Solitonic solutions for RK-RLW equation with different order of nonlinearity

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Abstract

In this paper we investigate the solutions of the important nonlinear partial differential equation Rosenau-Kawahara-RLW (RK-RLW). Hyperbolic-type solutions, harmonic-type solutions, and periodic-type solutions related to some values of parameters are pointed out. A part of these solutions are quite new, not being reported previously in literature. They describe the rich dynamics of the concerned system. Few graphical representations of these solutions support this statement.

Keywords: Rosenau-Kawahara-RLW equation, symbolic computations, soliton.

Introduction

Nonlinear phenomena are much more common in nature than linear ones and are commonly described by partial nonlinear differential equations (NPDEs). Therefore, solving them is a very important problem, because solutions give us important information on the dynamics of these phenomena. There is no clear and complete recipe for solving these equations, nor is it clear whether the equation can be integrable or not. The NPDEs do not have a single solution, but can admit several classes of solutions. These depend on the initial conditions or parameter values that appear in the equation. Some of these types of solutions are known as traveling wave solutions and there is a wide variety of methods that can be used to obtain them.

Among the direct methods that allow to obtain the traveling wave solutions we mention the following: the sin-cos method [1], the hyperbolic tangent method [2 – 4], tanh-coth method [5, 6] F-expansion method [7], elliptical function method [8], the imposition of specific conditions of integrability [9], the extended equation test method [10], the G'/G method [11], the truncation method [12], the exponential function method [13], or the functional expansion introduced in [14].

To explain how the generalized Rosenau-Kawahara-RLW equation was reached, we will briefly outline the steps taken.

Starting from the KdV equation:

$$u_t + 6u_x u + u_{3x} = 0 \tag{1}$$

And the RLW equation (regularized equation of long waves), or the BBM equation:

$$u_t + u_x + u_x u - u_{2xt} = 0,$$

It has been observed that the KdV equation cannot also describe wave-wave or wave-wall interactions, and therefore Rosenau proposed an equation called the Rosenau equation [15]:

$$u_t + u_x + u_x u + u_{4xt} = 0 \quad (2)$$

To this equation the term $-u_{2xt}$ is added and thus the equation Rosenau-RLW [16]:

$$u_t + u_x + u_x u + u_{4xt} - u_{2xt} = 0 \quad (3)$$

extended to the generalized Rosenau-RLW equation [17]:

$$u_t + u_x + u_x u^n + u_{4xt} - u_{2xt} = 0 \quad (4)$$

where $n > 0$ is an integer showing the power of non-linearity.

Returning to the Rosenau equation, the viscosity term of the KdV equation was introduced into it, thus obtaining the Rosenau-KdV equation [18]:

$$u_t + u_x + u_x u + u_{4xt} + u_{3x} = 0 \quad (5)$$

By joining the Rosenau-RLW equation (3) with the Rosenau-KdV equation (5), the Rosenau-KdV-RLW equation was obtained [19]:

$$u_t + u_x + u_x u + u_{4xt} + u_{3x} - u_{2xt} = 0 \quad (6)$$

This equation can be extended by increasing the power of nonlinearity by replacing the term $u_x u$ cu $u_x u^n$ sau $\left(\frac{u^{n+1}}{n+1}\right)_x$.

Solitary waves, shock waves, as well as conservation laws and asymptotic behavior for the extended or generalized Rosenau-KdV-RLW equation have been studied by [20].

On the other hand, the Kawahara equation, similar to the Rosenau-KdV equation with the distinction that instead of the term u_{4xt} contains the term $-u_{4x}$, it appeared in the modeling of shallow water waves with shallow tension [21]:

$$u_t + u_x + u_x u - u_{4x} + u_{3x} = 0 \quad (7)$$

By adding a new viscosity term $-u_{5x}$ to the Rosenau-KdV equation, Zuo [22] obtained the Rosenau-Kawahara equation:

$$u_t + u_x + u_x u + u_{4xt} + u_{3x} - u_{5x} = 0 \quad (8)$$

and studied the solitary solution and its periodic solution. Biswas [23] analyzed the solitary wave solution and the two invariants of the equation of this type.

