On generalized Kulish-Sklyanin models

A. Florian[1], V. S. Gerdjikov[2,3,4], A. Streche-Pauna[1]

[1]*Dept. of Physics, University of Craiova,*
St. Alexandru Ioan Cuza 13, 200585 Craiova, Romania E-mail address: maria.alina2009@yahoo.com, aureliaflorian@yahoo.com

[2]*Sankt-Petersburg State University of Aerospace Instrumentation*
St-Petersburg, B.Morskaya, 67A, St-Petersburg, 190000, Russia

[3]*Institute for Advanced Physical Studies*
Sofia Techno Park, Sofia 1111, Bulgaria

[4]*Institute of Mathematics and Informatics*
Bulgarian Academy of Sciences,
8 Acad. G. Bonchev str., 1113 Sofia, Bulgaria

E-mail address: vgerdjikov@gmail.com

Abstract

We consider a class of Lax operators L related to BD.I type symmetric spaces. They allow one to solve special classes of vector NLS and matrix equations known as generalizations of the Kulish-Sklyanin type model. We construct two types of soliton solutions applying the dressing Za'kharov-Shabat method and using projectors of rank 1 and two. We also construct the kernel of the resolvent for L and prove that the fundamental analytic solutions of L satisfy completeness relation on a subspace of the vector space $\mathcal{M}$. Finally we show that the diagonal of the resolvent of L is a generating functional of the hierarchy of Lax representations for the nonlinear evolution equations related to L.

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1 Introduction

The NLS equation [47] and the vector NLS equations [33] provide an important class of integrable models with important physical applications [33, 42, 44, 32, 6, 3, 1, 41, 30, 11, 12, 13, 20, 21, 27]. This paper is devoted to the Kulish-Sklyanin (KS) model [32] and its generalizations. KS model is an integrable 3-component vector NLS model. It belongs to a family of vector NLS models related to the symmetric spaces $SO(2r+1)/S(O(2)\otimes O(2r-1))$ of **BD.I**-type (see [29]) and can be written in the form:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger, \vec{q})\vec{q}(x,t) - (\vec{q}, s_0\vec{q})s_0\vec{q}^*(x,t) = 0, \qquad (1)$$

where $\vec{q}(x,t)$ is a $2r-1$-component vector function vanishing fast enough for $|x|\to\infty$ and the constant matrix s_0 is introduced in eq. (3) below. The Hamiltonian for this family of Kulish-Sklyanin models (1) is given by

$$H_{\rm KS} = \int_{-\infty}^{\infty} dx \left((\partial_x \vec{q}^\dagger, \partial_x\vec{q}) - (\vec{q}^\dagger, \vec{q})^2 + \frac{1}{2}(\vec{q}^\dagger, s_0\vec{q}^*)(\vec{q}^T, s_0\vec{q}) \right). \qquad (2)$$

and generalizes the Manakov models related to $SU(n+1)/S(U(n)\otimes U(1))$ [33]. We have studied many aspects of the KS model, including its applications to spin-1 Bose Einstein condensates [28, 2, 11, 13, 17, 20, 21], the construction of its soliton solutions [12, 14, 41, 17, 15], their reductions [34, 31, 23] and interactions [41, 18, 16].

The paper is a natural extension of [38] and is structured as follows. Section 2 contains preliminaries concerning the symmetric spaces of BD.I type, the structure of the Lax operators and their reductions. In Section 3 we formulate the generalized KS models and give a nontrivial example of its reduction. Section 4 contains the spectral properties of the Lax operators. In the next Section 5 we derive the soliton solutions by applying the dressing Zakharov-Shabat method. We outline that for the matrix KS models one may use projectors of rank 2 and higher which allows one to obtain one-soliton solutions with richer internal structure. In Section 6 we construct the kernel of the resolvent for L and prove that the fundamental analytic solutions of L satisfy completeness relation on a subspace of the vector space $\mathcal{M}$. Finally we show that the diagonal of the resolvent of L is a generating functional of the hierarchy of Lax representations for the nonlinear evolution equations related to L. We end with some discussion and conclusions.

2 Preliminaries

2.1 On BD.I-type symmetric spaces

For our specific purposes we will choose the simple Lie group $\mathcal{G}\simeq SO(2r+1)$, its Lie algebra $\mathfrak{g}\simeq so(2r+1)$. The orthogonality condition that we will use below is

$$X\in so(2r+1) \qquad \text{iff} \qquad X + S_0 X^T S_0^{-1} = 0, \qquad S_0 = \sum_{s=1}^{2r+1} (-1)^{s+1} E_{s,2r+2-s}, \qquad (3)$$

where E_{kn} is a $2r+1\times 2r+1$-matrix with $(E_{kn})_{pj} = \delta_{kp}\delta_{nj}$. This choice ensures that the Cartan subalgebra $\mathfrak{h}$ consists of diagonal matrices. The element $J\in\mathfrak{h}$ is chosen as:

$$J_k = \sum_{s=1}^{s} H_{e_s} = \begin{pmatrix} \openone_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\openone_k \end{pmatrix}. \qquad (4)$$

We will assume that the reader is familiar with the theory of simple Lie groups and algebras, see [29]. The system of positive roots of $so(2r+1)$ is well known [29]:

$$\Delta^+ = \{e_i \pm e_j, \quad 1\le i<j\le r, \qquad e_j, \quad 1\le j\le r\}.$$

Here we also mention that using J_k and the Cartan involution one can introduce a $\mathbb{Z}_2$-grading in $\mathfrak{g}$:

$$\begin{gathered} C_1 = \exp(\pi i J) = \begin{pmatrix} -\mathbb{1}_k & 0 & 0 \\ 0 & \mathbb{1}_{2r-2k+1} & 0 \\ 0 & 0 & -\mathbb{1}_k \end{pmatrix}, \qquad \mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, \\ \mathfrak{g}^{(0)} \equiv \{X \in \mathfrak{g}^{(0)} \colon C_1 X C_1^{-1} = X\}, \qquad \mathfrak{g}^{(1)} \equiv \{Y \in \mathfrak{g}^{(1)} \colon C_1 Y C_1^{-1} = -Y\}. \end{gathered} \tag{5}$$

