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Linear Stochastic processes with multiplicative noise.

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Abstract

Linear stochastic differential evolution equation are studied. An important result concerning the specific form of generalized central limit theorem on random walk on one dimensional affine group is generalized to arbitrary fininte dimensional affine group: under general assumption the exponent of algebraic decay of the probability distribution function is indepenent on the statistical properties of the random translation subgroup. The heavy tail exponent is related to thre critical index of a family of Lebesgue spaces, related to the asymptotic behavior for large times of the Banach space norm of the solution of the random evolurion equation. We prove that the heavy tail exponent of the inhomogenous stochastic linear equation can be related to the large time behavior of the solution of the homogenous part.

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Short title: Linear stochastic processes.

1 Introduction

The simplest random affine linear model of the one dimensional multiplicative processes [10] [11][12] predicts large fluctuations in the propagation of epidemics exactly in the critical case, when the mean value of the epidemic R factor is slightly less then unity.

Affine stochastic evolution equations were studied both in mathematical literature [2], [1, 3, 4, 5, 6], [11] as well as in physical literature [7, 8, 9, 10], [12]. This interest in affine stochastic evolution equations (ASEE) comes partly because they represents the simplest reduced models of large class of the natural or social processes they are the simplest, partially soluble, versions of the instability growth under the effect of external noise and must important, their relation with the occurence of heavy tail (HT) of the probability distribution function of the stationary solution. They are also related to models of the self-organized criticality [10], and to the renewal processes (see .[1, 3, 4]).

Despite their apparent formal simplicity, even in the classical examples of the discrete time ASEE, the affine iterated function systems, the stationary cumulative probability function in one or two-dimensional case, has complicated fractal structure [13, 14, 15].me

The apparently simple continuous time, one-dimensional [10] explains the simultaneous occurrence of the very small value of the heavy tail exponent and the large correlation time, approximate self-similarity, of the driving multiplicative noise in tokamak experiments.

Exact analytic results on the heavy tail exponent for one dimensional process was obtained in [8], [9], [10], [11], [12]. In this work we study higher N dimensional stochastic linear evolution equation, in finite dimensional space. We establish a relation between large time evolution of the solution of the homogenous linear stochastic equation and the heavy tail exponent of the stationary solution of the inhomogenous equation, in the case when the heavy tail index is greater than 1.

The occurrence of the HT in the stationary PDF is related to the following dynamical effect [6, 10], [11] . When the stationary PDF of the solution $X_t(\omega)$ of the ASEE has HT with exponent β_c, then for $t \to \infty$ the fractional order moments, or the related L^p norms, $\mathbb{E}[|X_t|^p]$ remains bounded for $0 < p < \beta_c$, respectively diverges for $p > \beta_c$. This is related to the "variance explosion" phenomena studied recently in the mathematical finance [6]. The objective of this article is to extend these results to higher dimensional random affine systems.

Previously explicit algebraic methods for computing β_c were elaborated in some special cases: in the framework of the discrete time models in [1, 8], with i.i.d. additive and multiplicative noise, and in the case of i.i.d. additive noise and multiplicative noise modelled by a finite state Markov process in [4] .

In the continuos time case, with the multiplicative noise modelled by a finite state Markov process [3], rigorous foundation of the computation of β_c was obtained. In Ref.[10] the multiplicative and additive noise were modelled by a superposition of Ornstein-Uhlenbeck processes and explicit formula for β_c was derived by asymptotic methods. In all of the cases the exponent β_c is independent of the additive term.

The ASEE model equation that is considered here is a class of n-dimensional random differential equation (RDE), which extends previous results [10]), by using new topological vector space methods. The additive and the multiplicative random terms in our model are stationary processes.

Recently there is an increasing interest in the study of the affine random processes due to the application in the epidemiogy. The simplest random affine linear model of the one dimensional multiplicative processes [10] [11][12] predicts large fluctuations in the propagation of epidemics exactly in the critical case, when the mean value of the epidemic R factor is slightly less then unity.

