbit around a massive object, its trajectory will be a geodesic in space-time curved by the mass of the central object.
2. Square Momentum Preservation: In general relativity, the geodesic equations have an interesting property: the square momentum of a particle, defined as p $p^{\mu}{}_{\mu}$, where p^{μ} is the particle's square momentum, is conserved along the geodesic. This property implies that particles moving under gravity will move along trajectories that preserve a certain magnitude of momentum, which affects the shape of the orbits and their stability.
3. Difference with Newtonian Laws of Motion: In the Newtonian theory of gravitation, the equations of motion are expressed as second-order ordinary differential equations. In general relativity, on the other hand, the geodesic equations are second-order covariant differential equations, which means that the trajectory of a particle is determined by the geometry of space-time in its environment. This leads to differences in the trajectories predicted by the two theories in certain scenarios, such as the orbit of Mercury, where general relativity predicts deviations from Newtonian orbits.
4. Trajectories under the Influence of Other Fields: Although the geodesic equations describe the motion of celestial bodies under the influence of gravity, they can also be extended to take into account other forces or fields present in space-time. For example, in the presence of electromagnetic fields, particles will follow trajectories that are solutions of the geodesic equations modified to include the electromagnetic interaction.

By incorporating the viscosity tensor Πμν into the geodesic equations, the effect of viscosity on the motion of particles in curved space-time is considered. The addition of this tensor modifies the geodesic equations and may have interesting implications for how celestial bodies follow trajectories under the combined influence of gravity and viscous drag.

The geodesic equations modified with the presence of the viscosity tensor would be expressed as follows:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{1}{2}g^{\mu\nu}\left(\frac{dg_{\alpha\nu}}{d\tau}\frac{dx^\alpha}{d\tau} + \frac{d\Pi_{\alpha\nu}}{d\tau}\frac{dx^\alpha}{d\tau}\right)$$

Where:

- x^{μ} represents space-time coordinates.
- τ is the affine parameter (a parameter that parameterises the trajectory).
- ${}^{\mu}{}_{\alpha\beta}$ Γare the Christoffel symbols defining the metric connections.
- $g^{\mu\nu}$ is the metric tensor describing the geometry of space-time.
- Παν is the viscosity tensor representing the viscous effects in space-time.

The presence of the second term on the right-hand side of the equation, which involves the viscosity tensor, modifies the acceleration of the particles compared to the case without viscosity. The drag force due to viscosity affects how the particles move in gravitationally curved space-time. When considering the viscosity tensor in the geodesic equations, it is possible that the particle trajectories will be different from those predicted by standard geodesic equations without viscosity. Viscous drag can alter the shape of the orbits, their stability and their time evolution. However, it is important to note that the incorporation of the viscosity tensor introduces additional complexity into the geodesic equations, which can make them difficult to solve analytically. In many cases, researchers resort to numerical methods to obtain approximate solutions and to understand how viscosity affects the trajectories of celestial bodies. An exact analytical solution for the geodesic equations with the viscosity tensor Παν in the context of general relativity can be extremely difficult and in many cases no general analytical solution has been found. However, we can make an approximate approach in a simplified case to illustrate how the viscosity tensor might affect trajectories. Consider the case of a flat space-time, where g $=\eta_{\mu\nu\mu\nu}$, which corresponds to the Minkowski metric, and where there is only one non-zero component of the viscosity tensor, i.e. Π_{00} (as a simplified example).

The geodesic equations for a massive body experiencing viscous drag would be given by:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} - \frac{1}{2}\eta^{ij}\frac{d\Pi_{00}}{d\tau}\frac{dx^j}{d\tau}$$

where i,j are spatial indices (integers: 1, 2, 3) and τ is the affine parameter.

In the case of viscous drag, the viscosity tensor Π00 could depend on the velocity of the body, and let us assume that it is linearly related to the velocity:

$$\Pi_{00} = -\eta\frac{dx^0}{d\tau}$$

where η is a coefficient of viscosity.

The geodesic equations would then reduce to:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 + \frac{\eta}{2}\eta^{ij}\frac{d^2x^j}{d\tau^2}$$

In this simplified case, viscosity manifests itself as an additional term in the equations of motion, similar to an acceleration-dependent force. However, it is important to note that this is only a rudimentary example and the actual solutions would depend on the specific metric and viscosity tensor, which can result in more complex geodesic equations. When viscosity is incorporated into the geodesic equations of general relativity, the viscosity tensor Πμν adds an additional term to the particle acceleration, which behaves similarly to an acceleration-dependent force. This viscous term is proportional to the acceleration of the particles and is related to the resistance to motion they experience due to the interaction with the viscous medium present in space-time.