The $\mathbb{Z}_2$-grading means that

$$[X_1, X_2] \in \mathfrak{g}^{(0)}, \qquad [X_1, Y_1] \in \mathfrak{g}^{(1)}, \qquad [Y_1, Y_2] \in \mathfrak{g}^{(0)}, \tag{6}$$

and provides the local structure of the symmetric space of BD.I-class $SO(2r+1)/(SO(2r-2k+1) \otimes SO(2k))$. We will see that this construction is directly related to the KS model for $k = 1$ and to its generalizations for $k > 1$. It is well known that the set of positive roots and the system of simple roots of B_r are:

$$\begin{aligned} \Delta^+_{B_r} &\equiv \{e_i - e_j, \quad e_i + e_j, \quad 1 \leq i < j \leq r; \qquad e_j, \quad 1 \leq j \leq r\} \\ \pi_{B_r} &\equiv \{\alpha_k = e_k - e_{k+1}, \quad \alpha_r = e_r, \quad 1 \leq k \leq r-1.\} \end{aligned} \tag{7}$$

Introducing J_k as in (4) we split the system of positive roots into two subsets:

$$\Delta^+_1 = \{\beta, \quad \beta(J_k) = 1\} \qquad \Delta^+_0 = \{\beta, \quad \beta(J_k) = 0 \mod 2\}.. \tag{8}$$

Below we will consider two cases with $k = 1$ and $k = 3$, $r = 4$:

$$\begin{aligned} k = 1, r \geq 3 \qquad & \begin{aligned} \Delta^+_0 &= \{e_i \pm e_j \quad 2 \leq i < j \leq r, \quad e_j \quad 2 \leq i \leq r\}, \\ \Delta^+_1 &= \{e_1 \pm e_j, \quad 2 \leq j \leq r, \quad e_1\}; \end{aligned} \\ k = 3, r = 4 \qquad & \begin{aligned} \Delta^+_0 &= \{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, e_1, e_2, e_3\}, \\ \Delta^+_1 &= \{e_1 \pm e_4, e_2 \pm e_4, e_3 \pm e_4, e_4\}; \end{aligned} \end{aligned} \tag{9}$$

The potential of the Lax operator is given by:

$$Q(x,t) = \sum_{\alpha \in \Delta^+_1} (q_\alpha E_\alpha + p_\alpha E_{-\alpha}) \in \mathfrak{g}^{(1)}. \tag{10}$$

2.2 The Riemann-Hilbert problem and Lax representations

The solution of the inverse scattering problem (ISP) for generic $n \times n$ Lax operators (13) was strongly influenced by the works of Shabat [36, 37] who introduced the notion of the fundamental analytic solution (FAS). Then the ISP can be reduced to a Riemann-Hilbert problem. The next step was done by Zakharov and Shabat [45, 46] who developed the dressing method for constructing the soliton solutions of the relevant NLEE. Below we briefly explain their ideas.

We will start by formulating the Riemann-Hilbert problem (RHP) for the KS model:

$$\begin{gathered} \xi^+(x,t,\lambda) = \xi^-(x,t,\lambda) G(x,t,\lambda), \\ i\frac{\partial G}{\partial x} - \lambda[J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^2[J, G(x,t,\lambda)] = 0. \end{gathered} \tag{11}$$

Given the sewing function $G(x,t,\lambda)$ find the functions $\xi^{\pm}(x,t,\lambda)$ taking values in the simple Lie group $\mathcal{G}$ and analytic for $\operatorname{Im}\lambda \gtrless 0$ such that eq. (11) holds. It is natural to impose also the normalization condition:

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}, \tag{12}$$

which ensures that the RHP has unique regular solution, on Figure 1 we .show the analyticity regions and the contours that will be used below.

Theorem 1 (Zakharov-Shabat theorem [45, 46]) *Let $\xi^{\pm}(x,t,\lambda)$ be a solution of the RHP whose sewing function satisfies the equations in (11). Then $\xi^{\pm}(x,t,\lambda)$ is a fundamental analytic solution (FAS) of the operators:*

$$\begin{aligned} \tilde{L}\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda[J,\xi^{\pm}(x,t,\lambda)] = 0, \\ \tilde{M}\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2[J,\xi^{\pm}(x,t,\lambda)] = 0, \end{aligned} \tag{13}$$

where $U(x,t)$ is λ-independent and $V(x,t,\lambda)$ is linear in λ:

$$U(x,t) = \lambda J - (\lambda\xi J\hat{\xi}(x,t,\lambda))_+, \qquad V(x,t,\lambda) = \lambda^2 J - (\lambda^2\xi J\hat{\xi}(x,t,\lambda))_+, \tag{14}$$

Proof 1 (Idea of the proof.) *Consider the functions*

$$\begin{aligned} g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}(\xi^{\pm})^{-1}(x,t,\lambda) + \lambda\xi^{\pm}(x,t,\lambda)J(\xi^{\pm})^{-1}(x,t,\lambda), \\ f^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}(\xi^{\pm})^{-1}(x,t,\lambda) + \lambda^2\xi^{\pm}(x,t,\lambda)J(\xi^{\pm})^{-1}(x,t,\lambda), \end{aligned} \tag{15}$$

and using the explicit x and t-dependence of $G(x,t,\lambda)$ prove that $g^{+}(x,t,\lambda) = g^{-}(x,t,\lambda)$ and $f^{+}(x,t,\lambda) = f^{-}(x,t,\lambda)$. Then, using eq. (12) we find that:

$$\lim_{\lambda\to\infty} g^{\pm}(x,t,\lambda) = \lambda J, \qquad \lim_{\lambda\to\infty} f^{\pm}(x,t,\lambda) = \lambda^2 J, \tag{16}$$

It remains to apply the great Liouville theorem that ensures that the functions $g^{+}(x,t,\lambda) = g^{-}(x,t,\lambda)$ (resp. $f^{+}(x,t,\lambda) = f^{-}(x,t,\lambda)$) are analytic on the whole complex λ-plane and therefore are linear (resp. quadratic) in λ. Of course the coefficients of these polynomials may depend on x and t.