2 The linear stochastic model and the results.

2.1 The framework

The driving stochastic processes are defined in a probability space $\{\Omega, \mathcal{F}, P\}$ with expectation value $\mathbb{E}_\omega[f(\omega)] = \mathbb{E}[f] = \int_\Omega f(\omega) dP(\omega)$. By ω will be denoted a generic element of the measure space Ω. Two driving $\mathcal{F}-$measurable stochastic, the finite dimensional vector space V valued process$\{b_t(\omega) : \mathbb{R} \times \Omega \to V \approx \mathbb{R}^n$ and operator valued process$\{A_t(\omega) : \mathbb{R} \times \Omega \to V^* \otimes V \approx \mathbb{R}^{n^2}$ defines formally the linear random differential equation (RDE) :

$$\frac{dX_t(\omega)}{dt} = A_t(\omega) X_t(\omega) + b_t(\omega) \tag{1}$$

$$X_0(\omega) = x_0 \tag{2}$$

We consider the case when the driving processes $b_t(\omega)$ and $A_t(\omega)$ are stationary and independent, consequently the operator $\mathbb{E}_\omega[A_t(\omega)]$ is constant, From previous study of the one dimensional systems, the problem 1-2has nontrivial stationary solution (or at least bounded solutions in the large time limit) if the eigenvalues of the operator $\mathbb{E}_\omega[A_t(\omega)]$ has strictly negative real parts.

The notations $X_t(\omega)$ or X_t, for the solutions of RDE will be reserved. Without loss of generality we consider deterministic initial condition $X_0(\omega) \equiv x_0$. The argument ω will be omitted when no confusion arises.

Associated to the random affine system 1 we will study the linear system for matrix valued random function $G(t_0, t, \omega)$

$$\frac{dG(t_0,t,\omega)}{dt} = A_t(\omega)G(t_0,t,\omega)) \tag{3}$$

$$G(t_0,t_0,\omega) = 1 = identity\ operator \tag{4}$$

The operator satisfies the following cocicle conditions [22]

$$G(t_1,t_2,\omega)G(t_2,t_{3,},\omega) = G(t_1,t_3,\omega) \tag{5}$$

$$[G(t_1,t_2,\omega)]^{-1} = G(t_2,t_1,\omega) \tag{6}$$

Formally, from equations 1, ...,6 result the following solution

$$X_t(\omega) = G(0,t,\omega)x_0 + \int_0^t G(t_1,t,\omega)b_{t_1}(\omega)\,dt_1 \tag{7}$$

$$X_t(\omega) \in V,\ b_{t_1}(\omega) \in V,\quad G(t_1,t,\omega) : V \longrightarrow V \tag{8}$$

For more clarity, the previous equation 7 will be rewritten as follows [18]

$$X_{t,k}(\omega) = \sum_{m=1}^{n} G_{k,m}(0,t,\omega)x_{0,m} + \int_0^t \sum_{m=1}^{n} G_{k,m}(t_1,t,\omega)b_{t_1,m}(\omega)\,dt_1 \tag{9}$$

2.1.1 Generalized L^p spaces [18], [19] , [20].

Consider the direct product of measure spaces, that give rise to anisotropic Banach space structure.

Let the measure space $(\Omega, \mathcal{A}, m)$ has the following product structure. The phase space Ω is split in two subspaces

$$\Omega_{pr} = \Omega_1 \times \Omega_2 \tag{10}$$

That means that the argument $\mathbf{x}$ of a measurable function can be represented as $\mathbf{x} = \{x_1, x_2\}$, so

$$f(\mathbf{x}) = f(x_1, x_2) \tag{11}$$

with $x_k \in \Omega_k$. We mention also that in general the component spaces Ω_k has the structure of $\mathbf{R}^n$ or more general infinite dimensional measure space. Each of the spaces Ω_k has their $\sigma-$algebra $\mathcal{A}_k$. The $\sigma-$algebra $\mathcal{A}$, that contains subsets of $\Omega_{pr} = \Omega_1 \times \Omega_2$ is defined as a tensor product: it is the largest $\sigma-$algebra on Ω such that all of the projections $\Omega \overset{p_k}{\to} \Omega_k$ are measurable.

