A Theoretical Investigation of an Extended Relativistic shell model of Mirror Nuclei ^{17}O and ^{17}F under Spin and Pseudo-spin Symmetries Conditions using the MQHP Model within the Framework of 3D-RNCS Symmetries

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Abstract

The present paper deals with the solutions of the 3-dimensional modified Dirac equation (MDE) for the extended relativistic interactions for nuclei ^{17}O and ^{17}F under the modified quadratic Hellmann potential (MQHP) model within Bopp's shift method and standard perturbation theory framework. The two mirror nuclei can be modeled as doubly magic isotopes $^{17}O = n + (N = Z = 8)$ and $^{17}F = p + (N = Z = 8)$, with one additional nucleon (valence) in the $1d_{5/2}$ level under the MQHP model in 3-dimensional relativistic non-commutative space (3D-RNCS) symmetries. The new relativistic energy eigenvalues for the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$ and the nY_j excited state is obtained by adopting Bopp's shift method and using the standard perturbation theory. The corresponding modified Hamiltonian operator has been calculated in 3D-RNCS symmetries. It is found that solutions of the new spectrum can be expressed by the discrete subatomic quantum numbers $(j, k, l(\tilde{l}), s(\tilde{s})$ and $m(\tilde{m}))$, the strength parameters (a, b), the range of studied potential α in addition to non-commutativity parameters (Θ, θ), which are induced by the effects of (space-space) non-commutativity properties. The total complete degeneracy of new relativistic energy levels for nuclei ^{17}O and ^{17}F under the MQHP model changed to become equal to the value $2n^2$ instead of the values n^2 in ordinary relativistic quantum mechanics, which is known in the literature.

Keywords: Dirac equation; mirror nuclei ^{17}O and ^{17}F; the quadratic Hellmann potential; non-commutative space phase, and Bopp's shift method.

1. Introduction

The Dirac equation (DE) is significant in relativistic quantum mechanics symmetries because the eigensolutions reveal crucial physical information about the quantum systems for relativistic quantum problems. This equation is well known to describe the motion of a spin-1/2 particle, such as an electron or positron, at high energy on the atomic scale, and research into this equation is currently a hot topic in particle physics and nuclear physics. However, this equation can be extended to describe other physics phenomena on subatomic scales because spin and pseudo-spin symmetries of the relativistic Hamiltonian have recently been empirically recognized in nuclear and hadronic spectroscopes. Therefore, the symmetries in the single-particle spectra of nuclei are the most critical concepts in nuclear structure [1–4]. The study of isotopes at the subatomic scale has attracted considerable interest in both theoretical and experimental physics, and it has many vital applications in both theoretical and practical research. Isotopes include the two nuclei ^{17}O and ^{17}F, which are suitable examples. Due to its significant experimental findings on binding energy, single-particle energy, etc.. To put the microscopic theory to the test in future investigations, it is helpful to calculate these quantities [5-7]. Mousavi *et al.* [6] solved the Schrödinger equation and the Dirac equation with the quadratic Hellmann potential model (QHP) using the parametric Nikiforov-Uvarov method and obtained energy eigenvalues and wave functions for the mirror nuclei of ^{17}O and ^{17}F in relativistic and nonrelativistic shell models. These isotopes can be modeled as a doubly magic isotope $^{17}O = n + (N = Z = 8)$ and $^{17}F = p + (N = Z = 8)$ with one additional nucleon (valence) in the $1d_{5/2}$ level. The ground-state spin and parity of (^{17}O and ^{17}F) are $j^{\Pi} = 5/2^{+}$, which correspond to the spin and parity of the level where the valence nucleon resides [6-7]. In the relativistic shell model with modified Eckart plus Hulthén potentials for the interaction between the core and a single nucleon, Mousavi *et al.* (2016) [8] analyzed various static features of ^{41}Ca and ^{41}Sc. They computed the energy values and wave function. To examine the interaction between the core and the single nucleon for elements ^{41}Ca and ^{40}Ca, Mousavi *et al.* [9] obtained the energy levels and charge radius for the stability line nucleus in 2017 using Eckart plus Hulthén potentials and a relativistic shell model that took into account a closed shell for each nucleus containing a double magic number and a single nucleon energy level. The main objective is to develop the research article for Mousavi *et al.* in [6] and expand it to the significant symmetry known by non-commutative quantum mechanics (NCQM) in the case of spin and pseudo-spin symmetry conditions to achieve a more accurate physical vision so that this study becomes valid in the field of nanotechnology. This is to achieve a comprehensive study highlighting the topological effects resulting from the deformation in space due to the mass effect of matter. The researchers believe that highlighting the impact of these new symmetries would give greater clarity and perception and could address some of the obstacles in which quantum mechanics failed, such as the problem of the normalization and unification of the cosmic four forces. On the other hand, one can explore the possibility of creating new applications and more profound interpretations in the subatomic and nanoscales using a new version of the QHP. We called it the modified quadratic Hellmann potential (MQHP) model because these combined potentials are significant nuclear potentials for a description of the interaction between nucleons, which has the following form:

$$\begin{cases} V_{hp}(\hat{r}) = V_{qh}(r) + V_{qh}^{ind}(r,a,b,\alpha)\vec{\mathbf{L}}\vec{\mathbf{\Theta}} + O(\Theta^2), \\ S_{hp}(\hat{r}) = S_{qh}(r) + V_{qh}^{ind}(r,a,b,\alpha)\vec{\tilde{\mathbf{L}}}\vec{\mathbf{\Theta}} + O(\Theta^2). \end{cases} \tag{1}$$

The attractive quadratic scalar potential $S_{qh}(r)$ and a repulsive potential $V_{qh}(r)$ QHP is given by [6,10,11]

$$\begin{cases} V_{qh}(r) = -\frac{a}{r} + \frac{be^{-\alpha r}}{r^2}, \\ S_{qh}(r) = -\frac{a_s}{r} + \frac{b_s e^{-\alpha r}}{r^2}. \end{cases} \tag{2}$$

While the induced potential $V_{qh}^{ind}(r,a,b,\alpha)$ expressed as (see the third section):

$$V_{qh}^{ind}(r,a,b,\alpha) = \frac{\alpha b\, exp(-\alpha r)}{2r^3} + \frac{b\, exp(-\alpha r)}{r^4} - \frac{a}{2r^3} \tag{3}$$

where a/a_s and b/b_s are strength parameters, while α is related to the range of the potential. We refer to the couplings ($\vec{\mathrm{L}}\vec{\Theta}$ and $\vec{\tilde{\mathrm{L}}}\vec{\Theta}$) in the next section. The new structure of 3D-RNCS symmetries based on new non-commutative canonical commutations relations in three representations of Schrödinger, Heisenberg, and Interactions pictures (SP, HP, and IP), respectively, as follows (In this research, we applied the system of natural units $c = \hbar = 1$) [12-17]:

$$\begin{cases} [\hat{x}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = [\hat{x}_{I\mu}(t) \overset{*}{,} \hat{p}_{I\nu}(t)] = i\hbar\delta_{\mu\nu} \Rightarrow |\Delta\hat{x}_\mu \Delta\hat{p}_\nu| \geq \hbar_{eff}\frac{\delta_{\mu\nu}}{2}, \\ [\hat{x}_\mu \overset{*}{,} \hat{x}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)] = [\hat{x}_{I\mu}(t) \overset{*}{,} \hat{x}_{I\nu}(t)] = i\delta_{\mu\nu}\hbar_{eff} \Rightarrow |\Delta\hat{x}_\mu \Delta\hat{x}_\nu| \geq \frac{|\theta_{\mu\nu}|}{2}. \end{cases} \tag{4}$$

where the indices $(\mu, \nu) \equiv \overline{1,3}$ and $\hbar_{eff} \approx \hbar$. This means that the principle of uncertainty for Heisenberg is generalized to include another new uncertainty related to the positions $(\hat{x}_\mu, \hat{x}_\nu)$ in addition to the ordinary uncertainty of $(\hat{x}_\mu, \hat{p}_\nu)$. The minimal parameter $\theta^{\mu\nu}$ is invertible antisymmetric real constant (3×3) matrices, and $(*)$ denotes the Weyl-Moyal star product, which is generalized between two arbitrary functions to the new form $(fg)(x) \to (\hat{f}\hat{g})(\hat{x}) \equiv (f * g)(x)$ in 3D-RNCS symmetries [18-25]:

$$(f * g)(x) \approx \left(fg - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f \partial_\nu^x g\right)(x) \tag{5}$$

The second term in the above equation presents the effects of (space-space) non-commutativity properties. However, the new operators: $\hat{\mathcal{R}}_{\mu H}(t) = (\hat{x}_\mu \vee \hat{p}_\mu)(t)$ and $\hat{\mathcal{R}}_{\mu I}(t) = (\hat{x}_{I\mu} \vee \hat{p}_{I\mu})(t)$ in (HP and IP, respectively) are dependent on the corresponding new operator $\hat{\xi}_S = \hat{x}_\mu \vee \hat{p}_\mu$ in SP from the following projection relations:

$$\begin{cases} \mathcal{R}_H(t) = exp(i/\hbar\hat{H}_{qh}T)\mathcal{R}_S\, exp(-i/\hbar\hat{H}_{qh}T) \\ \mathcal{R}_I(t) = exp(i/\hbar\hat{H}_{oqh}T)\mathcal{R}_S\, exp(-i/\hbar\hat{H}_{oqh}T) \end{cases} \Rightarrow \begin{cases} \hat{\mathcal{R}}_H(t) = exp(i/\hbar_{eff}\hat{H}_{nc}^{qh}T) * \hat{\mathcal{R}}_S * exp(-i/\hbar_{eff}\hat{H}_{nc}^{qh}T) \\ \hat{\mathcal{R}}_I(t) = exp(i/\hbar_{eff}\hat{H}_{nco}^{qh}T) * \hat{R}_S * exp(-i/\hbar_{eff}\hat{H}_{nco}^{qh}T) \end{cases} \tag{6}$$

Here $= t - t_0$, $R_{\mu S} = x_\mu \vee p_\mu$, $\mathcal{R}_{\mu H}(t) = (x_\mu \vee p_\mu)(t)$ and $\mathcal{R}_{\mu I}(t) = (x_{I\mu} \vee p_{I\mu})(t)$ are three representations in RQM symmetry, the operators $\hat{H}_{oqh}$ and $\hat{H}_{qh}$ are the free and global Hamiltonian in RQM for quadratic Hellmann potential, while $\hat{H}_{nco}^{qh}$ and $\hat{H}_{nc}^{qh}$ the corresponding Hamiltonians in the 3D-RNCS symmetries. The dynamics of new systems $\frac{d\langle\hat{R}_H(t)\rangle}{\mathrm{dt}}$ which is described by the modified Ehrenfest theory from the following motion equations in 3D-RNCS symmetries:

$$\frac{d\langle\mathcal{R}_H(t)\rangle}{\mathrm{dt}} = -(i/\hbar)\left[\mathcal{R}_H(t),\widehat{H}_{qh}\right] + \langle\frac{\partial\mathcal{R}_H(t)}{\partial t}\rangle \Rightarrow \frac{d\langle\hat{R}_H(t)\rangle}{\mathrm{dt}} = -(i/\hbar_{eff})\left[\hat{R}_H(t)\overset{*}{,}\widehat{H}_{nc}^{qh}\right] + \langle\frac{\hat{R}_H(t)}{\partial t}\rangle \tag{7}$$

It should be recalled that Heisenberg first proposed non-commutativity in 1930 [22], and then Syndre confirmed it in 1947 [23]. The organization of this paper, which is divided into six sections, is as follows: We quickly go over the DE with the quadratic Hellmann potential in the following section. The MQHP model and the modified spin-orbit operator for the mirror nuclei of (^{17}O and ^{17}F) under spin (pseudo-spin) symmetry conditions are obtained by using Bopp's shift approach in Section 3 to examine the MDE. The ground state and various excited states for the studied mirror nucleus, the magnetic Hamiltonians for the MQHP model, and their related spectra are covered in the following section. Section 5 analyzes the global relativistic energy in 3D-NRNCS symmetries and the corresponding Hamiltonian operator under the MQHP model in the presence of spin and (pseudo-spin) symmetry conditions. We studied the nonrelativistic energy limit and compared it with our previous study. Finally, a concluding summary and conclusions are given in the last section.