The explicit derivation of the Lax pairs is very effective if $\xi^{\pm}(x,t,\lambda)$ satisfy the canonical normalization (12). Indeed, in this case we can use the asymptotic expansion:

$$\xi^{\pm}(x,t,\lambda) = \exp\left(\mathcal{Q}(x,t,\lambda)\right), \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} Q_s. \tag{17}$$

Obviously such choice for $\xi^{\pm}(x,t,\lambda)$ involves a $\mathbb{Z}_2$ reduction, which can be formulated in several equivalent ways, e.g.:

$$\begin{aligned} &\text{a)} \qquad \xi^{+}(x,t,-\lambda) = (\xi^{-})^{-1}(x,t,\lambda), \qquad Q_1 \in \mathfrak{g}^{(1)}, \qquad Q_{2s} = 0, \\ &\text{b)} \qquad \xi^{+}(x,t,\lambda^*) = (\xi^{-})^{\dagger}(x,t,\lambda) \qquad \text{iff} \qquad Q_s = -Q_s^{\dagger}, \end{aligned} \tag{18}$$

In order to derive the relevant NLEE in terms of Q_1 we will use the formulae:

$$\xi^{\pm} J \hat{\xi}^{\pm}(x,t,\lambda) = J + \sum_{p=1}^{\infty} \frac{1}{p!} \operatorname{ad}_{\mathcal{Q}}^{p} J. \qquad \frac{\partial \xi^{\pm}}{\partial x} \hat{\xi}^{\pm}(x,t,\lambda) = \frac{\partial \mathcal{Q}}{\partial x} + \sum_{p=1}^{\infty} \frac{1}{(p+1)!} \operatorname{ad}_{\mathcal{Q}}^{p} \frac{\partial \mathcal{Q}}{\partial x}. \tag{19}$$

which allow us to express the Lax pair coefficients $U_s(x,t)$ and $V_s(x,t)$ in terms of Q_1 and its x-derivatives. In all our considerations we will need only the first few terms of these expansions; for more details see [23].

2.3 The reduction group

Following Mikhailov [34] we will also impose additional reductions using the famous reduction group G_R.

G_R is a finite group which preserves the Lax representation (13), i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations: i) $G_R \subset \operatorname{Aut}\mathfrak{g}$ and ii) $G_R \subset \operatorname{Conf}\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows:

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda), \tag{20}$$

where $C_k \in \operatorname{Aut}\mathfrak{g}$ and $\Gamma_k(\lambda) \in \operatorname{Conf}\mathbb{C}$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$. More specifically, below we will consider $\mathbb{Z}_2$-reductions of the form:

$$\text{a)} \quad B_1 U^{\dagger}(\kappa_1(\lambda)) B_1^{-1} = U(\lambda), \qquad B_1(V^{\dagger}(\kappa_1(\lambda)) B_1^{-1} = V(\lambda), \tag{21}$$

$$\text{b)} \quad B_2 U^{T}(\kappa_2(\lambda)) B_2^{-1} = -U(\lambda), \qquad B_2(V^{T}(\kappa_2(\lambda)) B_2^{-1} = -V(\lambda), \tag{22}$$

$$\text{c)} \quad B_3 U^{*}(\kappa_1(\lambda)) B_3^{-1} = -U(\lambda), \qquad B_3(V^{*}(\kappa_1(\lambda)) B_3^{-1} = -V(\lambda), \tag{23}$$

$$\text{d)} \quad B_4 U(\kappa_2(\lambda)) B_4^{-1} = U(\lambda), \qquad B_4(V(\kappa_2(\lambda)) B_4^{-1} = V(\lambda), \tag{24}$$

where the automorphisms B_k must of finite order. In the cases (21), (22) and (23) B_k must be of even order, which in general could be bigger than 2.

Beside the $\mathbb{Z}_2$-reductions we will impose additional $\mathbb{Z}$-reductions with $p > 2$:

$$\begin{gathered} AU(x,t,\kappa(\lambda))A^{-1} = U(x,t,\lambda), \qquad AV^{*}(x,t,\kappa(\lambda))A^{-1} = V(x,t,\lambda), \\ A(\xi^{+})(x,t,\kappa(\lambda))A^{-1} = (\xi^{+})^{-1}(x,t,\lambda), \qquad \kappa(\lambda) = \lambda\omega, \qquad \omega = e^{2\pi i/p}, \end{gathered} \tag{25}$$

where A is an automorphism of $\mathfrak{g}$ of order p. Typically we will use a realization of A as an element of the Weyl group of $\mathfrak{g}$.

3 Generalized Kulish-Sklyanin models

The sewing function of our RHP problem depends on two additional parameters x and t in a special way, which makes it convenient to analyze and solve special classes of nonlinear evolution equations (NLEE) in two-dimensional space-time, known as soliton equations. Here we also assume that the relevant symmetric space is $SO(2r+1)/(SO(2r-2k+1) \times SO(2k))$ with $k \geq 1$. We will omit the index k in J_k when it is clear from the context.

