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Gravitational field of spherical bodies with three homogeneous regions

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Abstract

The exact solution for the gravitational potential field and gravitational intensity field for spherical bodies in three homogeneous regions are calculated using the method of variable separation. Our results showed that the gravitational potential field in the second region and that in the third region for two homogeneous and three homogeneous regions respectively are not linear but that in the third region shows more curve. The gravitational intensity field for the two homogenous regions is less than the gravitational intensity field for the three homogenous regions. The results of the gravitational intensity field for the three homogenous regions is closer to the experimental value showing that the present result is more valid.

Keywords: Gravitational Intensity Field; Gravitational Potential Field; Three Homogeneous Regions.

1. Introduction

The exact solution for the gravitational potential field and gravitational intensity field for spherical bodies in one homogeneous region and two homogeneous regions are well known and their applications to obtain the variations of the Earth gravitational potential field and gravitational intensity field. It is well known also that the Earth is made up of three distinct regions i.e. the Inner Core, Outer Core and the Mantle. The Inner Core is solid in nature while the Outer Core is fluid in nature due to high temperature of the region. Studied carried so far only considered either a spherical body/bodies in one homogeneous region or two homogeneous regions. Since the core is made of both the inner and out core, there could be a variation in the properties of the core most especially that they do not exist in the same state. Hence, the need for an extension for the gravitational potential field and gravitational intensity field from two homogeneous regions to three homogenous regions become very imperative. It is noted that potential theory has a significant effect in the formulations of some quantities such as the gravitational acceleration. The gravitational acceleration g as a function of position (r) at any point in space is derived from the gravitational scalar field Φ. The gravitational acceleration and the gravitational scalar field can be expressed mathematically as [1, 2]

$$\underline{g}(r) = -\nabla\Phi(\underline{r}), \tag{1}$$

where $\underline{g}(r)$ is gravitational intensity field (or acceleration due to gravity), $\Phi(\underline{r})$ is gravitational potential field and ∇ is a del operator. From Newton's law of gravitation, the Newton's gravitational potential field equation is written as [3, 4]

$$\nabla^2\Phi(r) = -4\pi G\rho(r) \tag{2}$$

where G is the universal gravitational constant, ∇^2 is the Laplacian operator and $\rho(r)$ is the density of the space. This equation (2) was first discovered by French Mathematician, Simeon-Dennis Poisson and can be expressed in spherical coordinate to solve the gravitational scalar potential field variation of spherical body in one homogeneous region as well as the gravitational force field [5, 6]. Likewise, for a spherical body with two homogenous regions, the exact expression as an extension of the gravitational fields of spherical bodies of one homogeneous region was also derived for application in Mechanics [6, 7]. Figure 1 depicts a spherical body of two homogeneous regions. In each of this region, the position of r with respect to R_I and R_{II} are prominent.

In this paper, we developed the gravitational scalar potential field variation of a spherical body with three homogeneous regions as well as the gravitational force field. This is done theoretically and then applied mathematically to derive the gravitational intensity field of the Earth. For comparison with these approximations, we derive the exact perfect analytical gravitational scalar potential due to the distinct nature of the outer core. To the best of our understanding, this has not been considered in the literature.

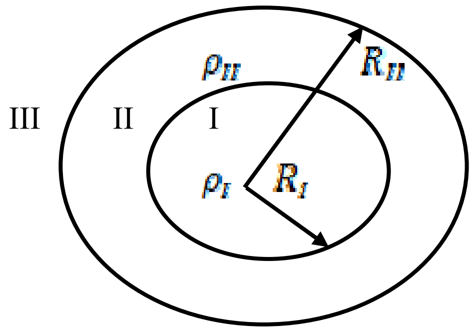

Figure 1: A spherical body of two homogeneous regions with densities ρ_I and ρ_{II}