2. Review of the DE for the quadratic Hellmann potential

Here, we present the basic concepts of DE under the quadratic Hellmann potential in outline form. We introduce the formalism of the DE for a spherically symmetric potential in 3-dimensional space reads for a single nucleon with the mass of M and relativistic energy $E_{nl}/E_{n\tilde{l}}^{p}$ moving in an attractive scalar potential $S_{qh}(r)$ and a repulsive potential $V_{qh}(r)$ as follows (Refs .[6, 23-25] gives a detailed description of this concepts):

$$(\alpha p + \beta(M + S(r)))\Psi(r,\theta,\phi) = \left(E - V_{qh}(r)\right)\Psi(r,\theta,\phi) \tag{8}$$

With $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & I_{2\times2} \end{pmatrix}$ while $(\sigma_1,\sigma_2,\sigma_3)$ are just the Dirac matrices. Thus, the corresponding ordinary Hamiltonian operator $\widehat{H}_{qh}$ can ban be expressed as:

$$\widehat{H}_{qh} = (\alpha p + \beta(M + S(r))) + V_{qh}(r) \tag{9}$$

The spinor $\Psi(r,\theta,\phi)$ can be written as [6, 23-25]:

$$\Psi(r,\theta,\phi) = \begin{pmatrix} f_{nk}(\vec{r}) \\ g_{n\tilde{k}}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{nk}(r)Y_{jm}^{l}(\theta,\phi) \\ iG_{n\tilde{k}}(r)Y_{j\tilde{m}}^{\tilde{l}}(\theta,\phi) \end{pmatrix} \tag{10}$$

Where, $F_{nk}(r)$ and $G_{n\tilde{k}}(r)$ are the upper and lower components of the Dirac spinor, $Y_{jm}^{l}(\theta,\phi)$ and $Y_{j\tilde{m}}^{\tilde{l}}(\theta,\phi)$ are the spin and pseudo-spin spherical harmonics, while k ($\tilde{k}$) is related to the total angular momentum quantum numbers for spin symmetry $l$and pseudo-spin symmetry $\tilde{l}$ as [25-26]:

$$k = \begin{cases} -(l+1) = -(j+12) \text{ if } ,(s_{1/2}, p_{3/2}, etc), \\ j = l + \frac{1}{2}, \text{ aligned spin } (k\langle 0) \\ +l = +(j+12) \text{ if}, (p_{1/2}, d_{3/2}, etc), \\ j = l - \frac{1}{2}, \text{ unaligned spin } (k\rangle 0) \end{cases} \tag{11}$$

and

$$\tilde{k} = \begin{cases} -\tilde{l} = -(j+12) \text{ if } (s_{1/2}, p_{3/2}, etc), \\ j = \tilde{l} - \frac{1}{2}, \text{ aligned -spin } (k\langle 0) \\ +(\tilde{l}+1) = +(j+12) \text{ if } ,(p_{1/2}, d_{3/2}, etc), \\ j = \tilde{l} + \frac{1}{2}, \text{ unaligned- spin } (k\rangle 0) \end{cases} \tag{12}$$

The two radial functions ($F_{nk}^{s}(r)$,$G_{n\tilde{k}}^{p}(r)$) are obtained by solving the following differential equations [27-29]:

$$\left(\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr}+\frac{k}{r}\right)}{M+E_{nk}-\Delta(r)} - \left(M + E_{nl} - \Delta^{qh}(r)\right)\left(M - E_{nk} + \Sigma^{qh}(r)\right)\right)F_{nk}^{s}(r) = 0 \tag{13}$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr}-\frac{k}{r}\right)}{M+E_{nk}-\Sigma(r)} - \left(M + E_{n\tilde{l}}^{p} - \Delta^{qh}(r)\right)\left(M - E_{nk} + \Sigma^{qh}(r)\right)\right)G_{nk}^{p}(r) = 0 \tag{14}$$

The QHP bound-state solutions for the spin symmetric and pseudo-spin symmetry when ($\frac{d\Delta^{qh}(r)}{dr} = 0$ and $\frac{d\Sigma^{qh}(r)}{dr} = 0$) are satisfied, respectively. The upper and lower components $F^{s}_{nk}(r)$ and $G^{p}_{nk}(r)$ of the Dirac spinor gives by [6]:

$$F^{s}_{nk}(r) = Nr^{(\sqrt{\chi_{0l}+1/4}+1/2)} exp\left(-\sqrt{\chi_{2n}}r\right) L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right) \tag{15}$$

and

$$G^{p}_{nk}(r) = N^{p}r^{(\sqrt{\chi^{p}_{0l}+1/4}+1/2)} exp\left(-\sqrt{\chi^{p}_{2n}}r\right) L_n^{2\sqrt{\chi^{p}_{0l}+1/4}}\left(\left(2+2\sqrt{\chi^{p}_{2n}}\right)r\right) \tag{16}$$

here N and N^{p} are the normalization constants, $\chi_{0k} = 2\mu + k(k+1)$, $\chi^{p}_{0k} = 2\mu + k(k-1)$, $\chi_{2n} = M^2 - E^2_{nl}$ and $\chi^{p}_{2n} = M^2 - E^{p2}_{nl}$. The relativistic positive energy eigenvalues for the MQHP model under the pseudo-spin-symmetry conditions are obtained as [6]:

$$(2n+1)\sqrt{M^2 - E^2_{nl}} - 2(E_{nl}+M)(a+2b\alpha) + 2\sqrt{(M^2 - E^2_{nl})(k(k+1)+2b(E_{nl}+M)+1/4)} = 0 \tag{17}$$

and

$$(2n+1)\sqrt{M^2 - E^{p2}_{nl}} - 2\left(E^{p}_{n\tilde{l}} - M\right)(a+2b\alpha) + 2\sqrt{\left(M^2 - E^{p2}_{nk}\right)\left(k(k-1)+2b\left(E^{p}_{nk} - M\right)+1/4\right)} = 0 \tag{18}$$

The lower component G^{s}_{nk} and the upper component $F^{p}_{nk}(r)$ of the Dirac spinor can be calculated as [6]:

$$(G^{s}_{nk}(r), F^{p}_{nk}(r)) = \frac{\left(\frac{d}{dr} \pm \frac{k}{r}\right)}{E^{s/p}_{n\tilde{l}} + M}(F^{s}_{nk}(r), G^{p}_{nk}(r))$$

3. NC Hamiltonian operator for relativistic MQHP model

3.1. Overview of the Bopp shift method

To find the MDE for the MQHP model in 3D-RNCS symmetries, we replace both the ordinary Hamiltonian operator $\hat{H}(p_i, x_i)$, ordinary spinor $\psi\left(\vec{r}\right)$, and ordinary energy E_{nk} with the NC Hamiltonian operator $\hat{H}(\hat{p}_i, \hat{x}_i)$, new spinor $\hat{\Psi}\left(\vec{r_n}\right)$, and new energy E^{qh}_{nc} and the ordinary product will be replaced by the star product $(*)$, respectively. This allows us to write the MED for MQHP as follows [31-44]:

$$\hat{H}(\hat{p}_i, \hat{x}_i)\hat{\Psi}\left(\vec{r_n}\right) = E^{qh}_{nc}\hat{\Psi}\left(\vec{r_n}\right) \Rightarrow H^{qh}_{nc} * \psi\left(\vec{r}\right) = E^{s/p}_{nl}\psi\left(\vec{r}\right) \tag{19}$$

her $(r_{noc} = r_n)$. Thus, in 3D-RNCS symmetries, the upper component $F^{s}_{nk}(r)$ and lower component $G^{p}_{nk}(r)$ of the Dirac spinor, which corresponds to spin and pseudo-spin symmetry, can be written in the following form:

$$\left(\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nl} - C^{s})\left(M - E_{nk} + \Sigma^{qh}(r)\right)\right) * F^{s}_{nk}(r) \tag{20}$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \left(M + E^{p}_{n\tilde{l}} - \Delta^{qh}(r)\right)(M - E_{nk} + C^{p})\right) * G^{p}_{nk}(r) = 0 \tag{21}$$

The Bopp shift method was discussed in detail in [45-47]. Here, we will mention the main points to remind the reader of the main idea of the Bopp shift method. Bopp shift method is usually used to transform the fundamental four equations deformed Klein-Gordon equation [48], the deformed Dirac equation [49], the deformed Schrödinger equation [50,51]., and the Duffin–Kemmer–Petiau equation [52] with the notion of star product to the Klein-Gordon equation, the Dirac equation and the Schrödinger equation with the idea of ordinary product. It is worth noting that the Bopp shift method permutes us to reduce the above equations to the simplest form:

$$\left(\frac{d^2}{dr^2} - \frac{k(k+1)}{{r_n}^2} - (M + E_{ni} - C^{s})(M - E_{nk} + \Sigma^{qh}(r_n))\right) F^{s}_{nk}(r) = 0 \tag{22}$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{{r_n}^2} - (M + E^{p}_{n\tilde{l}} - \Delta^{qh}(r_n))(M - E_{nk} + C^{p})\right) G^{p}_{nk}(r) = 0 \tag{23}$$

The new operator of the Hamiltonian operator $H^{qh}_{nc}(\hat{p}_i, \hat{x}_i)$ can be expressed in three general varieties: both NC space and NC phase (3D-RNCPS) symmetries, only NC space (3D-RNCS) symmetries, and only NC phase (3D-RNCP) symmetries, respectively:

$$\begin{cases} H_{nc}^{qh}(\hat{p}_\mu,\hat{x}_\nu) \equiv H\left(\hat{p}_\mu = p_\mu + i\frac{\overline{\theta}_{\mu\nu}}{2}x_\nu; \hat{x}_\mu = x_\mu - i\frac{\theta_{\mu\nu}}{2}p_\nu\right) \text{ for } \quad 3D-\text{RNCPS symmetries} \\ H_{nc}^{qh}(\hat{p}_\mu,\hat{x}_\nu) \equiv H\left(\hat{p}_\mu = p_\mu; \hat{x}_\mu = x_\mu - \frac{i\theta_{\mu\nu}}{2}p_\nu\right) \text{ for } \quad 3D-\text{RNCS} \quad \text{symmetries} \\ H_{nc}^{qh}(\hat{p}_\mu,\hat{x}_\nu) \equiv H\left(\hat{p}_\mu = p_\mu + i\frac{\overline{\theta}_{\mu\nu}}{2}x_\nu; \hat{x}_\mu = x_\mu\right) \text{ for } \quad 3D-\text{RNCP} \quad \text{symmetries} \end{cases} \tag{24}$$