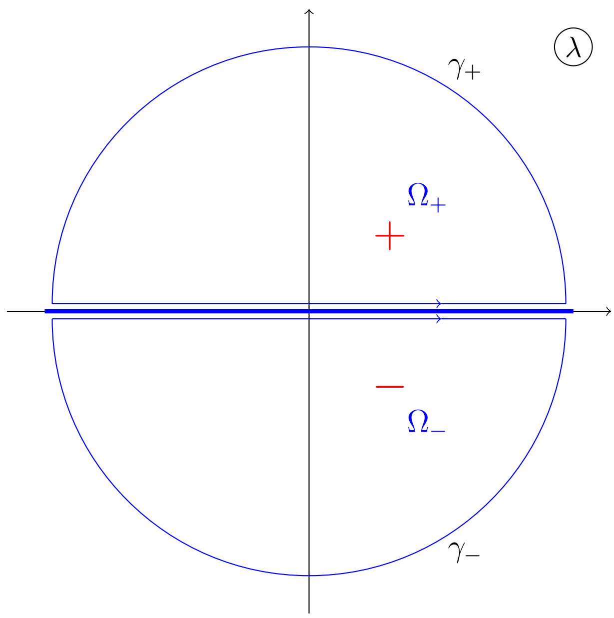

Figure 1: The continuous spectrum of a $L(\lambda)$ in thick blue, the analyticity regions $\Omega_\pm$ and the contours $\gamma_\pm$.

3.1 Lax representations

In the simplest case $s = 1$ $\vec{q}$ and $\vec{p}$ are vectors and the Lax pair is:

$$U^{(1)}(x,t,\lambda) = U_1(x,t) - \lambda J, \qquad V^{(1)}(x,t,\lambda) = iQ_{1,x} + V_2(x,t) + \lambda V_1(x,t) - \lambda^2 J, \tag{26}$$

where

$$U_1(x,t) = [J, Q_1(x,t)] = Q(x,t), \qquad V_1(x,t) = Q(x,t) \qquad V_2(x,t) = \frac{1}{2}\mathrm{ad}_{Q_1} Q(x,t), \tag{27}$$

$$Q_1(x,t) = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ -\vec{p} & 0 & s_0\vec{q} \\ 0 & -\vec{p}^T s_0 & 0 \end{pmatrix}, \qquad V_2(x,t) = \begin{pmatrix} (\vec{q},\vec{p}) & 0 & 0 \\ 0 & s_0\vec{q}\vec{p}^T s_0 - \vec{p}\vec{q}^T & 0 \\ 0 & 0 & -(\vec{q},\vec{p}) \end{pmatrix}. \tag{28}$$

As a result we get that this Lax pair leads to the well known KS model whose integrability has been known since 1981 [32]:

$$i\frac{\partial \vec{q}}{\partial t} + \frac{\partial^2 \vec{q}}{\partial x^2} + 2(\vec{q}^{\,\dagger}, \vec{q})\vec{q} - (\vec{q}^T s_0 \vec{q}) s_0 \vec{q}^{\,*} = 0, \qquad s_0 = \sum_{k=1}^{2r-1} (-1)^k E_{k,2r-k}. \tag{29}$$

where now E_{kn} is a $2r-1 \times 2r-1$-matrix with $(E_{kn})_{pj} = \delta_{kp}\delta_{nj}$. For applications of this model to Bose-Einstein condensates and detailed analysis for the inverse spectral transform see [21, 12, 9, 10, 13].

For $s > 1$ the compatibility condition becomes:

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \qquad V^{(1)}(x,t,\lambda) = iQ_{1,x} + V_2(x,t) + \lambda Q(x,t) - \lambda^2 J, \tag{30}$$

where

$$Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q} & 0 \\ \boldsymbol{p} & 0 & \tilde{\boldsymbol{q}} \\ 0 & \tilde{\boldsymbol{p}} & 0 \end{pmatrix}, \qquad V_2(x,t) = \begin{pmatrix} -\boldsymbol{q}\boldsymbol{p} & 0 & 0 \\ 0 & \boldsymbol{p}\boldsymbol{q} - \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}} & 0 \\ 0 & 0 & \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}} \end{pmatrix}. \tag{31}$$

The matrix S_0 from orthogonality condition (3) in this case equals $\begin{pmatrix} 0 & 0 & \boldsymbol{s}_1 \\ 0 & \boldsymbol{s}_2 & 0 \\ \boldsymbol{s}_1^{-1} & 0 & 0 \end{pmatrix}$. The blocks $\boldsymbol{s}_k$, $k = 1, 2$ are easily determined from (3) and satisfy $s_2^2 = \mathbb{1}$, $s_2^{-1} = \boldsymbol{s}_2$. Then $\tilde{\boldsymbol{q}} = -\boldsymbol{s}_2 \boldsymbol{q}^T \boldsymbol{s}_1$ and $\tilde{\boldsymbol{p}} = -\boldsymbol{s}_2 \boldsymbol{p}^T \boldsymbol{s}_1^{-1}$. As a result we get the generic Kulish-Sklyanin model [32]:

$$i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial^2 \boldsymbol{q}}{\partial x^2} + 2\boldsymbol{q}\boldsymbol{p}\boldsymbol{q} - \boldsymbol{q}\tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}} = 0, \qquad i\frac{\partial \boldsymbol{p}}{\partial t} - \frac{\partial^2 \boldsymbol{p}}{\partial x^2} - 2\boldsymbol{p}\boldsymbol{q}\boldsymbol{p} + \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}\boldsymbol{p} = 0, \tag{32}$$

After the additional reduction $\boldsymbol{p} = \boldsymbol{q}^\dagger$ we get:

$$i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial^2 \boldsymbol{q}}{\partial x^2} + 2\boldsymbol{q}\boldsymbol{q}^\dagger \boldsymbol{q} - \boldsymbol{q}\tilde{\boldsymbol{q}}\tilde{\boldsymbol{q}}^\dagger = 0. \tag{33}$$

3.2 Reduction of KS models – an example

Here we derive KS type models related to generic BD.I symmetric spaces $SO(2r+1)/(SO(2k) \times SO(2r-2k+1))$. These are rather complicated systems of equations for $k(2r-2k+1)$ functions of x and t. In this subsection we consider special case with $r = 4$, $k = 3$ and apply to it a special $\mathbb{Z}_6$-reduction. The result is a new type of 2-component NLS.