The measure m is also factorizable:

$$dm(\mathbf{x}) = dm(x_1, x_2) = dm_1(x_1)dm_2(x_2) \tag{12}$$

where the measures m_k are defined on the $\sigma-$algebras $\mathcal{A}_k$.
In other words, the measure space $(\Omega_{pr}, \mathcal{A}, m)$ is the tensor product

$$(\Omega_{pr}, \mathcal{A}, m) = \bigotimes_{j=1}^{2} (\Omega_j, \mathcal{A}_j, m_j) \tag{13}$$

The elementary probability $dP(\mathbf{x})$ is given by

$$dP(\mathbf{x}) = \rho(x_1, x_2) dm(\mathbf{x}) \tag{14}$$

where is some singular probability density function.

Consider a vector $\mathbf{p} = \{p_1, p_2\}$ of real numbers with $p_k \geq 1$. According to Ref.[18] , in close analogy to Ref.[19] , we define recursively the norm (depending on the measure m) $\|\rho\|_{\mathbf{p},m}$ as follows

$$f_1\,(x_1) := \left[\int\limits_{\Omega_2} [f(x_1, x_2)]^{p_2}\, dm_2(x_2) \right]^{1/p_2} = \|f(x_1, .)\|_{p_2} \tag{15}$$

$$\|f\|_{\mathbf{p},m} := \left[\int\limits_{\Omega_1} [f_1\,(x_1)]^{p_1}\, dm_1(x_1) \right]^{1/p_1} = \|f_1(.)\|_{p_2} \tag{16}$$

The corresponding anizotropic Banach spaces are defined according to the previous norms 15 and 16 . In the case of our physical problems, both of the measures $m_1(x_1)$, $m_2(x_2)$ are finite. The space $(\Omega_1, \mathcal{A}_1, m_1)$ is the probability space $\Omega = \Omega_1$ and

$$m_1(\Omega) = 1 \tag{17}$$

$$m_2(\Omega_2) = n < \infty \tag{18}$$

In our case $(\Omega_2, \mathcal{A}_2, m_2)$ is the discrete, finite probability space on the finite set $\Omega_2 = \{1, ..., n\}$ with the counting measure on $\{1, ..., n\}$. It is known [18] that the previous norm $\|f\|_{\mathbf{p},m}$ has all the properties of norm on functional spaces [18] . According to the norm defined by the finite dimensional vector space V, we define a norm related to the matrix valued random function G from equation 9 , that we denote by $N_{\mathbf{p},\mathbf{q}}(.)$ We associate to the (finite dimensional) random linear operator $G_{k,m}(t_1, t, \omega)$, a linear operator $\Gamma(t_1, t)$, acting as a multiplication operator in the space o random measurable functions, and acting as a matrix in the space V. So we have for all vector valued measurable function $\phi_k(\omega)$

$$[\Gamma(t_1, t)\phi]_k = \sum_{m=1}^{n} G_{k,m}(t_1, t, \omega)\phi_m(\omega) \tag{19}$$

We denote by $L^{\mathbf{p}}$ the set of measurable functions on Ω_{pr} : $f \in L^{\mathbf{p}} \iff \|f\|_{\mathbf{p},m} < \infty$. In general the operator Γ acts from the space $L^{\mathbf{q}}$ to $L^{\mathbf{p}}$ and define the corresponding norm by

$$N_{\mathbf{p},\mathbf{q}}(\Gamma(t_1, t)) = \sup_{\phi \neq 0} \frac{\|\Gamma(t_1, t)\phi\|_p}{\|\phi\|_{\mathbf{q}}} \tag{20}$$

So we have for all $\phi \in L^{\mathbf{q}}$ the following inequality

$$\|\Gamma(t_1, t)\phi\|_{\mathbf{p}} \leq \|\phi\|_{\mathbf{q}}\, N_{\mathbf{p},\mathbf{q}}(\Gamma(t_1, t)) \tag{21}$$