In recent work, we are interested in applying the second variety. Therefore, the modified Hamiltonian $H_{nc}^{qh}(\hat{p}_\mu,\hat{x}_\nu)$ defined as a function of ($\hat{x}_\mu = x_\mu - i\frac{\theta_{\mu\nu}}{2}p_\nu$ and $\hat{p}_\mu = p_\mu$) as follows:

$$H_{nc}^{qh}(\hat{p}_\mu,\hat{x}_\nu) = \alpha\hat{P} + \beta(M + S_{qh}(r_{noc})) + V_{qh}(\hat{r}) \tag{25}$$

where the MQHP $V_{qh}(\hat{r})$ is given by:

$$V_{qh}(r) \Rightarrow V_{qh}(r_n) = -\frac{a}{r_n} + \frac{b}{{r_n}^2}e^{-\alpha\hat{r}} \tag{26}$$

To obtain new centrifugal terms ($\frac{k(k+1)}{{r_n}^2}$, $\frac{k(k-1)}{{r_n}^2}$), $\Sigma(\hat{r})$ and $\Delta(\hat{r})$, we need to calculate ($-\frac{a}{\hat{r}}, \frac{b}{\hat{r}^2}, \frac{\text{bexp}(-\alpha r_n)}{{r_n}^2}$ and $\frac{k(k+1)}{{r_n}^2}$) can obtain for spin symmetry the following equations as follows in 3D-RNCS symmetries:

$$-\frac{a}{r_n} = -\frac{a}{r} - \frac{a}{2r^3}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2), \tag{27.1}$$

$$\frac{b}{{r_n}^2} = \frac{b}{r^2} + \frac{b}{r^4}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2), \tag{27.2}$$

$$\frac{\text{bexp}(-\alpha r_n)}{{r_n}^2} = \frac{\text{bexp}(-\alpha r)}{r^2} + \left(\frac{\alpha b}{2r^3} + \frac{b}{r^4}\right)\exp(-\alpha r)\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2), \tag{27.3}$$

and

$$\frac{k(k+1)}{{r_n}^2} = \frac{k(k+1)}{r^2} + \frac{k(k+1)}{r^4}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2). \tag{27.4}$$

Similarly, for pseudo-spin symmetry, the previous values can be expressed as

$$-\frac{a}{r_n} = -\frac{a}{r} - \frac{a}{2r^3}\vec{\tilde{\mathbf{L}}}.\vec{\mathbf{\Theta}} + O(\Theta^2), \tag{28.1}$$

$$\frac{b}{{r_n}^2} = \frac{b}{r^2} + \frac{b}{r^4}\vec{\tilde{\mathbf{L}}}.\vec{\mathbf{\Theta}} + O(\Theta^2), \tag{28.2}$$

$$\frac{\text{bexp}(-\alpha r_n)}{{r_n}^2} = \frac{\text{bexp}(-\alpha r)}{r^2} + \left(\frac{\alpha b}{2r^3} + \frac{b}{r^4}\right)\exp(-\alpha r)\vec{\tilde{\mathbf{L}}}.\vec{\mathbf{\Theta}} + O(\Theta^2), \tag{28.3}$$

and

$$\frac{k(k-1}{{r_n}^2} = \frac{k(k-1}{r^2} + \frac{k(k-1}{r^4}\vec{\tilde{\mathbf{L}}}.\vec{\mathbf{\Theta}} + O(\Theta^2). \tag{28.4}$$

We are substituting Eqs. (27) and (28) into Eqs. (22) and (23), we get an expression for two equations:

$$\left(\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nl} - C^s)\left(M - E_{nk} + \Sigma^{qh}(r)\right) - \Sigma_{pert}^{qh}(r)\right)F_{nk}^s(r) = 0 \tag{29}$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \left(M + E_{nk}^p + \Delta^{qh}(\hat{r})\right)(M - E_{nk} + C^p) - \Delta_{pert}^{qh}(r)\right)G_{n\tilde{k}}^p(r) = 0 \tag{30}$$

with

$$\Sigma_{pert}^{qh}(r) = \left[(M + E_{nl} - C^s)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right]\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2) \text{ for } \quad \text{spin symmetric case,} \tag{31.1}$$

and

$$\Delta_{pert}^{qh}(r) = \left[\left(M - E_{nk}^{p} + C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k-1)}{r^4}\right]\vec{\tilde{\mathbf{L}}}.\vec{\mathbf{\Theta}} + O(\Theta^2) \text{ for pseudo-spin symmetric case.} \quad (31.2)$$

Here $V_{qh}^{ind}(r,a,b,\alpha)$ is the induced potential that we have previously seen in Eq. (3). By comparing (Eqs. (13) and (14)) and (Eqs. (29) and (30)), we observe two additive potentials ($\Sigma_{pert}^{qh}(r)$ and $\Delta_{pert}^{qh}(r)$). Moreover, these terms are proportional to the infinitesimal non-commutativity couplings ($\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\vec{\tilde{\mathbf{L}}}\vec{\mathbf{\Theta}}$). From a physical point of view, this means that these two spontaneously generated terms ($\Sigma_{pert}^{qh}(r)$ and $\Delta_{pert}^{qh}(r)$) as a result, the topological properties of the deformation space-space can be considered very small compared to the fundamental terms ($\Sigma^{qh}(r)$ and $\Delta^{qh}(r)$), respectively. Furthermore, using the unit step function (also known as the Heaviside step function $\theta(z)$ or simply the theta function) to rewrite the two global induced two potentials ($\Sigma_{pert}^{qh}(r)$ and $\Delta_{pert}^{qh}(r)$) for spin and pseudo-spin symmetries corresponding to upper and lower components ($F_{nk}^{s}(s)$, $G_{nk}^{s}(s)$) and ($F_{nk}^{p}(s)$, $G_{nk}^{p}(s)$), respectively as

$$\Sigma_{qh}^{pert}(r) = \Sigma_{qh}^{pert}(r)\theta\left(E_{nc}^{qh-s}\right) - \Sigma_{qh}^{pert}(r)\theta\left(-\left|E_{nc}^{qh-s}\right|\right) = \begin{cases}\Sigma_{qh}^{pert}(r) & \text{for } F_{nk}^{s}(r)\\ -\Sigma_{qh}^{pert}(r) & \text{for } G_{nk}^{s}(r)\end{cases} \quad (32)$$

and

$$\Delta_{t_{qh}}^{pert}(r) = \Delta_{qh}^{pert}(r)\theta\left(E_{nc}^{qh-ps}\right) - \Delta_{sc}^{pert}(r)\theta\left(-\left|E_{nc}^{qh-p}\right|\right) = \begin{cases}\Delta_{qh}^{pert}(r) & \text{for } F_{nk}^{p}(r)\\ -\Delta_{qh}^{pert}(r) & \text{for } G_{nk}^{p}(r)\end{cases} \quad (33)$$

Here, the step function $\theta(z)$ is given by,

$$\theta(z) = \begin{cases}1 \text{ for } z \geq 0\\ 0 \text{ for } z < 0\end{cases} \quad (34)$$

The quadratic Hellmann potential is extended by including new additive potentials ($\Sigma_{pert}^{qh}(r)$ and $\Delta_{pert}^{qh}(r)$) expressed to the radial terms ($\frac{exp(-\alpha r)}{r^3}$, $\frac{exp(-\alpha r)}{r^4}$, $\frac{1}{r^3}$ and $\frac{1}{r^4}$) which are coupled with two couplings ($\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\vec{\tilde{\mathbf{L}}}\vec{\mathbf{\Theta}}$) to become the improved quadratic Hellmann potential in 3D-RNCS symmetries. The two global induced potentials ($\Sigma_{t_{sc}}^{pert}(r)$and $\Delta_{t-sc}^{pert}(r)$) represent the physical interaction between the system's physical properties that correspond to spin and pseudo-spin symmetries ($\vec{\mathrm{L}}$ and $\vec{\tilde{\mathrm{L}}}$) and the distance between diatomic molecules r with topological deformations of space-space characterized by non-commutativity vector $\vec{\Theta}$. The newly generated two effective potentials ($\Sigma_{sc}^{pert}(r)$ and $\Delta_{sc}^{pert}(r)$) are also proportional to the two infinitesimal couplings ($\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\vec{\tilde{\mathrm{L}}}\vec{\Theta}$). This allows us to consider the new additive parts of the effective potentials ($\Sigma_{sc}^{pert}(r)$ and $\Delta_{sc}^{pert}(r)$) as perturbation potentials compared with the main potentials ($\Sigma_{sc}(r)$ and $\Delta_{sc}(r)$) which are also known with the parent potential operator in the symmetries of MDT, that is, the two inequalities ($\Sigma_{sc}^{pert}(r) << \Sigma_{sc}(r)$ and $\Delta_{sc}^{pert}(r) << \Delta_{sc}(r)$) have been achieved. All physical justifications for applying the time-independent perturbation theory become satisfied when calculating the expectation values of previous radial terms. This allows us to give a complete prescription for determining the energy level of the generalized nY_j excited states. The aim is to derive the energy spectrum for a mirror nucleus (^{17}O and ^{17}F) with one additional nucleon (valence) in the $1d_{5/2}$ level and other excited states, for example, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$and the generalized excited states nY_j which are characterized by discrete quantum numbers $\left(n,j,l,\tilde{l},s,\tilde{s},m,\tilde{m}\right)^{th}$ in the presence of a potential given by (27) under spin (pseudo-spin) symmetry conditions.