Consider $SO(9)/(SO(6) \times SO(3))$. Then the subset of positive roots is split into

$$\Delta_0^+ \equiv \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, e_4\}, \qquad \Delta_1^+ \equiv \{e_1 \pm e_4, e_2 \pm e_4, e_3 \pm e_4, e_1, e_2, e_3\}, \tag{34}$$

The reduction is given by the Weyl-group element $w_4 = S_{e_1-e_2} S_{e_2-e_3} S_{e_4}$, i.e. $w_4^6 = \mathrm{Id}$. Obviously this Weyl group element leaves invariant Δ_0 and Δ_1 and the roots Δ_1 are split into four orbits:

$$\begin{aligned} &\mathcal{O}_1^\pm \quad : \quad \pm e_1 - e_4 \to \pm e_2 + e_4 \to \pm e_3 - e_4 \to \pm e_1 + e_4 \to \pm e_2 - e_4 \to \pm e_3 + e_4, \\ &\mathcal{O}_3 \quad : \quad e_1 \to e_2 \to e_3, \qquad \mathcal{O}_4 \quad : \quad -e_1 \to -e_2 \to -e_3. \end{aligned} \tag{35}$$

After applying another $\mathbb{Z}_2$-reductions, i.e. $Q = Q^\dagger$ one may expect a 2-component NLS. Realization of the automorphism w_4 is as follows $w_4(X) = A_1 X A_1^{-1}$:

$$A_1 = \begin{pmatrix} \boldsymbol{a}_1 & 0 & 0 \\ 0 & \boldsymbol{a}_2 & 0 \\ 0 & 0 & \boldsymbol{a}_3 \end{pmatrix}, \quad \boldsymbol{a}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{a}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{a}_3 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \tag{36}$$

The orthogonality condition is given by (3) with $S_0 = \begin{pmatrix} 0 & 0 & s_1 \\ 0 & -s_1 & 0 \\ s_1 & 0 & 0 \end{pmatrix}$, $s_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

The corresponding potential of the Lax operator is as in (28) with

$$\boldsymbol{q}(x,t) = \begin{pmatrix} q_1 & q_2 & -q_1 \\ -q_1 & -q_2 & q_1 \\ q_1 & q_2 & -q_1 \end{pmatrix}, \quad \boldsymbol{p}(x,t) = \begin{pmatrix} p_1 & -p_1 & p_1 \\ p_2 & -p_2 & p_2 \\ -p_1 & p_1 & -p_1 \end{pmatrix}. \tag{37}$$

The condition $Q = Q^\dagger$ reduces to $p_1 = q_1^*$ and $p_2 = q_2^*$. Then the NLEE becomes

$$\begin{aligned} &i\frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + 6(|q_1|^2 + |q_2|^2) q_1(x,t) - 3q_2^2 q_1^*(x,t) = 0, \\ &i\frac{\partial q_2}{\partial t} + \frac{\partial^2 q_2}{\partial x^2} + 3(4|q_1|^2 + |q_2|^2) q_2(x,t) - 6q_1^2 q_2^*(x,t) = 0, \end{aligned} \tag{38}$$

The corresponding Hamiltonian is:

$$H = 2\left|\frac{\partial q_1}{\partial x}\right|^2 + \left|\frac{\partial q_2}{\partial x}\right|^2 - \frac{3}{2}\left(2|q_1|^2 + |q_2|^2\right)^2 + 3\left(q_1 q_2^* - q_1^* q_2\right)^2. \qquad (39)$$

The change of variables: $v_1 = \sqrt{6} q_1$, $v_2 = \sqrt{3} q_2$ leads to:

$$\begin{gathered} i\frac{\partial v_1}{\partial t} + \frac{\partial^2 v_1}{\partial x^2} + (|v_1|^2 + 2|v_2|^2) v_1(x,t) - v_2^2 v_1^*(x,t) = 0, \\ i\frac{\partial v_2}{\partial t} + \frac{\partial^2 v_2}{\partial x^2} + (2|v_1|^2 + |v_2|^2) v_2(x,t) - v_1^2 v_2^*(x,t) = 0, \\ \boldsymbol{q}(x,t) = \frac{1}{\sqrt{6}}\begin{pmatrix} v_1 & \sqrt{2}v_2 & -v_1 \\ -v_1 & -\sqrt{2}v_2 & v_1 \\ v_1 & \sqrt{2}v_2 & -v_1 \end{pmatrix}, \qquad \boldsymbol{p} = \boldsymbol{q}^\dagger. \end{gathered} \qquad (40)$$

It is easy to check that assuming canonical Poisson brackets $\{v_k(x), v_j^*(y)\} = \delta_{kj}\delta(x-y)$ for v_j, the canonical Hamiltonian equations of motion:

$$i\frac{\partial v_j}{\partial t} = \frac{\delta H'}{\delta v_j^*}, \qquad j = 1, 2, \qquad (41)$$

with

$$H' = \left|\frac{\partial v_1}{\partial x}\right|^2 + \left|\frac{\partial v_2}{\partial x}\right|^2 - \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + \frac{1}{2}\left(v_1 v_2^* - v_1^* v_2\right)^2, \qquad (42)$$

coincides with (40).