Because $N_{\mathbf{p},\mathbf{p}}(\Gamma(t_1,t))$ is an operator norm, and from the stationarity of the stochastic process $A_t(\omega)$ results that $N_{\mathbf{p},\mathbf{p}}(\Gamma(t_1,t)) = N_{\mathbf{p},\mathbf{p}}(\Gamma(0,t_1-t))$ we have

$$N_{\mathbf{p},\mathbf{p}}(\Gamma(t_1,t)) = N_{\mathbf{p},\mathbf{p}}(\Gamma(0,t_1-t,)) \leq [N_{\mathbf{p},\mathbf{p}}(\Gamma(0,(t_1-t,)/m))]^m \tag{22}$$

We have the generalized Hölder inequality. Let $\mathbf{p} = \{p_1, p_2\}$ and $\mathbf{p}' = \{p_1', p_2'\}$ with

$$1/p_1 + 1/p_1' = 1 \tag{23}$$
$$1/p_2 + 1/p_2' = 1 \tag{24}$$

Then we have, for all $f \in L^{\mathbf{p}}$ and $g \in L^{\mathbf{p}'}$ the following inequality

$$\int\limits_{\Omega_{pr}} |f(\mathbf{x})g(\mathbf{x})| \, dm(\mathbf{x}) \leq \|f\|_{\mathbf{p},m} \|g\|_{\mathbf{p}',m} \tag{25}$$

Proposition 1 *In the case when* $\mathbf{p} = \{p_1, p_1\}$, $\mathbf{q} = \{q_1, q_1\}$, $\mathbf{r} = \{r_1, r_1\}$ *and*

$$1/p_1 + 1/q_1 = 1/r_1$$

then

$$\|fg\|_{\mathbf{r},m} \leq \|f\|_{\mathbf{p},m} \|g\|_{\mathbf{q},m}$$

Because the norms $\|.\|_{\mathbf{r},m}, \|.\|_{\mathbf{p},m} \|.\|_{\mathbf{q},m}$ *are equivalent for fixed* p_1, q_1, r_1, *there exists constant* C *depending only on* p, q, r *such that in general case*

$$\|fg\|_{\mathbf{r},m} \leq C \|f\|_{\mathbf{p},m} \|g\|_{\mathbf{q},m} \tag{26}$$

In the our case where the measures m_1, m_2 are all finite, and m_1 is a probability measure, then from 25 it follows

$$\int\limits_{\Omega_{pr}} |f(\mathbf{x})| \, dm(\mathbf{x}) \leq n^{1/p_1} \|f\|_{\mathbf{p},m} \tag{27}$$

Similar to the norm $\|f\|_{\mathbf{p},m}$ and corresponding $L^{\mathbf{p}}$, it is useful to define another norm $\|.\|^*_{\mathbf{p},m}$ too . We define

$$g_1(x_2) := \left[\int\limits_{\Omega_1} [f(x_1,x_2)]^{p_2} \, dm_1(x_1) \right]^{1/p_1} = \|f(.,x_2\|_{p_1}$$

$$\|g\|^*_{\mathbf{p},m} := \left[\int\limits_{\Omega_2} [g_1(x_2)]^{p_2} \, dm_2(x_2) \right]^{1/p_2} = \|g_1(.)\|_{p_2}$$

Remark 2 *In the following the index* m *will be omited, if no confusion appears. Let* $\mathbf{p} = \{p_{1,}p_2\}$, $\mathbf{p}' = \{p_{1,}p_2'\}$ *Because the vector space* V *is finite dimensional, for fixed* $\mathbf{p} \geqq 1$ *all of the norms are equivalent topologically, consequently there exists constants* $c_1,\ c_2$ *or* $c_1',\ c_2'$ *such that*

$$c_1 \|f\|_{\mathbf{p}'} \leq \|f\|_{\mathbf{p}} \leq c_2 \|f\|_{\mathbf{p}'} \tag{28}$$
$$c_1 \|g\|_{\mathbf{p}'} \leq \|g\|_{\mathbf{p}} \leq c_2 \|g\|_{\mathbf{p}'} \tag{29}$$

Remark 3 *In the case when $p = p_1 = p_2$, the anizotropic norms are equal to the classical norms.*

$$\|f\|_{\mathbf{p}} = \|f\|^*_{\mathbf{p}} = \left\{ \int\limits_{\Omega_1 \times \Omega_2} [f(x_1, x_2)]^p \, dm_1(x_1) dm_2(x_2 \right\}^{1/p} \tag{30}$$

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These class of norms will be usefull in the future works relates to the extensions of the our results.