4. The perturbed relativistic spin-orbit Hamiltonian and the corresponding spectrum for the MQHP model for the ground state and other excited states for the mirror nucleus ^{17}O and^{17}F under spin (pseudo-spin) symmetry conditions in 3D-RNCS symmetries

4.1. *The perturbed relativistic spin-orbit Hamiltonian for mirror nuclei ^{17}O and ^{17}Funder the MQHP in (RNC: 3D-RS) symmetries*

The results (26) can be rewritten in a more accessible physical form, and we replace both ($\vec{\mathrm{L}}\vec{\Theta}$ and $\vec{\tilde{\mathrm{L}}}\vec{\Theta}$) with ($\vec{\mathbf{L}}\vec{\mathbf{S}}$ and $\vec{\tilde{\mathbf{L}}}\vec{\tilde{\mathbf{S}}}$), respectively, and then the two perturbative terms $\Sigma_{pert}^{qh}(r)$ and $\Delta_{pert}^{qh}(r)$ for the spin (pseudo-spin) symmetry conditions, respectively, can be rewritten to the equivalent new form for MQHP as follows:

$$\begin{cases}\Sigma_{pert}^{qh}(r,\Theta,a,b,\alpha) = \left[\left(M + E_{nk} - C^{s}\right)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right]\Theta\vec{\mathbf{L}}.\vec{\mathbf{S}} & \text{for spin symmetric case,}\\ \Delta_{pert}^{qh}(r,\Theta,a,b,\alpha) = \left[\left(M - E_{nk}^{p} + C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k-1)}{r^4}\right]\Theta\vec{\tilde{\mathbf{L}}}.\vec{\tilde{\mathbf{S}}} & \text{for pseudo-spin symmetric case.}\end{cases} \quad (35)$$

Furthermore, the above perturbative terms $\Sigma_{pert}^{qh}(r,\Theta,a,b,\alpha)$ and $\Delta_{pert}^{qh}(r,\Theta,a,b,\alpha)$ can be rewritten to the following new equivalent form:

$$\begin{cases}\Sigma_{pert}^{qh}(r,\Theta,a,b,\alpha)=\left[(M+E_{nk}-C^{s})V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k+1)}{r^4}\right]\Theta G^2 \quad \text{for} \quad \text{spin symmetric case,}\\ \Delta_{pert}^{qh}(r,\Theta,a,b,\alpha)=\left[\left(M-E_{nk}^{p}+C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k-1)}{r^4}\right]\Theta \tilde{G}^2 \quad \text{for} \quad \text{pseudo-spin symmetric case.}\end{cases} \tag{36}$$

with

$$\begin{cases}\mathrm{G}^2=\frac{1}{2}\left(\vec{\mathrm{J}}^2-\vec{\mathrm{L}}^2-\vec{\mathrm{S}}^2\right)\\ \tilde{\mathrm{G}}^2=\frac{1}{2}\left(\vec{\mathrm{J}}^2-\vec{\tilde{\mathrm{L}}}^2-\vec{\tilde{\mathrm{S}}}^2\right)\end{cases} \tag{37}$$

To the best of our knowledge, we just replaced the coupling spin-orbit (pseudo-spin-orbit) $\vec{\mathbf{L}}.\vec{\mathbf{S}}$ and $\vec{\tilde{\mathbf{L}}}.\vec{\tilde{\mathbf{S}}}$ with the two expressions $\frac{1}{2}\left(\vec{\mathrm{J}}^2-\vec{\mathrm{L}}^2-\vec{\mathrm{S}}^2\right)$ and $\frac{1}{2}\left(\vec{\mathrm{J}}^2-\vec{\tilde{\mathrm{L}}}^2-\vec{\tilde{\mathrm{S}}}^2\right)$, respectively. In relativistic quantum mechanics. The set ($\mathrm{H}_{nc}^{qh}(\hat{p}_i,\hat{x}_i)$, $\vec{\mathrm{J}}^2$, $\vec{\mathrm{L}}^2$, $\vec{\tilde{\mathrm{L}}}^2$, $\vec{\mathrm{S}}^2$, $\vec{\tilde{\mathrm{S}}}^2$ and J_z) forms a complete of conserved physics quantities, and the spin-orbit quantum number k ($\tilde{k}$) is related to the quantum numbers for spin symmetry $l$and pseudo-spin symmetry $\tilde{l}$ as represented in Eqs. (11) and (12). In this case, we can form two diagonals (3×3) matrixes $\hat{H}_{so}^{qh}$ and $\hat{\tilde{H}}_{so}^{qh}$, for MQHP, respectively, in 3D-RNCS symmetries as:

$$\begin{cases}\left(\hat{H}_{so}^{qh}\right)_{11}(k_1)=\left[(M+E_{nk}-C^{s})V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k+1)}{r^4}\right]k_1\Theta\\ \text{for the states} \quad \left(s_{1/2},p_{3/2},etc\right), j=l+\frac{1}{2}, \text{aligned spin } (k\langle 0)\\ \left(\hat{H}_{so}^{qh}\right)_{22}(k_2)=\left[(M+E_{nk}-C^{s})V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k+1)}{r^4}\right]k_2\Theta\\ \text{for the states} \quad \left(p_{1/2},d_{3/2},etc\right), j=l-\frac{1}{2}, \text{unaligned spin } (k\rangle 0)\end{cases} \tag{38}$$

and

$$\begin{cases}\left(\hat{\tilde{H}}_{so}^{qh}\right)_{11}(\tilde{k}_1)=\left[\left(M-E_{nk}^{p}+C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k-1)}{r^4}\right]\tilde{k}_1\Theta\\ \text{for the states} \quad \left(s_{1/2},p_{3/2},etc\right), j=\tilde{l}-\frac{1}{2}, \text{aligned spin } (k\langle 0)\\ \left(\hat{\tilde{H}}_{so}^{qh}\right)_{22}(\tilde{k}_2)=\left[\left(M-E_{nk}^{p}+C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k-1)}{r^4}\right]\tilde{k}_2\Theta\\ \text{for the states} \quad \left(p_{1/2},d_{3/2},etc\right), j=\tilde{l}+\frac{1}{2}, \text{unaligned spin } (k\rangle 0)\end{cases} \tag{39}$$

while $\left(\hat{H}_{so}^{qh}\right)_{33}=\left(\hat{\tilde{H}}_{so}^{qh}\right)_{33}=0$ in the two above cases.

4.2. *The perturbed relativistic spin-orbit spectrum for mirror nucleus* (^{17}O and ^{17}F) *under MQHP in the presence of spin symmetry conditions in 3D-RNCS symmetries*

In this subsection, we will study the modifications to the energy levels ($E_{nc}^{per:u}(\Theta,k)$,$E_{nc}^{per:d}(\Theta,k)$ $E_{nc}^{per:u}(\Theta,\tilde{k})$ and $E_{nc}^{per:d}(\Theta,\tilde{k})$ for ($j=l\pm\frac{1}{2}$: spin-up/down for spin symmetry) and ($j=\tilde{l}\pm\frac{1}{2}$: spin-up/down for pseudo-spin symmetry), at the first order of the infinitesimal parameter Θ, for $\left(n,j,l,\tilde{l},s,\tilde{s},m,\tilde{m}\right)^{th}$ excited states under the spin (pseudo-spin) symmetry conditions, created by the effect of the relativistic spin-orbit operator**,** obtained by applying the standard perturbation theory, using Eqs. (10) and (36) as:

$$\begin{cases}E_{nc}^{per:u}(\Theta,k_1)\equiv\theta\left(E_{nc}^{qh}\right)k_1\Theta\int F_{nk}^{s\,*}(r)\left[(M+E_{nk}-C^{s})V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k+1)}{r^4}\right]F_{nk}^{s}(r)dr\\ E_{nc}^{per:d}(\Theta,k_2)\equiv\theta\left(E_{nc}^{qh}\right)k_2\Theta\int F_{nk}^{s\,*}(r)\left[(M+E_{nk}-C^{s})V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k+1)}{r^4}\right]F_{nk}^{s}(r)dr\end{cases} \tag{40}$$

and

$$\begin{cases}E_{nc}^{per:u}(\Theta,k_1)\equiv\theta\left(E_{nc}^{qh}\right)\tilde{k}_1\Theta\int G_{nk}^{s\,*}(r)\left[\left(M+E_{nk}^{p}-C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k-1)}{r^4}\right]F_{nk}^{s}(r)dr\\ E_{nc}^{per:d}(\Theta,k_2)\equiv\theta\left(E_{nc}^{qh}\right)\tilde{k}_2\Theta\int G_{nk}^{p\,*}(r)\left[\left(M+E_{nk}^{p}-C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k-1)}{r^4}\right]G_{nk}^{p}(r)dr\end{cases} \tag{41}$$

It is necessary to apply the orthogonality property of spherical harmonics:

$$\int Y_l^m(\theta,\phi) Y_{l'}^{m'}(\theta,\phi)\, sin(\theta)\, d\theta d\phi = \delta_{ll'}\delta_{mm'}$$

To obtain the explicit expressions of modified energy eigenvalues ($E_{nc}^{per:u}(\Theta,k_1)$,$E_{nc}^{per:d}(\Theta,k_2)$) for the MDE with the MQHP model under spin symmetry conditions, we use Eqs. (15) and (40):

$$E_{nc}^{per:u}(\Theta,k_1) \equiv N^2 k_1 \Theta (M + E_{nk} - C^s)$$
$$\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}+1)}\, exp\left(-2\sqrt{\chi_{2n}} r\right) \left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right)\right]^2 \left(V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right) dr \tag{42.1}$$

and

$$E_{nc}^{per:d}(\Theta,k_2) \equiv N^2 k_2 \Theta (M + E_{nk} - C^s)$$
$$\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}+1)}\, exp\left(-2\sqrt{\chi_{2n}} r\right) L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right) \left(V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right) dr \tag{42.2}$$

Now, we can rewrite the above equations to the simplified new form:

$$E_{nc}^{per:u}(\Theta,k_1,n,l,j) \equiv k_1 \Theta N^2 (M + E_{nk} - C^s) \sum_{\alpha=1}^{4} R_\alpha(n,l) \tag{43.1}$$

and

$$E_{nc}^{per:d}(\Theta,k_1,n,l,j) \equiv k_2 \Theta N^2 (M + E_{nk} - C^s) \sum_{\alpha=1}^{4} R_\alpha(n,l) \tag{43.2}$$

The expressions of the 4-factors $R_i\left(i=\overline{1,4}\right)$ are presented as follows:

$$\begin{cases} R_1(n,l) = \frac{\alpha b}{2}\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}-2)}\, exp\left(-\left(2\sqrt{\chi_{2n}}+\alpha\right)r\right) \left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr \\ R_2(n,l) = b\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}-3)}\, exp\left(-\left(2\sqrt{\chi_{2n}}+\alpha\right)r\right) \left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr \\ R_3(n,l) = -\frac{a}{2}\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}-2)}\, exp\left(-2\sqrt{\chi_{2n}} r\right) \left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr \\ R_4(n,l) = k(k+1)\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}-3)}\, exp\left(-2\sqrt{\chi_{2n}} r\right) \left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr \end{cases} \tag{44}$$

It should be noted that the ground state of the nuclei (^{17}O and ^{17}F) can be modeled as a doubly magic isotope ($^{17}O = n + (N = Z = 8)$ and $^{17}F = p + (N = Z = 8)$) with one additional nucleon (valence) in the $1d_{5/2}$ level. If identified with the typical state nY_j [53], the quantum numbers ($n = 1, Y \equiv d, j = 5/2$ and $l = 2$). Besides $k(k-1) = 6$, which gives $k = 3$. We have :

$$L_{n=1}^{2\sqrt{\chi_{02}(,k=3)+1/4}}\left(\left(2+2\sqrt{\chi_{21}(k=3)}\right)r\right) = -\Omega r + \Lambda$$

with

$$\left.\begin{array}{l} \chi_{21}(l=2) = M^2 - E_{12}^2 \\ \chi_{02}(k=3) = 2b\alpha + 12 \\ \Omega = 2\left(2+2\sqrt{\chi_{21}(k=3)}\right) \\ \Lambda = 2\sqrt{\chi_{02}(l=2)+1/4}+1 \end{array}\right\},$$