The field equations in terms of the gravitational potential field, universal gravitational constant (G) and gravitational intensity field in regions I, II and III for two homogeneous regions are given as [6]

$$\nabla^2\Phi^I(r) = 4\pi G\rho^I; \;\; 0 \le r \le R_I, \tag{3}$$

$$\nabla^2\Phi^{II}(r) = 4\pi G\rho_{II}; \;\; R_I \le r \le R_{II}, \tag{4}$$

$$\nabla^2\Phi^{III}(r) = 0; \;\; r \ge R_{II}. \tag{5}$$

$$g_I(r) = -\frac{GM_I r}{R_I^3}, \tag{6}$$

$$g_{II}(r) = -\frac{GM_{II}}{R_{II}^2 - R_I^2}\left(\frac{R_I^2}{r^2} - r\right) - \frac{GM_I}{r^2}, \tag{7}$$

$$g_{III}(r) = -\frac{G}{r^2}\left(M_{II} + M_I\right). \tag{8}$$

The details/solutions to the equations in region I, region II and region III for two homogeneous bodies can be found in ref. [6].

2.0: Mathematical Analysis

In this section, we compute the gravitational potential field and gravitational intensity field of a spherical body for three homogeneous regions.

2.1: Gravitational potential field of spherical bodies with three homogeneous regions

For a spherical body with three distinct homogeneous regions of densities ρ_I, ρ_{II} and ρ_{III} the field equations in region I, II, III and IV are given by

$$\nabla^2\Phi^I(r,\theta) = 4\pi G\rho^I;\;\; 0 \le r \le R_I, \tag{9}$$

$$\nabla^2\Phi^{II}(r,\theta) = 4\pi G\rho_{II};\;\; R_I \le r \le R_{II}, \tag{10}$$

$$\nabla^2\Phi^{III}(r,\theta) = 4\pi G\rho_{III};\;\; R_{II} \le r \le R_{III}, \tag{11}$$

$$\nabla^2\Phi^{IV}(r,\theta) = 0;\;\; r \ge R_{III}. \tag{12}$$

The general solutions of the equations above can be found in appendix A

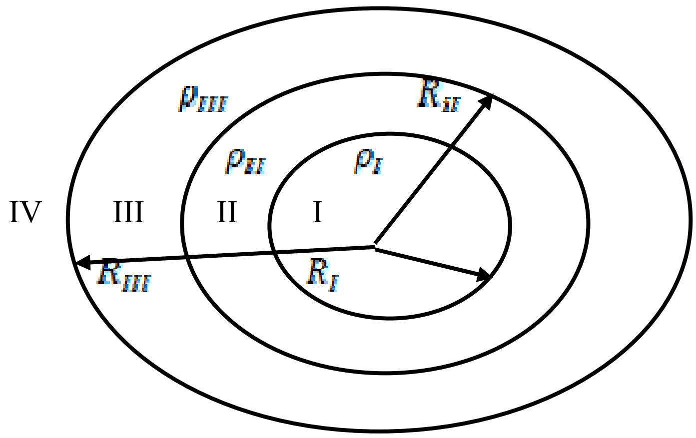

Figure 2: A Spherical body with three distinct homogeneous regions.

The three homogeneous regions satisfied the same conditions in both one and two homogeneous regions. Hence, we obtain the continuity equations for Φ^I, Φ^{II}, and Φ^{III} for $l=0$, and $l>0$ in the following form

(i): Continuity of Φ^I across boundaries at $r = R_{I,}$ then by taking the coefficient of $P_l(\cos\theta)$; when $l=0$;

$$A_0 + \frac{2}{3}\pi G\rho_I R_I^2 = B_0 + \frac{C_0}{R_I} + \frac{2}{3}\pi G\rho_{II} R_I^2, \tag{13}$$

and for $l>0$, we have

$$A_1 R_I^l = B_1 R_I^l + \frac{C_0}{R_I^{l+1}};\; l = 1,2,3,... \tag{14}$$

(ii): Continuity of the derivative Φ^I across boundaries at $r = R$ when $l = 0$, the coefficient of $P_0(\cos\theta)$ is taken as;

$$\frac{4}{3}\pi G\rho_I R_I = -\frac{C_0}{R_I^2} + \frac{4}{3}\pi G\rho_{II} R_I, \quad (15)$$

and for $l > 0,$ we have

$$lA_1 R_I^{l-1} = lB_1 R_I^{l-1} - \frac{(l+1)C_l}{R_I^{l+2}};\ l = 1, 2, 3, \ldots \quad (16)$$