In the following we introduce some conventions related to the conditions on validity of the our results, that are the consequence of the stationarity. We call that the stochastic process associated to the affine random process that results from condition1 is **regular**, if for all $\mathbf{p}$ with $p_1 > 0$, for all $\phi \in L^{\mathbf{q}}$ all with $q_1 = \infty$, there exists $0 < \beta < \infty$ and a monotone increasing function $\gamma(\mathbf{p})$ such that for fixed p_2 and $p_1 < \beta$ we have $\gamma(\mathbf{p}) < 0$ and for $p_1 > \beta$ we have $\gamma(\mathbf{p}) > 0$ and the following bounds

$$N_{\mathbf{p},\mathbf{p}}(\Gamma(t_1, t)) < d_1 \exp\left[\gamma(\mathbf{p})(t_2 - t_1)\right] \tag{31}$$

$$\|\Gamma(t_1, t)\phi\|_{\mathbf{p}} \le d_2 \|\phi\|_{\mathbf{q}} N_{\mathbf{p},\mathbf{p}}(\Gamma(t_1, t)) \tag{32}$$

here d_1, d_2 are some constants. Observe that in equation 32, compared with equation 21 there is a new constant and the indices $N_{\mathbf{p},\mathbf{q}}$ are modified on $N_{\mathbf{p},\mathbf{p}}$. It is easy to prove that there exists a large class of stochastic processes with this property

2.2 The main theorem.

Theorem 4 *Suppose first that the stochastic process $b_t(.)$ is stationary and $b_t(.) \in L^{\mathbf{q}}$ with $q_1 = \infty$. In addition suppose that the process $G(t_1, t_2, .)$ is regular. Then for sufficiently large $t_2 - t_1$ the solution of the equation 7 9 is bounded by*

$$\|X_t(\omega)\| \le d_1 \exp\left[\gamma(\mathbf{p})t\right] \|x_0\| + A/\gamma \left[\exp(\gamma t) - 1\right] \tag{33}$$

where $\gamma = \gamma(\mathbf{p})$ and $A > 0$.

Proof. We use the regularity conditions 31, 32 , as well as the equations 7 9 and we obtain

$$\|X_t(\omega)\|_{\mathbf{p}} \le \|G(0, t, .)x_0\|_{\mathbf{p}} + \int\limits_0^t \|G(t_1, t, .)b_{t_1}(.)\|_{\mathbf{p}} \, dt_1 \tag{34}$$

$$= X \qquad + \qquad Y \tag{35}$$

by using equation 21 we obtain the first term in inequality 33 $X = d_1 \exp\left[\gamma(\mathbf{p})t\right] \|x_0\|$. In the second term, by using the Hölder inequality 21 we obtain

$$Y \le \int\limits_0^t N\left[G(t_1, t_2, .)\right]_{\mathbf{p}} \|b_{t_1}(.)\|_{\mathbf{q}} \, dt_1 \tag{36}$$

. By using the stationarity of $b_{t_1}(.)$ and regularity condition 32 we obtain

$$Y \le \|b_{t_1}(.)\|_{\mathbf{q}} \, C d_2 d_1 \int\limits_0^t \exp\left[\gamma(\mathbf{p})(t_2 - t_1)\right] dt_1$$

which proves the theorem. ■

3 Conclusions.

In a class of general finite dimensional random affine differential equations the large time behavior of the solution was studied. Under general assumptions we proved that the computation of the heavy tail properties of the stationary solution can be reduced to the study of the large time behavior of the generalized $L^{p_{1,2}}$ norm [18] of the solution of the homogenous random linear equation. A critical exponent β_c was defined such that the norm of the solution, of order p_1 remains bounded if $p_1 < \beta_c$ and diverges, on a massive set of initial conditions, when $p_1 > \beta_c$. When heavy tail exists then β_c is the heavy tail exponent. The speed of convergence/divergence, for large time, of the norm of order p_1 of the solution is exponential, depending on $p - \beta_c$. The generalization to the case when the vector space V is a more general Banach space with infinite dimension remains a challenging open problem.

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