The generalized Rosenau-Kawahara equation:

$$u_t + u_x + u_x u^n + u_{4xt} + u_{3x} - u_{5x} = 0 \quad (9)$$

has been extensively studied and new conservation laws have been found based on Lie symmetry analysis and soliton solutions [20, 24]. In 2015, J. Zuo [25] obtained solutions for this equation using the hyperbolic secant method.

By coupling the generalized Rosenau-RLW equation (4) with the generalized Rosenau-Kawahara equation (9), the generalized Rosenau-Kawahara-RLW equation is obtained [21]:

$$u_t + u_x + u_x u^n + u_{4xt} + u_{3x} - u_{2xt} - u_{5x} = 0 \quad (10)$$

where $n > 0$ is an integer showing the power of non-linearity.

We will study this equation in the form of:

$$u_t - \alpha u_{2xt} + \beta u_{4xt} + \gamma u_x + \delta u^n u_x + \tau u_{3x} + \lambda u_{5x} = 0 \quad (11)$$

We can reduce the number of degrees of fredom in the previous equation, by transforming the nonlinear differential equation with partial derivatives (NPDE) into a ordinary nonlinear differential equation (NODE), by using as unique variable $\xi = x - vt$:

$$(\gamma - v)u_{\xi} + (\alpha v + \tau)u_{3\xi} + (\lambda - \beta v)u_{5\xi} + \frac{\delta}{n+1}(u^{n+1})_{\xi} = 0 \qquad (12)$$

By integrating (5) in relation to ξ and vanishing the integration constant, we get:

$$(\gamma - v)u + (\alpha v + \tau)u_{2\xi} + (\lambda - \beta v)u_{4\xi} + \frac{\delta}{n+1}u^{n+1} = 0 \qquad (13)$$

2 Description of the method

In this section we will describe an efficient direct method namely sin-cos method which allows to obtain few types of solutions for NPDEs.

This method involves looking for a solution of the ordinary equation in the form of a sinus or cosinus function at a power. The general form of this solution is:

$$u(\xi) = A\cos^{\eta}(B\xi) \qquad (14)$$

Unknown parameters A, B, n will be determined so that the general solution (14) verifies the equation.

3 Application of the RK-RLW equation

We can apply the sin-cos method for the equation (13). Thus, we can express a general solution to this equation in the form (14).

We calculate the derivatives of this solution until order IV and then insert them into the equation (13).

From the balance of the maximum powers of the $\cos$ function we find:

$$\eta - 4 = \eta(n+1) \Rightarrow \eta = -\frac{4}{n} \qquad (15)$$

We group the terms according to his powers $\cos(B\xi)$ and equaling zero coefficients. We get an algebraic system of 3 equations with unknowns A, B, v:

$$\cos^{\eta}(B\xi): (\gamma - v)A - (\alpha v + \tau)AB^2\eta^2 + (\lambda - \beta v)AB^4\eta^4 = 0 \qquad (16)$$

$$\cos^{\eta-2}(B\xi): (\alpha v + \tau)AB^2\eta(\eta - 1) - 2(\lambda - \beta v)AB^4\eta(\eta - 1)(\eta^2 - 2\eta + 2) = 0$$

$$\cos^{\eta-4}(B\xi): (\lambda - \beta v)AB^4\eta(\eta - 1)(\eta - 2)(\eta - 3) + \frac{\delta}{n+1}A^{n+1} = 0$$