(iii): Continuity of Φ^{II} across boundaries at $r = R_{II}$ when $l = 0,$ the coefficient of $P_0(\cos\theta)$ is taken as;

$$B_0 + \frac{C_0}{R_{II}} + \frac{2}{3}\pi G\rho_{II} R_{II}^2 = D_0 + \frac{N_0}{R_{II}^2} + \frac{2}{3}\pi G\rho_{III} R_{II}^2, \quad (17)$$

and for $l > 0,$ we have

$$B_1 R_{II}^l + \frac{C_l}{R_{II}^{l+1}} = D_l R_{II}^l + \frac{N_l}{R_{II}^{l+2}}.\ l = 1, 2, 3, \ldots \quad (18)$$

(iv): Continuity of the derivative Φ^{II} across boundaries at $r = R_{II}$ when $l = 0$, the coefficient of $P_0(cos\theta)$ is taken as;

$$-\frac{C_0}{R_{II}^2} + \frac{4}{3}\pi G\rho_{II} R_{II} = -\frac{2N_0}{R_{II}^3} + \frac{4}{3}\pi G\rho_{III} R_{II}, \quad (19)$$

and for $l > 0,$ we have

$$lB_1 R_{II}^{l-1} - (l+1)\frac{C_l}{R_{II}^{l+2}} = lD_l R_{II}^{l-1} - (l+1)\frac{N_l}{R_I^{l+2}}.\ l = 1, 2, 3, \ldots \quad (20)$$

(v): Continuity Φ^{III} across boundaries at $r = R_{III}$; When $l = 0$, the coefficient of $P_0(cos\theta)$ is taken as;

$$D_0 + \frac{F_0}{R_{III}^2} + \frac{2}{3}\pi G\rho_{III} R_{III}^2 = \frac{M_0}{R_{III}^2}, \quad (21)$$

and for $l > 0,$ we have

$$D_1 R_{III}^l + \frac{N_l}{R_{III}^{l+2}} = \frac{M_l}{R_{III}^{l+2}},\ l = 1, 2, 3, \ldots \quad (22)$$

(vi): Continuity of the derivative Φ^{III} across boundaries at $r = R_{III}$ when $l = 0$, the coefficient of $P_0(cos\theta)$ is taken as;

$$-\frac{2N_0}{R_{III}^2} + \frac{4}{3}\pi G\rho_{III} R_{III} = -\frac{2M_0}{R_{III}^2}, \quad (23)$$

and for $l > 0,$ we have

$$lD_1 R_{III}^{l-1} - \frac{(l+2)N_l}{R_{III}^{l+2}} = -\frac{(l+2)M_l}{R_{III}^{l+2}},\ l = 1, 2, 3, \ldots \quad (24)$$

The equations above have different constants whose values can be determine as follows

$$C_0 = \frac{4}{3}\pi G(\rho_{II} - \rho_I)R_I^3, \quad (25)$$

$$N_0 = \frac{2}{3}\pi G R_{II}\left[(\rho_{II} - \rho_I)R_I^3 + (\rho_{III} - \rho_{II})R_{II}^3\right], \quad (26)$$

$$M_0 = \frac{2}{3}\pi G\left\{R_{II}\left[(\rho_{II} - \rho_I)R_I^3 + (\rho_{III} - \rho_{II})R_{II}^3\right] - \rho_{II} R_{III}^3\right\}. \quad (27)$$

At this point, we can now define the following terms in the regions

$$\Phi^{I}(r) = A_0 + \frac{2\pi G\rho_I R_I}{3} r^2, \tag{28}$$

$$\Phi^{II}(r) = B_0 + \frac{C_0}{r} + \frac{2\pi G\rho_I R_I^3}{3\left(R_{II}^2 - R_I^2\right)} r^2, \tag{29}$$

$$\Phi^{III}(r) = D_0 + \frac{N_0}{r^2} + \frac{2\pi G\rho_{III}(R_{III}^3 - R_{II}^3)}{3\left(R_{III}^2 - R_{II}^2\right)} r^2, \tag{30}$$

$$\Phi^{IV}(r) = \frac{2\pi G\left\{R_{II}\left[(\rho_{II} - \rho_I)R_I^3 + (\rho_{III} - \rho_{II})R_{II}^3\right] - \rho_{II}R_{III}^3\right\}}{3r^2}. \tag{31}$$