4 Spectral properties of the Lax operators

4.1 The case $SO(2r+1)/(SO(2r-1) \times SO(2))$

Here we will outline the methods of solving the direct and the inverse scattering problem (ISP) for L. We will use the Jost solutions which are defined by, see [10, 11, 12, 13] and the references therein

$$\lim_{x\to-\infty} \phi(x,t,\lambda) e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,t,\lambda) e^{i\lambda Jx} = \mathbb{1} \qquad (43)$$

and the scattering matrix $T(\lambda, t) \equiv \psi^{-1}\phi(x,t,\lambda)$. Due to the special choice of J and to the fact that the Jost solutions and the scattering matrix take values in the group $SO(2r+1)$ we can use the following block-matrix structure of $T(\lambda,t)$ and its inverse $\hat{T}(\lambda,t)$:

$$T(\lambda,t) = \begin{pmatrix} m_1^+ & -\vec{B}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{b}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \qquad \hat{T}(\lambda,t) = \begin{pmatrix} m_1^- & \vec{b}^{-T} & c_1^- \\ -\vec{B}^+ & \hat{\mathbf{T}}_{22} & s_0\vec{B}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix}, \qquad (44)$$

where $\vec{b}^\pm(\lambda,t)$ and $\vec{B}^\pm(\lambda,t)$ are $2r-1$-component vectors, $\mathbf{T}_{22}(\lambda)$ and $\hat{\mathbf{T}}_{22}(\lambda)$ are $2r-1 \times 2r-1$ blocks and $m_1^\pm(\lambda)$, $c_1^\pm(\lambda)$ are scalar functions satisfying $c_1^\pm = 1/2(\vec{b}^\pm \cdot s_0\vec{b}^\pm)/m_1^\mp$.

Important tools for reducing the ISP to a Riemann-Hilbert problem (RHP) are the fundamental analytic solution (FAS) $\chi^{\pm}(x,t,\lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda,t)$

$$\chi^{\pm}(x,t,\lambda)=\phi(x,t,\lambda)S_J^{\pm}(t,\lambda)=\psi(x,t,\lambda)T_J^{\mp}(t,\lambda)D_J^{\pm}(\lambda). \qquad (45)$$

Here $S_J^{\pm}$ and $T_J^{\pm}$ are upper- and lower-block-triangular matrices, while $D_J^{\pm}(\lambda)$ are block-diagonal matrices with the same block structure as $T(\lambda,t)$ above. Skipping the details we give the explicit expressions of the Gauss factors in terms of the matrix elements of $T(\lambda,t)$

$$\begin{aligned}
S_J^{\pm}(t,\lambda)&=\exp\left(\pm\sum_{\beta\in\Delta_1^+}\tau_\beta^{\pm}(\lambda,t)E_{\pm\beta}\right), \qquad T_J^{\pm}(t,\lambda)=\exp\left(\mp\sum_{\beta\in\Delta_1^+}\rho_\beta^{\pm}(\lambda,t)E_{\pm\beta}\right),\\
D_J^{+}&=\begin{pmatrix} m_1^+ & 0 & 0\\ 0 & \mathbf{m}_2^+ & 0\\ 0 & 0 & 1/m_1^+\end{pmatrix}, \qquad D_J^{-}=\begin{pmatrix} 1/m_1^- & 0 & 0\\ 0 & \mathbf{m}_2^- & 0\\ 0 & 0 & m_1^-\end{pmatrix},
\end{aligned} \qquad (46)$$

where $\vec{b}^+=(T_{1,2},\dots,T_{1,2r})^T$

$$\begin{aligned}
\vec{\tau}^+(\lambda,t)&=\frac{\vec{b}^-}{m_1^+}, \quad \vec{\tau}^-(\lambda,t)=\frac{\vec{B}^+}{m_1^-}, \qquad \boldsymbol{m}_2^+=\boldsymbol{T}_{22}+\frac{\vec{b}^+\vec{b}^{-,T}}{2m_1^+}=\hat{\boldsymbol{T}}_{22}+\frac{s_0\vec{b}^-\vec{b}^{+,T}s_0}{2m_1^+},\\
\vec{\rho}^+(\lambda,t)&=\frac{\vec{b}^+}{m_1^+}, \quad \vec{\rho}^-(\lambda,t)=\frac{\vec{b}^-}{m_1^-}, \qquad \boldsymbol{m}_2^-=\hat{\boldsymbol{T}}_{22}+\frac{\vec{B}^+\vec{B}^{-,T}}{2m_1^-}=\hat{\boldsymbol{T}}_{22}+\frac{s_0\vec{B}^-\vec{B}^{+,T}s_0}{2m_1^-},.
\end{aligned} \qquad (47)$$

If $Q(x,t)$ evolves according to (29) then the scattering matrix and its elements satisfy the following linear evolution equations

$$i\frac{d\vec{B}^{\pm}}{dt}\pm\lambda^2\vec{B}^{\pm}(t,\lambda)=0, \qquad i\frac{d\vec{b}^{\pm}}{dt}\pm\lambda^2\vec{b}^{\pm}(t,\lambda)=0, \qquad i\frac{dm_1^{\pm}}{dt}=0, \qquad i\frac{d\mathbf{m}_2^{\pm}}{dt}=0, \qquad (48)$$

so the block-diagonal matrices $D^{\pm}(\lambda)$ are generating functionals of the integrals of motion. The fact that all $(2r-1)^2$ matrix elements of $m_2^{\pm}(\lambda)$ for $\lambda\in\mathbb{C}_{\pm}$ generate integrals of motion reflects the super-integrability of the model and is due to the degeneracy of the dispersion law determined by $\lambda^2 J$. We remind that $D_J^{\pm}(\lambda)$ allow analytic extension for $\lambda\in\mathbb{C}_{\pm}$ and that their zeroes and poles determine the discrete eigenvalues of L. We will use also another set of FAS:

$$\chi^{\prime,\pm}(x,t,\lambda)=\chi^{\pm}(x,t,\lambda)\hat{D}_J^{\pm}(\lambda). \qquad (49)$$