The solutions of the equation system are:

$$B = \pm\sqrt{\frac{(\alpha v + \tau)}{2(\lambda - \beta v)(\eta^2 - 2\eta + 2)}} \qquad (17)$$

$$A = \left[-\frac{(n+1)\eta(\eta - 1)(\eta - 2)(\eta - 3)(\alpha v + \tau)^2}{4\delta(\lambda - \beta v)(\eta^2 - 2\eta + 2)^2} \right]^{1/n}$$

$$v=\frac{-b\pm\sqrt{\Delta'}}{a}$$

with notations:

$$\Delta'=b^2-ac$$
$$a=\alpha^2\eta^2(\eta^2-2)^2-4\beta(\eta^2-2\eta+2)^2$$
$$b=\alpha\tau\eta^2(\eta^2-2)^2+2(\eta^2-2\eta+2)^2(\gamma\beta+\lambda)$$
$$c=\eta^2\tau^2(\eta^2-2)^2-4\gamma\lambda(\eta^2-2\eta+2)^2$$

Case 1:

We choose parameters:

$$n=2\Rightarrow\eta=-2 \tag{18}$$
$$\alpha=2,\tau=\beta=\gamma=\lambda=\delta=1$$
$$v_1=7\,,A_1=3\frac{\sqrt{15}}{2},B_1=i\frac{\sqrt{2}}{4}$$
$$v_2=\frac{1}{3},\ A_2=i\frac{\sqrt{15}}{2},\ B_2=\frac{\sqrt{2}}{4}$$

The master equation will be:

$$(1-v)u+(2v+1)u_{2\xi}+(1-v)u_{4\xi}+\frac{1}{3}u^3=0 \tag{19}$$

And she will have the solutions:

$$u_1(x,t)=\frac{3}{2}\sqrt{15}\cos^{-2}\left(i\frac{\sqrt{2}}{4}(x-7t)\right)=\frac{3}{2}\sqrt{15}\,\mathrm{sech}^2\left(\frac{\sqrt{2}}{4}(x-7t)\right) \tag{20}$$

$$u_2(x,t)=\sqrt{-\frac{15}{4}}\cos^{-2}\left(\frac{\sqrt{2}}{4}\left(x-\frac{t}{3}\right)\right)=i\sqrt{\frac{15}{4}}\sec^{-2}\left(\frac{\sqrt{2}}{4}\left(x-\frac{t}{3}\right)\right) \tag{21}$$

Graphic representation of the solution for $u_1(x,t)$ is given in Figure 1 and represents a bright soliton, and in Figure 2 the complex solution is plotted $u_2(x,t)$.

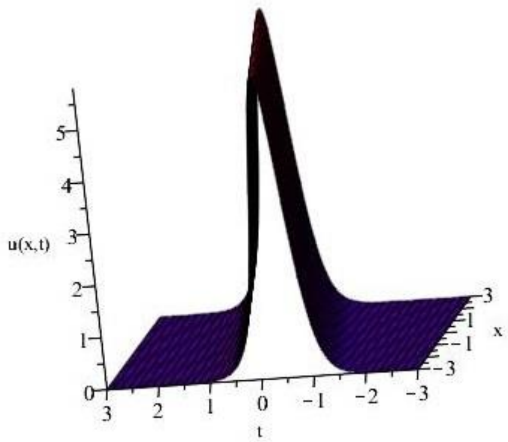

Figure 1: Graphic representation of the solution $u_1(x,t)$ for $x=-4,..,4$ and $t=-3,...,3$

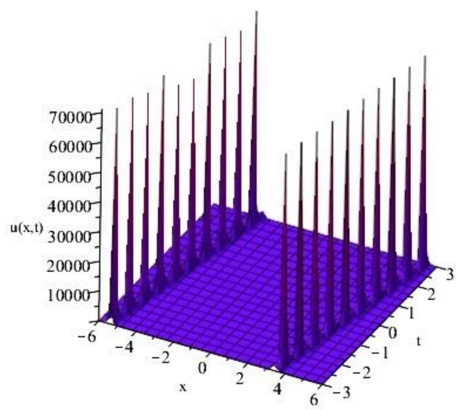

Figure 2: Graphic representation of the solution $u_2(x,t)$ for $x=-6,..,6$ and $t=-3,...,3$