2.2 Gravitational Intensity Fields

By definition the gravitational field intensity g is given as [1]

$$g(\underline{r}) = -\nabla\Phi(\underline{r}), \tag{32}$$

Following the definition and the extension from two homogeneous regions to three homogeneous regions, the gravitational intensity field in spherical coordinate can be written as follows

$$g_I(\underline{r}) = -\frac{4\pi G\rho_I R_I}{3} r, \tag{33}$$

$$g_{II}(\underline{r}) = \frac{C_0}{r^2} - \frac{4\pi G\rho_{II}(R_{II}^3 - R_I^3)}{3(R_{II}^2 - R_I^2)} r, \tag{34}$$

$$g_{III}(\underline{r}) = \frac{2N_0}{r^2} - \frac{4\pi G\rho_{III}(R_{III}^3 - R_{II}^3)}{3(R_{III}^2 - R_{II}^2)} r, \tag{35}$$

and

$$g_{IV}(\underline{r}) = \frac{4\pi G\left\{R_{II}\left[(\rho_{II} - \rho_I)R_I^3 + (\rho_{III} - \rho_{II})R_{II}^3\right] - \rho_{II}R_{III}^3\right\}}{3r^2}. \tag{36}$$

3. Result

Table 1: The three major Earth layers [7]

Layer	Radius	Density
Inner Core	1228km	12.8x10^3kgm^{-3}
Outer Core	3488km	11.05x10^3kgm^{-3}
Mantle	6378km	4.5x10^3kgm^{-3}

Table 2: Values of constants used in this study

A_0 =	-7.4385e^7
B_0 =	-7.3279e^7
C_0 =	-5.0930e^{11}
D_0 =	-5.1144e^7
E_0=	-1.3701e^{20}
F_0 =	-1.1773e^{21}
A_0^I	-12039.84 x 10^4
B_0^I	-8506.07 x 10^{10}

C_0'	-8174.77 x 10^{10}
D_0'	-44297.25 x 10^{10}

where A_0', B_0', C_0', and D_0', represent the constants in two homogeneous region while A_0, B_0, C_0, D_0, M_0 and N_0 represent the constants in three homogeneous regions respectively.

Table 3: Gravitational potential field of the Earth as $0 \leq r \leq \infty$ for the three homogeneous regions (Inner core, Outer core and Mantle)

R(m)		$\Phi(J/kg)$
0	$\Phi^0(r)$	-7.4385e^7
R_1	$\Phi^I(r)$	-7.1688e^7
R_2	$\Phi^{II}(r)$	-5.4644e^7
R_3	$\Phi^{III}(r)$	-2.8940e^7
∞	$\Phi^{IV}(r)$	0

Table 4: Gravitational potential field of the Earth as $0 \leq r \leq \infty$ for the two homogeneous regions (Core and Mantle).

R(m)	$\Phi'(J/kg)$
0	-12039.84e^4
R_1'	-10020.55e^4
R_2'	-6954.03e^4
∞	0

Table 5: Masses of the various layers the earth in three homogeneous regions (Inner core, Outer core and Mantle).

$M_I(kg)$	9.9287e^{22}
$M_{II}(kg)$	187.85e^{22}
$M_{III}(kg)$	409.06e^{22}

Table 6: the gravitational field intensity of the earth as $0 \leq r \leq \infty$ for the two homogeneous regions represented by $\underline{g}^1(r)m/sec^2$

Earth (Core and Mantle)	
R(m)	$\underline{g}^1(r)m/sec^2$
0	0
R_1'	-11.64
R_2'	-10.9
∞	0

Table 7: The gravitational field intensity of the earth as $0 \leq r \leq \infty$ for the three homogeneous regions Earth (Inner core, Outer core and Mantle) represented by $\underline{g}(r)m/sec^2$