Which allows us to find the 4-factors $R_i\left(i=\overline{1,4}\right)$ as follows:

$$\begin{cases} R_1(1,l=2) = \frac{\alpha b}{2}\int_0^{+\infty} r^{(\delta_{02}-2)}\, exp(-(2\sqrt{\chi_{21}}+\alpha)r)\, [-\Omega r + \Lambda]^2 dr \\ R_2(1,l=2) = b\int_0^{+\infty} r^{(\delta_{02}-3)}\, exp(-(2\sqrt{\chi_{21}}+\alpha)r)\, [-\Omega r + \Lambda]^2 dr \\ R_3(1,l=2) = -\frac{a}{2}\int_0^{+\infty} r^{(\delta_{02}-2)}\, exp(-2\sqrt{\chi_{21}} r)\, [-\Omega r + \Lambda]^2 dr \\ R_4(1,l=2) = 12\int_0^{+\infty} r^{(\delta_{02}-3)}\, exp(-2\sqrt{\chi_{21}} r)\, [-\Omega r + \Lambda]^2 dr \end{cases} \tag{45}$$

Here

$$\delta_{02} = 2\sqrt{\chi_{02}(k=3)+1/4} \tag{46}$$

A direct simplification gives the 4-factors $R_i(1, l=2)$ $\left(i=\overline{1,4}\right)$ as follows:

$$\begin{cases} R_1(1,l=2)=\frac{\alpha b}{2}\int_0^{+\infty} exp(-(2\sqrt{\chi_{21}}+\alpha)r)\,(\Omega^2 r^{\delta_{02}+1-1}+2\Omega\Lambda r^{\delta_{02}-1}+\Lambda^2 r^{\delta_{02}-1-1})dr \\ R_2(1,l=2)=b\int_0^{+\infty} exp(-(2\sqrt{\chi_{21}}+\alpha)r)\,(\Omega^2 r^{\delta_{02}-1}+2\Omega\Lambda r^{\delta_{02}-1-1}+\Lambda^2 r^{\delta_{02}-2-1})dr \\ R_3(1,l=2)=-\frac{a}{2}\int_0^{+\infty} exp(-2\sqrt{\chi_{21}}r)\,(\Omega^2 r^{\delta_{02}+1-1}+2\Omega\Lambda r^{\delta_{02}-1}+\Lambda^2 r^{\delta_{02}-1-1})dr \\ R_4(1,l=2)=12\int_0^{+\infty} exp(-2\sqrt{\chi_{21}}r)\,(\Omega^2 r^{\delta_{02}-1}+2\Omega\Lambda r^{\delta_{02}-1-1}+\Lambda^2 r^{\delta_{02}-2-1})dr \end{cases} \tag{47}$$

Utilizing the following particular integral is practical [54]:

$$\int_0^{+\infty} z^{\varepsilon-1}\, exp(-\beta z^d)\, dz=\frac{\beta^{-\varepsilon/d}}{d}\Gamma(\varepsilon/d) \tag{48}$$

Here $\Gamma(\varepsilon/d)$ is the Gamma function and $(Re\,\beta\rangle 0$, $Re\,\varepsilon\rangle 0$, $d\rangle 0)$. Simple calculations can yield the following clear results:

$$\begin{cases} R_1(n=1,l=2)=\frac{\alpha b}{2}\left(\Omega^2{\beta_1}^{-(\delta_{02}+1)}\Gamma(\delta_{02}+1)+2\Omega\Lambda{\beta_1}^{-\delta_{02}}\Gamma(\delta_{02})+\Lambda^2{\beta_1}^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)\right) \\ R_2(n=1,l=2)=b\left(\Omega^2{\beta_1}^{-\delta_{02}}\Gamma(\delta_{02})+2\Omega\Lambda{\beta_1}^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)+\Lambda^2{\beta_1}^{-(\delta_{02}-2)}\Gamma(\delta_{02}-2)\right) \\ R_3(n=1,l=2)=-\frac{a}{2}\left(\Omega^2{\beta_2}^{-(\delta_{02}+1)}\Gamma(\delta_{02}+1)+2\Omega\Lambda{\beta_2}^{-\delta_{02}}\Gamma(\delta_{02})+\Lambda^2{\beta_2}^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)\right) \\ R_4(n=1,l=2)=12\left(\Omega^2{\beta_2}^{-\delta_{02}}\Gamma(\delta_{02})+2\Omega\Lambda{\beta_2}^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)+\Lambda^2{\beta_2}^{-(\delta_{02}-2)}\Gamma(\delta_{02}-2)\right) \end{cases} \tag{49}$$

with $\beta_1=2\sqrt{\chi_{21}}+\alpha$ and $\beta_2=2\sqrt{\chi_{21}}$. This allows us to obtain the exact modifications $E_{nc}^{per:u}(\Theta,k_1=-3{,}1,l=2,j=5/2)$ and $E_{nc}^{per:d}(\Theta,k_2=2{,}1,l=2,j=5/2)$ of the ground state for the nuclei (^{17}O and ^{17}F) with one additional nucleon (valence) in the $1d_{5/2}$ level for spin symmetry conditions:

$$\begin{cases} E_{nc}^{per:u}(\Theta,k_1=-3{,}1,l=2,j=5/2)\equiv -3\Theta N^2(M+E_{nk}-C^s)R_{11}(1,l=2) \\ E_{nc}^{per:d}(\Theta,k_2=2{,}1,l=2,j=5/2)\equiv 2\Theta N^2(M+E_{nk}-C^s)R_{11}(1,l=2) \end{cases} \tag{50}$$

with

$$R_{11}(1,l=2)=\sum_{\alpha=1}^{4}R_\alpha(1,l=2) \tag{51}$$

The $2S_{1/2}$ level corresponds to the one additional nucleon (valence) containing the first excited state. Therefore, the additional nucleon or the single nucleon (neutron or proton) corresponds to the subatomic quantum numbers ($n=2, Y\equiv S$, $j=1/2$ and $l=0$), then, to obtain the exact modifications $E_{nc}^{per:u}(\Theta,k_1=-1{,}2,l=0,j=1/2)$ and $E_{nc}^{per:d}(\Theta,k_2=0{,}2,l=0,j=1/2)$ for the first excited state, we replace $L_n^{2\sqrt{\chi_{0l}+1/4}}\left((2+2\sqrt{\chi_{2n}})r\right)$ in Eq. (44) by $L_{n=2}^{2\sqrt{\chi_{00}+1/4}}\left((2+2\sqrt{\chi_{22}})r\right)=h_1r^2+h_2r+h_3$ with $h_1=\frac{1}{2}(2+2\sqrt{\chi_{22}})^2$, $h_2=-\left(2\sqrt{\chi_{00}+1/4}+2\right)(2+2\sqrt{\chi_{22}})$, $h_3=\frac{1}{2}\left(2\sqrt{\chi_{00}+1/4}+1\right)\left(2\sqrt{\chi_{00}+1/4}+2\right)$ while $\chi_{00}(l=0,k=0)=2b\alpha$, or $\chi_{00}(l=0,k=1)=2b\alpha+2$ and $\chi_{22}(l=0)=M^2-E_{20}^2$, we obtain the following results:

$$\begin{cases} E_{nc}^{per:u}(\Theta,k_1=-1{,}2,l=0,j=5/2)\equiv -\Theta N^2(M+E_{nk}-C^s)R_{12}(2,l=0) \\ E_{nc}^{per:d}(\Theta,k_2=0{,}1,l=2,j=5/2)\equiv 0 \end{cases} \tag{52}$$

with

$$R_{12}(2,l=0)=\sum_{\alpha=1}^{4}R_\alpha(2,l=0) \tag{53}$$

We can express the 4-factors $R_1(2,l=0)$, $R_2(2,l=0)$, $R_3(2,l=0)$ and $R_4(2,l=0)$ as follows:

$$\begin{cases} R_1(2,l=0)=\frac{\alpha b}{2}\int_0^{+\infty} r^{(\delta_{00}-2)}\, exp(-(2\sqrt{\chi_{22}}+\alpha)r)\,(h_1r^2+h_2r+h_3)^2dr \\ R_2(2,l=0)=b\int_0^{+\infty} r^{(\delta_{00}-3)}\, exp(-(2\sqrt{\chi_{22}}+\alpha)r)\,(h_1r^2+h_2r+h_3)^2dr \\ R_3(2,l=0)=-\frac{a}{2}\int_0^{+\infty} r^{(\delta_{00}-2)}\, exp(-2\sqrt{\chi_{22}}r)\,(h_1r^2+h_2r+h_3)^2dr \\ R_4(2,l=0)=2\int_0^{+\infty} r^{(\delta_{00}-2)}\, exp(-2\sqrt{\chi_{22}}r)\,(h_1r^2+h_2r+h_3)^2dr \end{cases} \tag{54}$$

Here $\delta_{00}=2\sqrt{\chi_{00}+\frac{1}{4}}$. A simple calculation gives the $4-$ factors $R_i(2,l=0)$ $\left(i=\overline{1,4}\right)$ as follows :

$$\begin{cases} R_1(2,l=0)=\frac{\alpha b}{2}\int_0^{+\infty}exp(-(2\sqrt{\chi_{22}}+\alpha)r)\left(z_1r^{\delta_{00}+3-1}+z_2r^{\delta_{00}+2-1}+z_3r^{\delta_{00}+1-1}+z_4r^{\delta_{00}-1}+z_5r^{\delta_{00}-2}\right)dr \\ R_2(2,l=0)=b\int_0^{+\infty}exp(-(2\sqrt{\chi_{22}}+\alpha)r)\left(z_1r^{\delta_{00}+2-1}+z_2r^{\delta_{00}}+z_3r^{\delta_{00}-1}+z_4r^{\delta_{00}-1-1}+z_5r^{\delta_{00}-2-1}\right)dr \\ R_3(2,l=0)=-\frac{a}{2}\int_0^{+\infty}exp(-2\sqrt{\chi_{22}}r)\left(z_1r^{\delta_{00}+3-1}+z_2r^{\delta_{00}+2-1}+z_3r^{\delta_{00}+1-1}+z_4r^{\delta_{00}-1}+z_5r^{\delta_{00}-1-1}\right)dr \\ R_4(2,l=0)=2\int_0^{+\infty}exp(-2\sqrt{\chi_{22}}r)\left(z_1r^{\delta_{00}+3-1}+z_2r^{\delta_{00}+2-1}+z_3r^{\delta_{00}+1-1}+z_4r^{\delta_{00}-1}+z_5r^{\delta_{00}-1-1}\right)dr \end{cases} \quad (55)$$

with $(g_1,g_2,g_3,g_4,g_5)=({h_1}^2,2h_1h_2,2h_1h_3+{h_2}^2,2h_3h_2,{h_3}^2)$. Applying the special integral from Eq. (48) allows it straightforward to obtain the four factors $R_1(2,l=0)$, $R_2(2,l=0)$, $R_3(2,l=0)$ and $R_4(2,l=0)$ as follows:

$$\begin{cases} R_1(2,l=0)=\frac{\alpha b}{2}\begin{Bmatrix} g_1{\lambda_1}^{-(\delta_{00}+3)}\Gamma(\delta_{00}+3)+g_2{\lambda_1}^{-(\delta_{00}+2)}\Gamma(\delta_{00}+2)+g_3{\lambda_1}^{-(\delta_{00}+1)}\Gamma(\delta_{00}+1) \\ +g_4{\lambda_1}^{-\delta_{00}}\Gamma(\delta_{00})+g_5{\lambda_1}^{-(\delta_{00}+3)-1}\Gamma(\delta_{00}-1) \end{Bmatrix}, \\ R_2(2,l=0)=b\begin{Bmatrix} g_1{\lambda_1}^{-(\delta_{00}+2)}\Gamma(\delta_{00}+2)+g_2{\lambda_1}^{-(\delta_{00}+1)}\Gamma(\delta_{00}+1)+g_3{\lambda_1}^{-\delta_{00}}\Gamma(\delta_{00}) \\ +g_4{\lambda_1}^{-(\delta_{00}-1)}\Gamma(\delta_{00}-1)+g_5{\lambda_1}^{-(\delta_{00}-4)}\Gamma(\delta_{00}-2) \end{Bmatrix}, \\ R_3(2,l=0)=-\frac{a}{2}\begin{Bmatrix} g_1{\lambda_2}^{-(\delta_{00}+3)}\Gamma(\delta_{00}+3)+g_2{\lambda_2}^{-(\delta_{00}+2)}\Gamma(\delta_{00}+2)+g_3{\lambda_2}^{-(\delta_{00}+1)}\Gamma(\delta_{00}+1) \\ +g_4{\lambda_2}^{-\delta_{00}}\Gamma(\delta_{00})+g_5{\lambda_2}^{-(\delta_{00}-1)}\Gamma(\delta_{00}-1) \end{Bmatrix}, \\ R_4(2,l=0)=2\begin{Bmatrix} g_1{\lambda_2}^{-(\delta_{00}+3)}\Gamma(\delta_{00}+3)+g_2{\lambda_2}^{-(\delta_{00}+2)}\Gamma(\delta_{00}+2)+g_3{\lambda_2}^{-(\delta_{00}+1)}\Gamma(\delta_{00}+1) \\ +g_4{\lambda_2}^{-\delta_{00}}\Gamma(\delta_{00})+g_5{\lambda_2}^{-(\delta_{00}-1)}\Gamma(\delta_{00}-1) \end{Bmatrix}. \end{cases} \quad (56)$$

with $\lambda_1=2\sqrt{\chi_{22}}+\alpha$ and $\lambda_2=2\sqrt{\chi_{22}}$. The second excited state corresponds to the nuclei ^{17}O and $^{17}F$with an additional nucleon (valence) in the $1d_{3/2}$ level, thus, the additional nucleon or the single nucleon (neutron or proton) corresponds to subatomic quantum numbers ($n=1, Y\equiv d$, $j=3/2$ and $l=1$), we replace $L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}(l)}\right)r\right)$ in Eq. (44) by $L_{n=1}^{2\sqrt{\chi_{01}(k=2)+1/4}}\left(\left(2+2\sqrt{\chi_{21}(l=1)}\right)r\right)=\Omega_2r+\Lambda_2$, with $\Omega_2=\left(2+2\sqrt{\chi_{21}(l=1)}\right)$, $\Lambda_2=2\sqrt{\chi_{01}(k=2)+1/4}+1$, $\chi_{01}(k=2)=2b\alpha+6$ and $\chi_{21}(l=1)=M^2-{E_{11}}^2$, then, the exact modifications of the energy levels $E_{nc}^{per:u}(\Theta,k_1=-2,n=1,l=1,j=3/2)$ and $E_{nc}^{per:d}(\Theta,k_2=1,n=1,l=1,j=5/2)$ are given by:

$$\begin{cases} E_{nc}^{per:u}(\Theta,k_1=-2,1,l=1,j=3/2)\equiv-2\Theta N^2(M+E_{nk}-C^s)R_{11}(1,l=1) \\ E_{nc}^{per:ud}(\Theta,k_2=1,1,l=1,j=5/2)\equiv\Theta N^2(M+E_{nk}-C^s)R_{11}(1,l=1) \end{cases} \quad (57)$$

with $R_{11}(n=1,l=1)=R_{11}(n=1,l=2)(\Omega\to\Omega_2$ and $\Lambda\to\Lambda_2)$. Now, the $(n,k,j,l)^{th}$ excited states of the nuclei (^{17}O and ^{17}F) with one additional nucleon (valence) in the nY_j level, under the MQHP model under spin symmetry conditions, in global quantum group symmetry 3D-RNCS is given by:

$$\begin{cases} E_{nc}^{per:u}(\Theta,k_1=-(l+1),n,l,j)\equiv-(l+1)\Theta N^2(M+E_{nk}-C^s)R_{1n}(n,l) \\ E_{nc}^{per:d}(\Theta,k_2=l,n,l,j)\equiv l\Theta N^2(M+E_{nk}-C^s)R_{1n}(n,l) \end{cases} \quad (58)$$

with

$$R_{1n}(n,l)=\sum_{\alpha=1}^{4}R_\alpha(n,l) \quad (59)$$

4.3 *The perturbed relativistic spin-orbit spectrum for mirror nucleus* (^{17}O and ^{17}F)*under the MQHP model in the presence of the pseudo-spin symmetry conditions in 3D-RNCS symmetries*

In this subsection, in the case of deformation Dirac theory symmetries, we find the energy levels $E_{nc}^{per:d}u\left(\Theta,\tilde{k}_1=-\tilde{l},n,\tilde{l},j=\tilde{l}+1/2\right)$ and $E_{nc}^{per:d}\left(\Theta,k_2=-(\tilde{l}+1),n,\tilde{l},j=\tilde{l}-1/2\right)$ which produced by the relativistic pseudo (spin-orbit) effect under the pseudo-spin symmetry conditions can be determined by applying the same procedures as before, and to avoid repetition, we make the following steps:

$$\begin{cases} \tilde{N}\leftrightarrow N,k_1\to\tilde{k}_1,k_2\to\tilde{k}_2 \\ k(k+1)\leftrightarrow k(k-1) \end{cases} \quad (60)$$

Allow us to obtain ($E_{nc}^{per:u}\left(\Theta,\tilde{k}_1=-\tilde{l},n,\tilde{l},j=\tilde{l}+1/2\right)$, $E_{nc}^{per:d}\left(\Theta,k_2=-(\tilde{l}+1),n,\tilde{l},j=\tilde{l}-1/2\right)$) as follows, respectively:

$$\begin{cases} E_{nc}^{per:u}\left(\Theta,\tilde{k}_1=-\tilde{l},n,\tilde{l},j=\tilde{l}+1/2\right)\equiv-\tilde{l}\Theta N^{p2}\left(M-E_{nk}^p+C^p\right)R_{1n}\left(n,\tilde{l}\right) \\ E_{nc}^{per:d}\left(\Theta,k_2=-(\tilde{l}+1),n,\tilde{l},j=\tilde{l}-1/2\right)\equiv-(\tilde{l}+1)\Theta N^{p2}\left(M-E_{nk}^p+C^p\right)R_{1n}\left(n,\tilde{l}\right) \end{cases} \quad (61)$$

4.4 *The perturbed relativistic magnetic spectrum for mirror nucleus* (^{17}O and ^{17}F) *under the MQHP model in the presence of spin (pseudo-spin) symmetry conditions in 3D-RNCS symmetries*

Having obtained the exact modifications of the energy levels ($E_{nc}^{per:u}(\Theta, k_1 = -(l+1), n, l, j = l+1/2)$ and $E_{nc}^{per:d}(\Theta, k_2 = l, n, l, j = l-1/2)$), ($E_{nc}^{per:u}(\Theta, \tilde{k}_1 = -\tilde{l}, n, \tilde{l}, j = \tilde{l}+1/2)$ and $E_{nc}^{per:d}(\Theta, k_2 = -(\tilde{l}+1), n, \tilde{l}, j = \tilde{l}-1/2)$) under the spin /(pseudo) symmetry conditions, respectively, for the nuclei (^{17}O and ^{17}F) with one additional nucleon (valence) in the $1d_{5/2}$ level (the subatomic quantum numbers are $n = 1$, $Y \equiv d$, $j = 5/2$ and $l = 2$, $m = \overline{-2,+2}$). The first excited state $2S_{1/2}$ (the subatomic quantum numbers are $n = 2, Y \equiv S$, $j = 5/2$ and $l = 0$, $m = 0$). The second excited state $1d_{3/2}$ (the subatomic quantum numbers are $n = 1, Y \equiv d$, $j = 3/2$ and $l = 1$, $m = 0, \pm 1$). In addition to the generalized $(n, j, l, \tilde{l}, s, \tilde{s}, m, \widetilde{m})^{th}$ excited states nX_j (the subatomic quantum numbers are n,j, l and $m = \overline{-l,+l}$), which are produced by the effect of the NC spin-orbit Hamiltonian operator. We are now considering another interesting physically meaningful phenomenon, which is also produced from the perturbative terms of the MQHP model related to the influence of an external uniform magnetic field ($\Theta \to \chi B$); it is sufficient to apply the following two replacements to describe these phenomena:

$$\left[(M + E_{nk} - C^s)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right]\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} \to \chi\left[(M + E_{nk} - C^s)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right]\vec{\mathbf{B}}.\vec{\mathbf{L}} \quad \text{for spin symmetry} \tag{62}$$

and

$$\left[(M - E_{nk}^p + C^p)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k-1)}{r^4}\right]\vec{\tilde{\mathbf{L}}}.\vec{\mathbf{\Theta}} \to \chi\left[(M - E_{nk}^p + C^p)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k-1)}{r^4}\right]\vec{\mathbf{B}}.\vec{\tilde{\mathbf{L}}} \quad \text{for p- spin symmetry} \tag{63}$$

Here χ is an infinitesimal real proportional constant, and we choose the magnetic field $\vec{B} = B\vec{k}$, which allows us to introduce the modified new magnetic Hamiltonian $\widehat{H}_m^{qh}(r,a,b,\alpha,\chi)$ in 3D-RNCS symmetries as:

$$\widehat{H}_m^{qh}(r,a,b,\alpha,\chi) = \chi\begin{cases}\left[(M + E_{nk} - C^s)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k+1)}{r^4}\right]\left(\vec{B}\vec{J} - \vec{B}\,\vec{S}\right) & \text{for spin symmetry} \\ \left[(M - E_{nk}^p + C^p)V_{qh}^{ind}(r,a,b,\alpha) + \frac{k(k-1)}{r^4}\right]\left(\vec{B}\vec{J} - \vec{B}\vec{\tilde{S}}\right) & \text{for pseudo-spin symmetry}\end{cases} \tag{64}$$

Here ($\vec{\mathbf{B}}.\vec{\mathbf{S}}$ and $\vec{\mathbf{B}}.\vec{\tilde{\mathbf{S}}}$) are present the new Zeeman effect and the pseudo-Zeeman effect in 3D-RNCS symmetries. To obtain exact contributions of the magnetic modifications of energy levels $E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, m,, a, b, \alpha)$ and $E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, \widetilde{m}, a, b, \alpha)$ for the MQHP model under the spin (pseudo-spin) symmetry conditions, respectively, which are produced automatically by the effect of $\widehat{H}_m^{qh}(r,a,b,\alpha,\chi)$, we make the following two simultaneous replacements:

$$\tilde{k}_1 \to \widetilde{m} \quad , k_1 \to m \quad \text{and} \quad \Theta \to \chi\,\mathrm{B} \tag{65}$$