Case 2 We choose the parameters:

$$n=1 \Rightarrow \eta=-4 \tag{22}$$

$$\alpha=2, \tau=\beta=\gamma=\lambda=\delta=1$$

$$\Rightarrow v_3=\frac{7}{25},\ A_3=-\frac{21}{10},\ B_3=\frac{\sqrt{6}}{12};v_4=19, A_4=\frac{105}{2}, B_4=i\frac{\sqrt{6}}{12}$$

The equation becomes:

$$(1-v_{3,4})u+(2v_{3,4}+1)u_{2\xi}+(1-v_{3,4})u_{4\xi}+\frac{1}{2}u^2=0 \tag{23}$$

With particular solutions:

$$u_3(x,t)=-\frac{21}{10}\cos^{-4}\left(\frac{\sqrt{6}}{12}\left(x-\frac{7}{25}t\right)\right) \tag{24}$$

$$u_4(x,t)=\frac{105}{2}\cosh^{-4}\left(\frac{\sqrt{6}}{12}(x-19t)\right) \tag{25}$$

They are shown in Figure 3 and Figure 4 respectively.

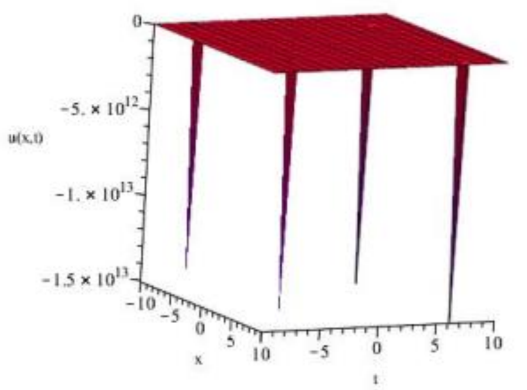

Figure 3: Graphic representation of the solution $u_3(x,t)$

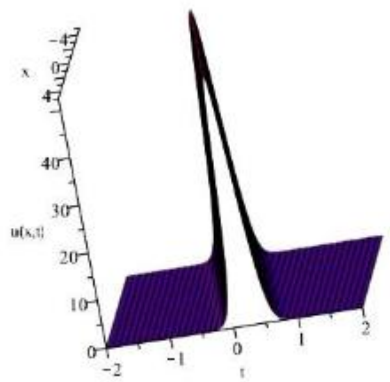

Figure 4: Graphic representation of the solution $u_4(x,t)$

Case 3: We choose the parameters:

$$n=3 \Rightarrow \eta=-4/3 \tag{26}$$

$$\alpha=2, \tau=\beta=\gamma=\lambda=\delta=1$$

$$\Rightarrow v_5=\frac{19}{49}, A_5=\left(-\frac{39}{7}\right)^{1/3}, B_5=\frac{3\sqrt{10}}{20}; v_6=\frac{13}{3}, A_6=\left(\frac{91}{3}\right)^{1/3}, B_6=i\frac{3\sqrt{10}}{20}$$

The solutions:

$$u_5(x,t)=\left(-\frac{39}{7}\right)^{1/3}\cos^{-4/3}\left(\frac{3\sqrt{10}}{20}\left(x-\frac{19}{49}t\right)\right) \tag{27}$$

$$u_6(x,t)=\left(\frac{91}{3}\right)^{1/3}\cosh^{-4/3}\left(\frac{3\sqrt{10}}{20}\left(x-\frac{13}{3}t\right)\right) \tag{28}$$

are shown in Figure 5 and Figure 6 respectively.