R(m)	$\underline{g}(r)m/sec^2$
0	0
R_I	-4.3916
R_{II}	-10.8103
R_{III}	-9.0750
∞	0

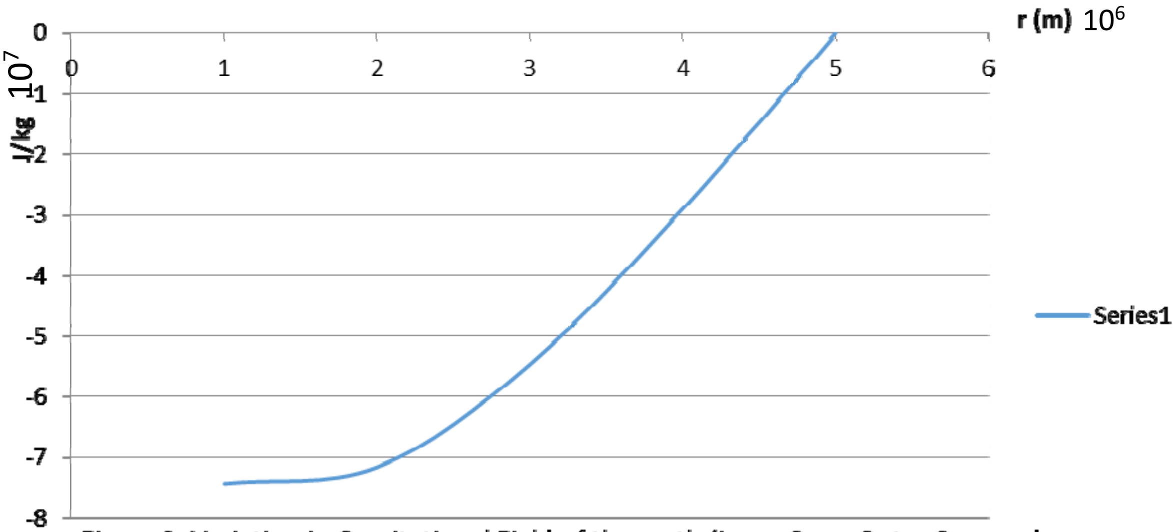

Figure 3: Variation in Gravitational Field of the earth (Inner Core, Outer Core and Mantle)