Thus, the relativistic magnetic modifications ($E_{mag}^{qh}(\chi, n = 1, l = 2, j = 5/2, (m = \pm 2, 0, \pm 1))$, $E_{mag}^{qh}\left(\chi, n = 1, \tilde{l} = 2, j = 5/2, (\widetilde{m} = \pm 2, 0, \pm 1)\right)$), ($E_{mag}^{qh}(\chi, n = 1, l = 0, j = 1/2, m = 0)$, $E_{mag}^{qh}(\chi, n = 1, \tilde{l} = 0, j = 1/2, \widetilde{m} = 0)$), ($E_{mag}^{qh}\left(\chi, n = 1, l = 1, j = \frac{3}{2}, (m = 0, \pm 1)\right)$, $E_{mag}^{qh}\left(\chi, n = 1, \tilde{l} = 1, j = \frac{3}{2}, (\widetilde{m} = 0, \pm 1)\right)$) and ($E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, j, l, m, a, b, \alpha)$, $E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, j, \tilde{l}, \widetilde{m}, a, b, \alpha)$) corresponding ($1d_{5/2}$, $1s_{1/2}$, $1d_{1/2}$ and nY_j) in 3D-RNCS symmetries, respectively, can be determined from the following relations:

$$1d_{5/2} \to \begin{cases}E_{mag}^{qh}(\chi, n = 1, l = 2, j = 5/2, (m = \pm 2, 0, \pm 1)) \equiv \chi N^2 R_{11}(n = 1, l = 2)Bm & \text{for spin symmetry} \\ E_{mag}^{qh}\left(\chi, n = 1, \tilde{l} = 2, j = 5/2, (\widetilde{m} = \pm 2, 0, \pm 1)\right) \equiv \chi N^{p2} R_{11}(n = 1, l = 2)B\widetilde{m} & \text{for pseudo-spin symmetry}\end{cases} \tag{66}$$

$$1s_{1/2} \to \begin{cases}E_{mag}^{qh}(\chi, n = 1, l = 0, j = 1/2, m = 0) \equiv 0 & \text{for spin symmetry} \\ E_{mag}^{qh}(\chi, n = 1, \tilde{l} = 0, j = 1/2, \widetilde{m} = 0) \equiv 0 & \text{for pseudo-spin symmetry}\end{cases} \tag{67}$$

$$1d_{1/2} \rightarrow \begin{cases} E_{mag}^{qh}\big(\chi, n=1, l=1, j=3/2, (m=0,\pm1)\big) \equiv \chi N^2 R_{11}(n=1, l=1)\mathrm{Bm} & \text{for spin symmetry} \\ E_{mag}^{qh}\big(\chi, n=1, \tilde{l}=1, j=3/2, (\tilde{m}=0,\pm1)\big) \equiv \chi N^{p2} R_{11}(n=1, l=1)\mathrm{B}\tilde{m} & \text{for pseudo-spin symmetry} \end{cases} \quad (68)$$

and

$$nY_j \rightarrow \begin{cases} E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, j, l, m, a, b, \alpha) = \chi N^2 R_{1n}(n,l) Bm & \text{for spin symmetry} \\ E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, j, \tilde{l}, \tilde{m}, a, b, \alpha) = N^{p2} R_{1n}(n,\tilde{l}) B\tilde{m} & \text{for pseudo-spin symmetry} \end{cases} \quad (69)$$

where $\tilde{m}$ and m are the angular momentum quantum numbers ($-\tilde{l} \leq \tilde{m} \leq +\tilde{l}$ and $-l \leq m \leq +l$), which allow us to fix $(2\tilde{l}+1)$ and $(2l+1)$ values of $E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, j, \tilde{l}, \tilde{m}, a, b, \alpha)$ and $E_{\mathrm{mag}}^{\mathrm{qh}}(\chi, n, j, l, m, a, b, \alpha)$ under pseudo-spin (spin) symmetry conditions, respectively.

5. The perturbed modified global spectrum for the mirror nucleus (^{17}O and ^{17}F) under the MQHP model in the presence of spin and (pseudo-spin) symmetry conditions in 3D-RNCS symmetries

In the previous sub-sections, we have obtained the solutions of the MDE for the nuclei (^{17}O and ^{17}F) with one additional nucleon (valence) in the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$, and the generalized $\left(n, l, \tilde{l}, m, \tilde{m}\right)^{th}$ excited states nX_j under the MQHP model using Bopp's shift method and standard perturbation theory. The energy eigenvalues were calculated in 3D-RNCS symmetries, under-spin (and pseudo-spin) symmetry conditions for two perturbed principal physics interesting phenomena corresponding to the perturbed spin-orbit interaction and modified Zeeman effect. Now, we will use the physical superposition principle to find the corrective total energy resulting from the topological effects of space, which correspond to the previously nominated excited states (the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$) based on our original results presented in Eqs. ((50), (52), (57), (58)) and Eqs. ((66), (67), (68), (69)), in addition to ordinary energies E_{nl} and $E_{n\tilde{l}}^{p}$ for quadratic Hellmann potential, which is presented in Eqs. (17) and (18) for spin and pseudo-spin symmetry in three-dimensional relativistic quantum mechanics symmetries.

5.1 The corrective total energy resulting from the topological effects of space for spin symmetry:

For spin symmetry, the modified relativistic eigenenergies $\left(E_{\mathrm{nc}}^{uqh}, E_{\mathrm{nc}}^{dqh}\right)(n=1, (m=0,\pm1,\pm2), j=5/2, l=2)$, $\left(E_{\mathrm{nc}}^{uqh}, E_{\mathrm{nc}}^{dqh}\right)(n=2, (m=0), j=1/2, l=0)$, $\left(E_{\mathrm{nc}}^{uqh}, E_{\mathrm{nc}}^{dqh}\right)(n=1, (m=0,\pm1), j=3/2, l=1)$ and $\left(E_{\mathrm{nc}}^{uqh}, E_{\mathrm{nc}}^{dqh}\right)\left(n, \left(m=\overline{-l,+l}\right), j, l\right)$ with spin-1/2 for single nucleon are obtained in this paper based on our original results presented in previously mentioned equations as follows:

$$1d_{5/2} \rightarrow \begin{cases} E_{\mathrm{nc}}^{uqh}\left(1, \left(m=\overline{-2,+2}\right), j=5/2, l=2\right) = E_{12} + N^2(-3\Theta + \chi Bm) R_{11}(1, l=2) \\ E_{\mathrm{nc}}^{dqh}\left(1, \left(m=\overline{-2,+2}\right), j=5/2, l=2\right) = E_{12} + N^2(2\Theta + \chi Bm) R_{11}(1, l=2) \end{cases} \quad (70)$$

$$1s_{1/2} \rightarrow \begin{cases} E_{\mathrm{nc}}^{uqh}(2, (m=0), j=1/2, l=0) = E_{20} - \Theta N^2 R_{12}(2, l=0) \\ E_{\mathrm{nc}}^{uqh}(2, (m=0), j=1/2, l=0) = E_{20} \end{cases} \quad (71)$$

$$1d_{1/2} \rightarrow \begin{cases} E_{\mathrm{nc}}^{uqh}(1, (m=0,\pm1), j=3/2, l=1) = E_{11} + N^2(-2\theta\Theta + \chi Bm) R_{11}(1, l=1) \\ E_{\mathrm{nc}}^{dqh}(1, (m=0,\pm1), j=3/2, l=1) = E_{11} + N^2(\Theta + \chi Bm) R_{11}(1, l=1) \end{cases} \quad (72)$$

and

$$nY_j \rightarrow \begin{cases} E_{nc}^{uqh}\left(n, \left(m=\overline{-l,+l}\right), j, l\right) = E_{nl} + N^2(-(l+1)\Theta + \chi Bm) R_{1n}(n,l) \\ E_{nc}^{dqh}\left(n, \left(m=\overline{-l,+l}\right), j, l\right) = E_{nl} + N^2(l\Theta + \chi Bm) R_{1n}(n,l) \end{cases} \quad (73)$$

where $E_{12}(k=0)$, $E_{20}(k=2)$ and $E_{11}(k=2)$ are the energy of the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$ for mirror nuclei ^{17}O and ^{17}F in the symmetries of relativistic quantum mechanics under quadratic Hellmann potential, which are determined from the following equations:

$$3\sqrt{M^2 - E_{12}^2} - 2(E_{12} + M)(a + 2b\alpha) + 2\sqrt{(M^2 - E_{12}^2)(2b(E_{12} + M) + 1/4)} = 0 \quad (74)$$

$$5\sqrt{M^2 - E_{20}^2} - 2(E_{20} + M)(a + 2b\alpha) + 2\sqrt{(M^2 - E_{nl}^2)(6 + 2b(E_{20} + M) + 1/4)} = 0 \quad (75)$$

$$5\sqrt{M^2 - E_{11}^2} - 2(E_{11} + M)(a + 2b\alpha) + 2\sqrt{(M^2 - E_{11}^2)(6 + 2b(E_{nl} + M) + 1/4)} = 0 \quad (76)$$

5.1 The corrective total energy resulting from the topological effects of space for pseudo-spin symmetry:

For the case of pseudo-spin symmetry, the modified relativistic eigenenergies of the single nucleon (neutron or proton) $E_{\mathrm{nc}}^{uqh}\left(n,\left(\widetilde{m}=\overline{-\tilde{l},+\tilde{l}}\right),j,\tilde{l}\right)$ and $E_{\mathrm{nc}}^{dqh}\left(n,\left(m=\overline{-\tilde{l},+\tilde{l}}\right),j,l\right)$ which correspond to the up and down polarities for the generalized $\left(n,j,k,\tilde{l},\widetilde{m}\right)^{th}$ excited states nX_j under the MQHP model in 3D-RNCS symmetries:

$$nY_j\rightarrow\begin{cases}E_{\mathrm{nc}}^{uqh}\left(n,\left(\widetilde{m}=\overline{-\tilde{l},+\tilde{l}}\right),j,\tilde{l}\right)=E_{n\tilde{l}}^{p}+N^{p2}\left(-\tilde{l}\Theta+\chi B\widetilde{m}\right)R_{1n}\left(n,\tilde{l}\right)\\E_{\mathrm{nc}}^{dqh}\left(n,\left(m=\overline{-\tilde{l},+\tilde{l}}\right),j,l\right)=E_{n\tilde{l}}^{p}+N^{p2}\left(\left(\tilde{l}+1\right)\Theta+\chi B\widetilde{m}\right)R_{1n}\left(n,\tilde{l}\right)\end{cases}\tag{77}$$