The FAS for real λ are linearly related

$$\begin{aligned}
\chi^{+}(x,t,\lambda)&=\chi^{-}(x,t,\lambda)G_J(\lambda,t), \qquad G_{0,J}(\lambda,t)=S_J^{-}(\lambda,t)S_J^{+}(\lambda,t),\\
\chi^{\prime,+}(x,t,\lambda)&=\chi^{\prime,-}(x,t,\lambda)G_J^{\prime}(\lambda,t), \qquad G_{0,J}^{\prime}(\lambda,t)=T_J^{+}(\lambda,t)T_J^{-}(\lambda,t).
\end{aligned} \qquad (50)$$

One can rewrite eq. (50) in an equivalent form for the FAS $\xi^{\pm}(x,t,\lambda)=\chi^{\pm}(x,t,\lambda)e^{i\lambda Jx}$ and $\xi^{\prime,\pm}(x,t,\lambda)=\chi^{\prime,\pm}(x,t,\lambda)e^{i\lambda Jx}$ which satisfy also the relation

$$\lim_{\lambda\to\infty}\xi^{\pm}(x,t,\lambda)=\mathbb{1}, \qquad \lim_{\lambda\to\infty}\xi^{\prime,\pm}(x,t,\lambda)=\mathbb{1}. \qquad (51)$$

Then for $\operatorname{Im}\lambda = 0$ these FAS satisfy:

$$\begin{aligned} \xi^+(x,t,\lambda) &= \xi^-(x,t,\lambda) G_J(x,\lambda,t), \qquad & G_J(x,\lambda,t) &= e^{-i\lambda Jx} G_{0,J}(\lambda,t) e^{i\lambda Jx}, \\ \xi'^{,+}(x,t,\lambda) &= \xi'^{,-}(x,t,\lambda) G'_J(x,\lambda,t), \qquad & G'_J(x,\lambda,t) &= e^{-i\lambda Jx} G'_{0,J}(\lambda,t) e^{i\lambda Jx}. \end{aligned} \tag{52}$$

Obviously the sewing function $G_j(x,\lambda,t)$ is uniquely determined by the Gauss factors of $T(\lambda,t)$. In view of eq. (46) we arrive to the following

Lemma 2 *Let the potential $Q(x,t)$ be such that the Lax operator L has no discrete eigenvalues. Then as minimal set of scattering data which determines uniquely the scattering matrix $T(\lambda,t)$ and the corresponding potential $Q(x,t)$ one can consider either one of the sets $\mathfrak{T}_i$, $i=1,2$*

$$\mathfrak{T}_1 \equiv \{\vec{\rho}^+(\lambda,t), \vec{\rho}^-(\lambda,t), \quad \lambda\in\mathbb{R}\}, \qquad \mathfrak{T}_2 \equiv \{\vec{\tau}^+(\lambda,t), \vec{\tau}^-(\lambda,t), \quad \lambda\in\mathbb{R}\}. \tag{53}$$

Proof 2 *i) From the fact that $T(\lambda,t)\in SO(2r+1)$ one can derive that*

$$\frac{1}{m_1^+ m_1^-} = 1 + (\vec{\rho}^+, \vec{\rho}^-) + \frac{1}{4}(\vec{\rho}^+, s_0\vec{\rho}^+)(\vec{\rho}^-, s_0\vec{\rho}^-) \tag{54}$$

for $\lambda\in\mathbb{R}$. Using the analyticity properties of $m_1^\pm$ we can recover them from eq. (54) using Cauchy-Plemelji formulae. Given $\mathfrak{T}_i$ and $m_1^\pm$ one easily recovers $\vec{b}^\pm(\lambda)$ and $c_1^\pm(\lambda)$. In order to recover $\mathbf{m}_2^\pm$ one again uses their analyticity properties, only now the problem reduces to a RHP for functions on $SO(2r+1)$. The details will be presented elsewhere.

ii) Given $\mathfrak{T}_i$ one uniquely recovers the sewing function $G_J(x,t,\lambda)$. In order to recover the corresponding potential $Q(x,t)$ one can use the fact that the RHP (52) with canonical normalization has unique regular solution $\chi^\pm(x,t,\lambda)$. Given $\chi^\pm(x,t,\lambda)$ we recovers $Q(x,t)$ via:

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda\left(J - \chi^\pm J\widehat{\chi}^\pm(x,t,\lambda)\right) = \lim_{\lambda\to\infty}\lambda\left(J - \chi'^{,\pm} J\widehat{\chi}'^{,\pm}(x,t,\lambda)\right).. \tag{55}$$

which is well known.

We impose also the standard reduction, namely assume that $Q(x,t) = Q^\dagger(x,t)$, or in components $p_k = q_k^*$. As a consequence we have $\vec{\rho}^-(\lambda,t) = \vec{\rho}^{+,*}(\lambda,t)$ and $\vec{\tau}^-(\lambda,t) = \vec{\tau}^{+,*}(\lambda,t)$.