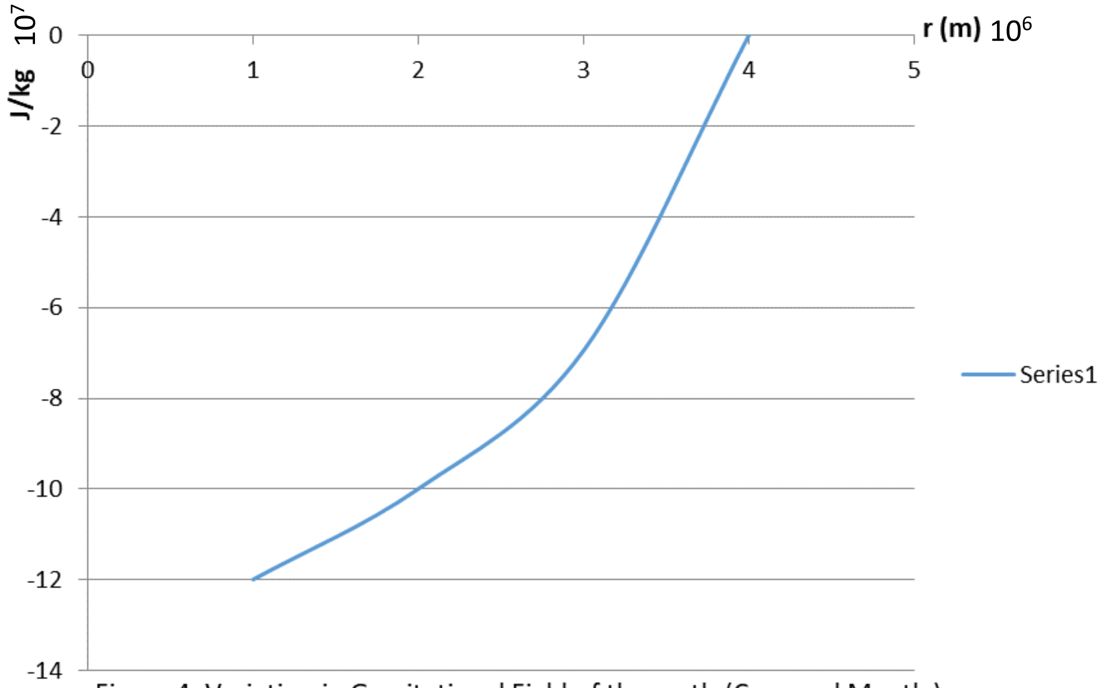

Figure 4: Variation in Gravitational Field of the earth (Core and Mantle)

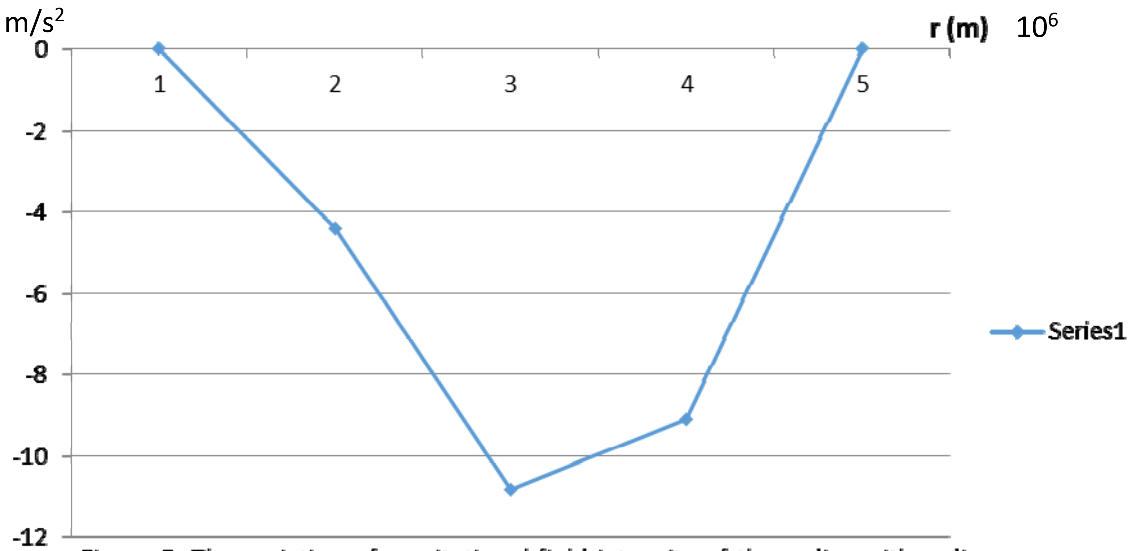

Figure 5: The variation of gravitational field intensity of the radius with radius (r) for the inner core, outer core and the mantle