Now, it is crucial to construct the Hamiltonian operator $\widehat{H}_{nc}^{qh}$ for the MQHP model based on previously obtained results. Naturally, the first term in the modified Hamiltonian operator represents the kinetic energy and the potential energy in the ordinary commutative space $\widehat{H}_{qh}$ of the nuclei (^{17}O and ^{17}F) which is presented by Eq. (19). The second term $\widehat{H}_{so}^{qh}(k_1,k_2)$ or $\widehat{\widetilde{H}}_{so}^{qh}\left(\tilde{k}_1,\tilde{k}_2\right)$ represents, the induced spin-orbit parts for the pseudo-spin symmetry conditions and spin symmetry, and the last term is the modified new magnetic Hamiltonian $\widehat{H}_{mag}^{qh}(r,a,b,\alpha,\chi)$ (see Eq. (64)). Thus, we have obtained the global new Hamiltonian operator $\widehat{H}_{nc}^{qh}$ in 3D-RNCS symmetries as follows:

$$\widehat{H}_{nc}^{qh}=\widehat{H}_{qh}+\begin{cases}\widehat{H}_{so}^{qh}(k_1,k_2)+\chi\left[\left(M+E_{nk}-C^{s}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k+1)}{r^4}\right]\left(\vec{\mathbf{B}}\vec{\mathbf{J}}-\vec{\mathbf{B}}\vec{\mathbf{S}}\right)\text{ for spin symmetry}\\\widehat{\widetilde{H}}_{so}^{qh}\left(\tilde{k}_1,\tilde{k}_2\right)+\chi\left[\left(M-E_{nk}^{p}+C^{p}\right)V_{qh}^{ind}(r,a,b,\alpha)+\frac{k(k-1)}{r^4}\right]\left(\vec{\mathbf{B}}\vec{\mathbf{J}}-\vec{\mathbf{B}}\vec{\tilde{\mathbf{S}}}\right)\text{ for pseudo-spin symmetry}\end{cases}\tag{78}$$

This way, we can obtain the complete energy spectra for the MQHP model in 3D-RNCS symmetries. Now, the following accompanying constraint relations are given:

- The original spectrum contains only one value of energy in ordinary three-dimensional spaces, which Eqs. (17) and (18) present,
- As mentioned in the previous subsection, the quantum numbers m and $\widetilde{m}$ satisfied the two intervals: $-l\leq m\leq +l$ and $-\tilde{l}\leq\widetilde{m}\leq+\tilde{l}$, thus we have $(2l+1)$ and $(2\tilde{l}+1)$ values, respectively,
- We also have two polarities corresponding to the values ($j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$) and ($j=\tilde{l}+\frac{1}{2}$ and $j=\tilde{l}-\frac{1}{2}$) for spin and (pseudo-spin) symmetry conditions. This allows us to deduce the important original results: Every state in usually 3-dimensional spaces will be replaced by $2(2l+1)$ and $2\left(2\tilde{l}+1\right)$sub-states. Then, the degenerated states for mirror nuclei (^{17}O and ^{17}F) changed to the new values:

$$\begin{cases}\underbrace{\sum_{l=0}^{n-1}(2l+1)=n^2}_{\text{3D-RQM}}\rightarrow\underbrace{2\sum_{l=0}^{n-1}(2l+1)}_{\text{3D-RNCS}}\equiv 2n^2\\\underbrace{\sum_{\tilde{l}=0}^{n-1}\left(2\tilde{l}+1\right)=n^2}_{\text{3D-RQM}}\rightarrow\underbrace{2\sum_{\tilde{l}=0}^{n-1}\left(2\tilde{l}+1\right)}_{\text{3D-RNCS}}\equiv 2n^2\end{cases}\tag{79}$$

in 3D-RNCS symmetries. Finally, we resume our original results in this article; the first is the induced spin (pseudo-spin)-orbit Hamiltonian operators ($\widehat{H}_{so}^{qh}(k_1,k_2)$and $\widehat{\widetilde{H}}_{so}^{qh}\left(\tilde{k}_1,\tilde{k}_2\right)$) and corresponding eigenvalues ($E_{nc}^{per:u}(\Theta,k,j,n,l)$, $E_{nc}^{per:d}(\Theta,k,j,n,l)$) and ($E_{nc}^{per:d}\left(\Theta,k,j,n,\tilde{l}\right)$, $E_{nc}^{per:u}\left(\Theta,k,j,n,\tilde{l}\right)$), respectively as:

$$\widehat{H}_{so}^{qh}\Psi_{nk}(r,\theta,\phi)\equiv\begin{cases}\left(\widehat{H}_{so}^{qh}\right)_{11}(k_1)\left(\frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\phi)\right)=E_{nc}^{per:d}(\Theta,k,j,n,l)\left(\frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\phi)\right)\\\left(\widehat{H}_{so}^{qh}\right)_{22}(k_2)\left(\frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\phi)\right)=E_{nc}^{per:u}(\Theta,k,j,n,l)\left(\frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\phi)\right)\\\left(\widehat{H}_{so}^{qh}(k_1,k_2)\right)_{33}\left(\frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\phi)\right)=0\end{cases}\tag{80}$$

and

$$\widehat{\widetilde{H}}_{so}^{qh}\Psi(r,\theta,\phi) \equiv \begin{cases} \left(\widehat{\widetilde{H}}_{so}^{qh}\right)_{11}\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\right) = E_{nc}^{per:d}\left(\Theta,k,j,n,\tilde{l}\right)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\right) \\ \left(\widehat{\widetilde{H}}_{so}^{qh}\right)_{22}\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\right) = E_{nc}^{per:u}\left(\Theta,k,j,n,\tilde{l}\right)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\right) \\ \left(\widehat{\widetilde{H}}_{so}^{qh}\right)_{33}\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\right) = 0 \end{cases} \tag{81}$$

The second original result is the induced modified new magnetic Hamiltonian operator $\widehat{H}_{mag}^{qh}(r,a,b,\alpha,\chi)$ and corresponding eigenvalues $E_{mag}^{qh}(\chi,n,j,l,m)$ and $E_{mag}^{qh}\left(\chi,n,j,\tilde{l},\widetilde{m}\right)$, respectively as:

$$\widehat{H}_{mag}^{kp}(r,a,b,\alpha,\chi)\Psi(r,\theta,\phi) \equiv \begin{pmatrix} E_{mag}^{qh}(\chi,n,j,k,l,m)\frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\phi) \\ E_{mag}^{qh}\left(\chi,n,j,k,\tilde{l},\widetilde{m}\right)i\frac{G_{n\tilde{k}}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi) \end{pmatrix} \tag{82}$$

Now, it is essential to return to the case of the nonrelativistic limit to obtain the nonrelativistic energy in 3-dimensional nonrelativistic non-commutative space (3D-NRNCS) symmetries; we apply the transformation known in the works of literature as follows:

$$\begin{cases} E_{\mathrm{nc}}^{uqh}\left(n,\left(m=\overline{-l,+l}\right),j,k,l\right) - M \to E_{\mathrm{nc\text{-}}uqh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) \\ E_{\mathrm{nc}}^{uqh}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) + M \to 2\mu \end{cases} \tag{83}$$

and

$$\begin{cases} E_{\mathrm{nc}}^{dqh}\left(n,\left(m=\overline{-l,+l}\right),j,k,l\right) - M \to E_{\mathrm{nc\text{-}}dqh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) \\ E_{\mathrm{nc}}^{dqh}\left(n,\left(m=\overline{-l,+l}\right),j,k,l\right) + M \to 2\mu \end{cases} \tag{84}$$

where $E_{\mathrm{nc\text{-}}uqh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right)$ and $E_{\mathrm{nc\text{-}d}qh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right)$ are the nonrelativistic energy in the 3D-NRNCS symmetries, inserting the above transformation into Eq. (73) yields the following:

$$\begin{cases} E_{\mathrm{nc\text{-}u}qh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) = 2E_{nl}^{nr} - 2(l+1)\Theta N^2 T R_{1n}(n,l) + 2\chi N^2 R_{1n}(n,l)Bm \\ E_{\mathrm{nc\text{-}d}qh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) = 2E_{nl}^{nr} + 2l\Theta N^2 R_{1n}(n,l) + 2\chi N^2 R_{1n}(n,l)Bm \end{cases} \tag{85}$$

Here E_{nl}^{nr} denotes the nonrelativistic energy in 3D-RQM symmetries given by:

$$E_{nl}^{nr} = -2\mu\frac{(a+b\alpha)^2}{2n+1+\sqrt{2\mu b+l(l+1)+1/4}} \tag{86}$$

In the case of the nonrelativistic limit, Eq. (85) becomes a Schrödinger equation with an interaction potential of $2V_{qp}(r)$. To aim for $V_{qp}(r)$, not $2V_{qp}(r)$in the interaction potential under the nonrelativistic limit, we apply the same procedure as has been applied by Alhaidari *et al.* [55] and Xiang-Jun Xie *et al.* for the Morse potential [56] to rescale the vector potential $V_{mp}(r)$and scalar potential $S_{mp}(r)$ and rewrite Eq. (85) in the form of:

$$\begin{cases} E_{\mathrm{nc\text{-}}uqh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) = E_{nl}^{nr} + N^2(-(l+1)\Theta + \chi Bm)R_{1n}(n,l) \text{ for } j = l+\frac{1}{2} \\ E_{\mathrm{nc\text{-}d}qh}^{NR}\left(n,\left(m=\overline{-l,+l}\right),j,l\right) = E_{nl}^{nr} + N^2(l\Theta + \chi Bm)R_{1n}(n,l) \quad \text{ for } j = l-\frac{1}{2} \end{cases} \tag{87}$$

These results are in excellent agreement with our work's results in a nonrelativistic study [57]. The above equation represents the nonrelativistic energy spectrum for the extended nonrelativistic shell model in 3D-RNCS symmetries. It is worth mentioning that (in the limit $(\Theta,\chi) \to (0,0)$, we obtain the commutative results obtained in [6] with the influence of quadratic Hellmann potential related to the relativistic and nonrelativistic energies eigenvalues in 3D-RQM and 3D-NRQM symmetries known in the works of literature.

6. Conclusions

In this paper, we have performed exact analytical bound state solutions: the energy spectra of mirror nuclei (^{17}O and ^{17}F) and the corresponding NC Hermitian Hamiltonian operator for MDE in 3-dimensional relativistic non-commutative space symmetries for the modified quadratic Hellmann potential model within Bopp's shift method and standard perturbation theory framework under the spin and pseudo-spin-symmetry conditions. It is found that the energy eigenvalues of the ground state $1d_{5/2}$ (the first excited state

$2S_{1/2}$, the second excited state $1d_{3/2}$ and the nY_j the excited state depends on the dimensionality of the problem (a, b, α), new subatomic quantum numbers ($j = \tilde{l} \pm 1/2, j = l \pm 1/2, \tilde{s} = \pm 1/2, l, \tilde{l}, \tilde{m}$and m) in addition to the two infinitesimal parameters (Θ,χ), and we also showed that the obtained energy spectra degenerate and every old state will be replaced by $2(2\tilde{l} + 1)$ and $2(2l + 1)$ sub-states under the pseudo spin symmetry and spin symmetry conditions, respectively, for $(n, j, l, \tilde{l}, s, \tilde{s}, m, \tilde{m})^{th}$ excited states. Therefore, with the realization of this work, we have shown that the first term in the modified Hamiltonian operator represents a new Hamiltonian operator in 3D-RNCS symmetries composed of the main part in 3D-RQM symmetries and the perturbed two parts, which are induced with the deformation space-space effect; we call it the perturbed spin (pseudo-spin) orbit interaction, while the other is the perturbed modified Zeeman effect (Eq. (78)). For the MQHP, our computed nonrelativistic energy eigenvalues for mirror nuclei (^{17}O and ^{17}F) are in excellent agreement with existing results in the literature.

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