In the example I mentioned above, where we assume that there is only one non-zero component of the viscosity tensor Π_{00} and that it is linearly related to the velocity, the geodesic equation simplifies as:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 + \frac{\eta}{2}\eta^{ij}\frac{d^2x^j}{d\tau^2}$$

Where the term viscous

$$\frac{\eta}{2}\eta^{ij}\frac{d^2x^j}{d\tau^2}$$

is added to the acceleration

$$-\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2$$

which is normally found in geodesic equations without viscosity.

This viscous term represents the viscous drag faced by the particle as it moves in the curved space-time. The magnitude of the term depends on the coefficient of viscosity η and on the acceleration

$$\frac{d^2x^j}{d\tau^2}$$

of the particle in that spatial direction.

When the acceleration is large, the viscous term becomes more significant and affects the trajectory more. That is, at high accelerations, the viscosity is more effective and can noticeably change the particle trajectory compared to geodesic equations without viscosity. On the other hand, when the acceleration is small, the contribution of the viscous term may be negligible and the trajectories will more closely resemble those of standard geodesic equations without viscosity.

In general, viscosity introduces a modification in the equations of motion that depends on acceleration and thus affects the behaviour of the trajectories of celestial bodies in gravitationally curved space-time. Viscous drag can alter the shape, stability and time evolution of trajectories, which adds an additional layer of complexity to the study of how bodies move in general relativity. In the case of small acceleration, can we relate this addition of viscosity to MOND? Could a MOND scheme with viscosity in spacetime be possible? Suppose we are working in a galactic system where MOND is relevant because of the low accelerations present in the outer regions of the galaxy. In these regions, the gravitational forces are dominated by the baryonic mass, and MOND proposes that the effective gravitational force F_{MOND} is a modified function of the acceleration a and the standard Newtonian acceleration aN:

$$F_{\text{MOND}} = \mu\left(\frac{a}{a_N}\right) F_N$$

Where μ(x) is an interpolating function that is fitted to match Newtonian predictions at high accelerations (a "aN) and has asymptotic behaviour at low accelerations (a "aN).

Now, let us consider the effect of the viscous medium in curved space-time. Assume that the viscosity tensor Πμν depends on the particle velocity and is linearly related to the velocity:

$$\Pi_{\mu\nu} = -\eta u_\mu u_\nu$$

Where η is the coefficient of viscosity and u_μ is the quadravelocity of the particle.

With these considerations, the equations of motion for the particle would be given by:

$$m\frac{du^\mu}{d\tau} = -\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2}g^{\mu\nu}\left(\frac{dg_{\alpha\nu}}{d\tau}u^\alpha + \frac{d\Pi_{\alpha\nu}}{d\tau}u^\alpha\right)$$

Where m is the particle mass, uμ is the quadratic velocity, τ is the affine parameter and $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols defining the metric connections. Solving this equation of motion analytically is very complex due to the presence of MOND and the viscosity tensor. The interpolating function μ(x) of MOND introduces additional non-linearities, and the viscosity tensor adds terms dependent on the derivatives of the metric and velocity. The analytical solution for this system may not be feasible and it is likely that numerical approaches and approximations will have to be used to study how the combination of MOND and viscosity affects particle trajectories in curved space-time.

This equation of motion incorporates both the MOND modifications to gravity and the viscous effect of the medium in curved space-time. However, it is worth reiterating that solving this equation analytically can be extremely complicated due to the presence of MOND and the viscosity tensor, and the interpolating function μ(x) introduces additional non-linearities. In practice, numerical methods and approximations are likely to be required to study how these two ideas interact and affect particle trajectories in curved space-time.

Perturbation techniques can be used to study the effect of viscosity as a small correction to the MOND equations of motion. This can lead to approximate solutions for specific cases where viscosity is a minor influence and can be treated as a perturbation in the equations.

Another possible approach is to consider limiting cases or simplified situations where certain mathematical or physical approximations are valid. For example, in regions where the accelerations are sufficiently high, the MOND term could be reduced to the standard Newtonian law, and the viscosity tensor could be considered as a small additional correction. In such cases, more tractable approximate analytical solutions could be obtained.