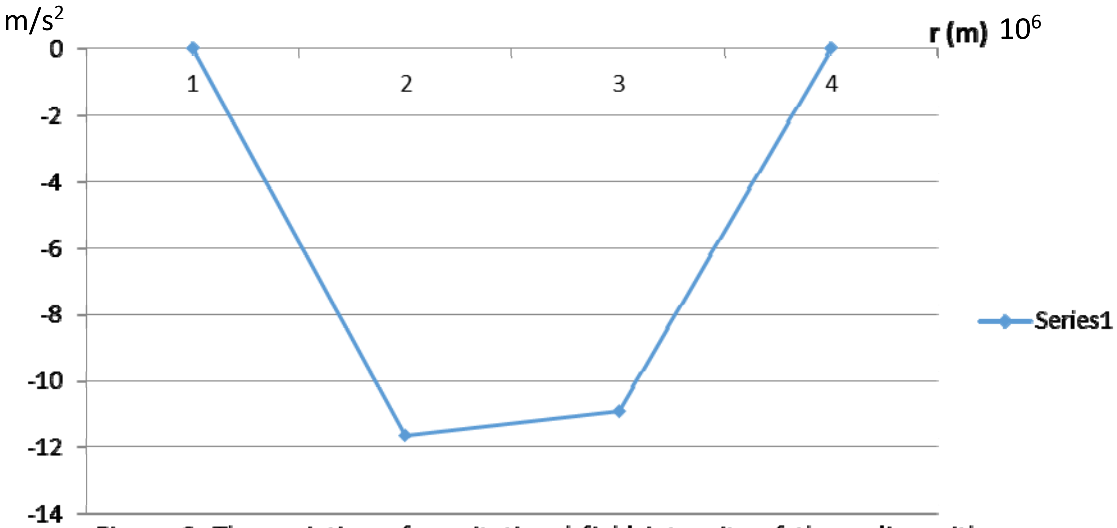

Figure 6: The variation of gravitational field intensity of the radius with radius (r) for the core and the mantle

4. Discussion

In Table 1, we presented the radius and density of the three major layers of the Earth. The density decreases as the radius of the layer increases. The inner core with the smallest radius has the highest density while the Mantle with the highest radius has the smallest density. The values of the constants used in this study are given in Table 2. Table 3 shows the gravitational potential field for three homogeneous regions. The gravitational potential field varies directly with the regions. Our result also shows that at infinity, the gravitational potential field is also infinity. The gravitational potential field for two homogeneous regions is presented in Table 4. In this case, the core is not considered as inner and outer parts. The masses of the layers of the earth for three homogeneous regions are presented in Table 5. It is noted that

$M_I < M_{II} < M_{III}$. In Tables 6 and 7, we presented gravitational field intensity of the earth for two and three homogeneous regions respectively. The result indicated that the gravitational field intensity of the three homogeneous regions is greater than that of the two homogenous regions. Figure 1 depicts a spherical body of two homogeneous regions with densities ρ_I and ρ_{II} while Figure 2 shows a spherical body of three homogeneous regions. The variation of the Gravitational Field of the earth for three homogeneous regions (Inner Core, Outer Core and Mantle) and two homogeneous regions (Core and Mantle) are shown in Figure 3 and Figure 4 respectively. In both cases, the gravitational field increases monotonically and have its maximum at zero. However, the gravitational field for the three homogeneous regions has a higher minimum. The variation of gravitational field intensity of the radius with radius (r) for the three homogeneous region (inner core, outer core and the mantle) and two homogeneous region (core and mantle) are presented in Figure 5 and Figure 6 respectively. In each case, the gravitational field intensity decreases and then increases. The gravitational field intensity for the three homogenous regions has a higher turning point. It is observed that the gravitational field intensity for the three homogeneous regions is more curve compared to that of the two homogenous region and shows depression at the boundary between the third and fourth regions before tending to zero as $r \to \infty$.

5. Conclusion

In this study, the gravitational field potential and the gravitational field intensity for three homogeneous regions were examined. Our study revealed that the gravitational field potential for three homogeneous regions is higher than that of the two homogenous regions at the origin. This is due to the density of the outer core which was not considered in the two homogeneous regions. The density of the outer core increases the potential of the three homogeneous regions. The gravitational field intensity is non-linear for both the two and three homogeneous regions. The gravitational intensity field for the two homogenous regions has a value of -11.64 g(r) m/s^2 which is less than the gravitational intensity field for the three homogenous regions whose value is -10.81 g(r) m/s^2. The value of gravitational intensity field for the three homogenous regions is closer to the experimental value of 10 m/s^2. This shows that the present study gives a better result.

Appendix A

$$\Phi^{I}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \frac{2}{3}\pi G \rho_I r^2, \tag{A1}$$

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l(\cos\theta) + \frac{2}{3}\pi G \rho_{II} r^2, \tag{A2}$$

$$\Phi^{III}(r,\theta) = \sum_{l=0}^{\infty} \left(D_l r^l + \frac{N_l}{r^{l+2}} \right) P_l(\cos\theta) + \frac{2}{3}\pi G \rho_{III} r^2, \tag{A3}$$

$$\Phi^{IV}(r,\theta) = \sum_{l=0}^{\infty} \left(\frac{M_l}{r^{l+2}} \right) P_l(\cos\theta). \tag{A4}$$

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