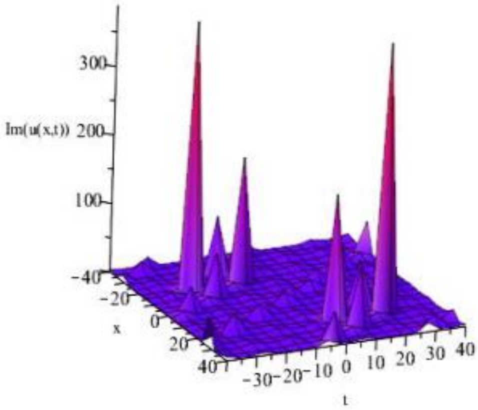

Figure 5: Graphic representation of the solution $u_5(x,t)$

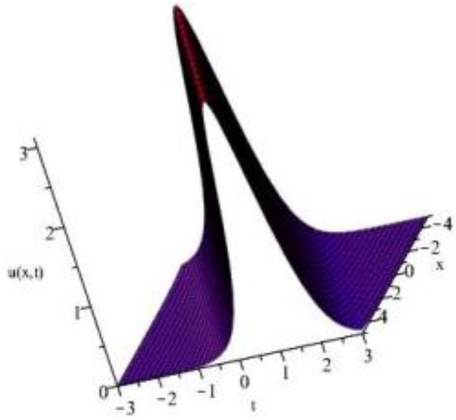

Figure 6: Graphic representation of the solution $u_6(x,t)$

Case 4: We choose the parameters:

$$n = 4 \Rightarrow \eta = -1 \tag{29}$$
$$\alpha = 2, \tau = \beta = \gamma = \lambda = \delta = 1$$
$$\Rightarrow v_7 = \frac{13}{4}, A_7 = (30)^{1/4}, B_7 = \frac{i\sqrt{3}}{3}; v_8 = \frac{7}{16}, A_8 = (-\frac{15}{2})^{1/4}, B_8 = \frac{\sqrt{3}}{3}$$

The solutions:

$$u_7(x,t) = (30)^{1/4} \cosh^{-1}\left(\frac{\sqrt{3}}{3}\left(x - \frac{13}{4}t \right) \right) \tag{30}$$

$$u_8(x,t) = \left(-\frac{15}{2} \right)^{1/4} \cos^{-1}\left(\frac{\sqrt{3}}{3}\left(x - \frac{7}{16}t \right) \right) \tag{31}$$

are shown in Figure 7 and Figure 8 respectively.

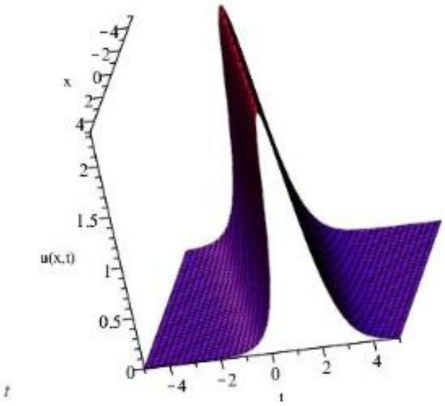

Figure 7: Graphic representation of the solution $u_7(x,t)$

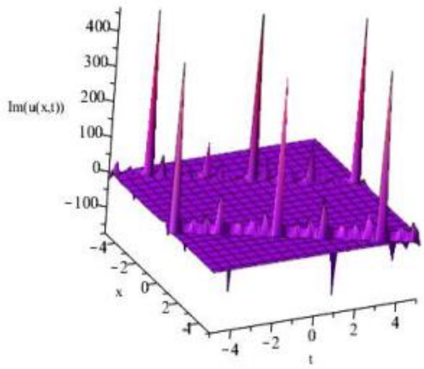

Figure 8: Graphic representation of the solution $u_8(x,t)$

4.Concluding remarks

This paper addressed the dynamics of shallow water waves by the RK-RLW equation with power law nonlinearity.

We used the direct sin-cos method to get solutions for the RK-RLW equation**.** From the graphs shown in the Figures 1 – 8 it is noted that the solutions $u_1(x,t)$, $u_4(x,t)$, $u_6(x,t)$, and $u_7(x,t)$ are solitonic solutions representing a bright soliton, the solution $u_5(x,t)$ is multi-peakons solution, and the solution $u_8(x,t)$ is a periodic solution.

By the symbolic computation technique, the sin-cos method has proven itself to be simple and efficient. Therefore, it very simple applys to a large variety of NPDEs.

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