Let us consider a specific scenario in which MOND-modified gravity is relevant in regions where accelerations are low, such as in the outer regions of a galaxy. Furthermore, let us assume that the viscosity in space-time is small enough that it can be treated as a small additional correction to the equations of motion. These assumptions allow us to make a perturbative approximation to describe the motion of the particle.

Let us start with MOND's equations of motion, which as indicated above can be expressed as:

$$m\frac{du^{\mu}}{d\tau} = -\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} + \mu\left(\frac{a}{a_N}\right)m\frac{d^2x^{\mu}}{d\tau^2}$$

Where u_μ is the particle quadravelocity, τ is the affine parameter, x^μ are the space-time coordinates, $\Gamma^\mu{}_{\alpha\beta}$ are the Christoffel symbols defining the metric connections and aN is the Newtonian acceleration.

Next, let us incorporate the viscosity tensor $\Pi\mu\nu$ as a small correction. Recall that the viscosity tensor is linearly related to the quadravelocity:

$$\Pi_{\mu\nu} = -\eta u_{\mu}u_{\nu}$$

To maintain our perturbative approximation, let us assume that the viscosity tensor has a much smaller contribution than the other forces present, and we can treat it as a perturbation. Then, we can add the viscosity tensor term to the MOND equations of motion as:

$$m\frac{du^{\mu}}{d\tau} = -\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} + \mu\left(\frac{a}{a_N}\right)m\frac{d^2x^{\mu}}{d\tau^2} + \delta\Pi^{\mu}_{\alpha}u^{\alpha}$$

Where $\delta\Gamma^\mu{}_\alpha$ represents the perturbation of the viscosity tensor. This perturbation can be related to the viscosity through the coefficient η and the velocity of the particle.

Finally, there is a pending discussion; the four-velocity u_μ provides a fundamental description of the motion of particles in curved space-time, representing both their direction and magnitude of movement relative to proper time. In the context of this exploration involving MOND with the inclusion of viscosity, the four-velocity parameter serves as a key parameter in understanding how particles navigate through the gravitational field of massive bodies while also experiencing viscous effects. The four-velocity encapsulates the trajectory followed by a particle as it traverses through space-time, accounting for the modifications introduced by MOND and viscosity. In regions of low accelerations, where MOND becomes relevant, the four-velocity may reflect the altered gravitational dynamics proposed by MOND theory, capturing deviations from classical newtonian predictions. The inclusion of viscosity in the Einstein field equations introduces also additional complexities to the motion of particles. The four-velocity now also accounts for the resistance to motion experienced by particles due to the presence of a viscous medium in this space-time concept. As particles move through this viscous medium, their trajectories are influenced by drag forces, altering their paths compared to scenarios without viscosity.

This approximation is simplified and based on specific assumptions, so the results obtained may be approximate and valid only under certain specific conditions.

Conclusion

The author delves into the possibility of merging the Theory of Modified Newtonian Dynamics (MOND) with the incorporation of viscosity in curved spacetime, within the framework of general relativity. MOND proposes a modification to the laws of gravity at low accelerations, aiming to explain the observed dynamics in galactic regions without invoking the existence of dark matter; The author's primary interest is in exploring the conceptual core of MOND, not in its application as a dark matter criticising tool. On the other hand, the incorporation of viscosity through the viscosity tensor $\Pi\mu\nu$ is considered to investigate viscous effects on particle motion in curved spacetime. While these concepts are independent, there is a potential to explore possible connections between MOND and viscosity, as both influence the motion of particles in curved spacetime. However, to date, a well-established and widely accepted formulation that coherently combines these two ideas has yet to be developed.

Taking a mathematical approach, the author encounters a complex and nonlinear system due to the presence of the MOND interpolating function $\mu(x)$ and the viscosity tensor $\Pi\mu\nu$ in the equations of motion. Consequently, obtaining accurate analytical solutions is challenging. This approach considers a scenario where MOND is relevant in regions of low accelerations, and viscosity is treated as a small additional correction. Perturbative techniques and series expansions are employed to explore how these two ideas might affect the motion of a particle in curved spacetime. However, it is emphasized that this approach is simplified, and its results need to be validated and compared with more precise observations and analyses.

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