4.2 The case $SO(9)/(SO(3)\times SO(6))$

Effects on the scattering data:

- $T(\lambda)$ belongs to $SO(9)$, therefore $T^{-1} = S_0 T^T(\lambda) S_0$;
- $T(\lambda)$ is unitary matrix $T^\dagger(\lambda^*) = T^{-1}(\lambda)$;
- $T(\lambda)$ is invariant with respect to the automorphism A_1

We parametrize $T(t,\lambda)$ using the same block-matrix structure as for $Q(x,t)$ and J (31):

$$T(\lambda) = \begin{pmatrix} \boldsymbol{m}^+ & -\boldsymbol{b}^- & \boldsymbol{c}^- \\ \boldsymbol{b}^+ & \boldsymbol{T}_{22} & -\boldsymbol{B}^- \\ \boldsymbol{c}^+ & \boldsymbol{B}^+ & \boldsymbol{m}^- \end{pmatrix}, \qquad T^{-1}(\lambda) = \begin{pmatrix} \boldsymbol{s}_1\boldsymbol{m}^{-,T}\boldsymbol{s}_1 & \boldsymbol{s}_1\boldsymbol{B}^{-,T}\boldsymbol{s}_1 & \boldsymbol{s}_1\boldsymbol{c}^{-,T}\boldsymbol{s}_1 \\ -\boldsymbol{s}_1\boldsymbol{B}^{+,T}\boldsymbol{s}_1 & \boldsymbol{s}_1\boldsymbol{T}_{22}^T\boldsymbol{s}_1 & \boldsymbol{s}_1\boldsymbol{b}^{-,T}\boldsymbol{s}_1 \\ \boldsymbol{s}_1\boldsymbol{c}^{+,T}\boldsymbol{s}_1 & -\boldsymbol{s}_1\boldsymbol{b}^{+,T}\boldsymbol{s}_1 & \boldsymbol{s}_1\boldsymbol{m}^{-,T}\boldsymbol{s}_1 \end{pmatrix}, \tag{56}$$

$$T^{-1} = S_0 T^T(\lambda) S_0, \qquad T^\dagger(\lambda^*) = T^{-1}(\lambda), \qquad T(\lambda) = A_1 T(\lambda) A_1^{-1}, \tag{57}$$

i.e.

$$\begin{aligned} \boldsymbol{m}^{+,\dagger}(\lambda^*) &= \boldsymbol{s}_1 \boldsymbol{m}^{-,T}(\lambda) \boldsymbol{s}_1, \qquad \boldsymbol{c}^{+,\dagger}(\lambda^*) = \boldsymbol{s}_1 \boldsymbol{c}^{-,T}(\lambda) \boldsymbol{s}_1, \\ \boldsymbol{b}^{\pm,\dagger}(\lambda^*) &= \boldsymbol{s}_1 \boldsymbol{B}^{\mp,T}(\lambda) \boldsymbol{s}_1, \qquad \boldsymbol{B}^{\pm,\dagger}(\lambda^*) = \boldsymbol{s}_1 \boldsymbol{b}^{\mp,T}(\lambda) \boldsymbol{s}_1. \end{aligned} \tag{58}$$

and

$$\begin{aligned} \boldsymbol{m}^+ &= \boldsymbol{a}_1 \boldsymbol{m}^+ \boldsymbol{a}_1^{-1}, \qquad \boldsymbol{b}^- = \boldsymbol{a}_1 \boldsymbol{b}^- \boldsymbol{a}_2^{-1}, \qquad \boldsymbol{c}^- = \boldsymbol{a}_1 \boldsymbol{c}^- \boldsymbol{a}_3^{-1}, \\ \boldsymbol{b}^+ &= \boldsymbol{a}_2 \boldsymbol{b}^+ \boldsymbol{a}_1^{-1}, \qquad \boldsymbol{T}_{22} = \boldsymbol{a}_2 \boldsymbol{T}_{22} \boldsymbol{a}_2^{-1}, \qquad \boldsymbol{B}^- = \boldsymbol{a}_2 \boldsymbol{B}^- \boldsymbol{a}_3^{-1}, \\ \boldsymbol{c}^+ &= \boldsymbol{a}_3 \boldsymbol{c}^+ \boldsymbol{a}_1^{-1}, \qquad \boldsymbol{B}^+ = \boldsymbol{a}_3 \boldsymbol{B}^+ \boldsymbol{a}_2^{-1}, \qquad \boldsymbol{m}^- = \boldsymbol{a}_3 \boldsymbol{m}^- \boldsymbol{a}_3^{-1}. \end{aligned} \tag{59}$$

The spectral properties of these operators are analyzed similarly as above. The substantial difference is that now the continuous spectrum of L fills up the real axis and has multiplicity 6. It is not difficult to generalize the above lemma 2 also for this more general Lax operator. The substantial difference here is that now $\boldsymbol{m}^{\pm}$ are matrix-valued analytic functions. Therefore, instead of eq. (54) we will have to use the relevant RHP for recovering them from the minimal sets of scattering data. These details come out of the scope of the present paper.

5 Construction of the soliton solutions

The involution:

$$U^\dagger(x,t,\kappa_1\lambda^*) = U(x,t,\lambda), \qquad Q(x,t) = Q^\dagger(x,t), \qquad \kappa_1 = 1, \tag{60}$$

means that the Jost solutions must satisfy:

$$\phi^\dagger(x,t,\lambda^*) = \phi^{-1}(x,t,\lambda), \qquad \psi^\dagger(x,t,\lambda^*) = \psi^{-1}(x,t,\lambda), \tag{61}$$

so for the scattering matrix we have

$$T^\dagger(t,\lambda^*) = T^{-1}(t,\lambda), \tag{62}$$

and for the Gauss factors:

$$S^{-\dagger}(\lambda^*) = \hat{S}^+(\lambda), \qquad T^{-\dagger}(\lambda^*) = \hat{T}^-(\lambda), \qquad D^{-\dagger}(\lambda^*) = \hat{D}^+(\lambda), \tag{63}$$

As a consequence, if $\lambda_1^+ \in \mathbb{C}_+$ is a zero of $D^+(\lambda)$ and an eigenvalue of L then $(\lambda_1^+)^* \in \mathbb{C}_-$ is a zero of $D^-(\lambda)$ and is also an eigenvalue of L.

There are several versions (realizations) of the Zakharov-Shabat dressing method [45, 46, 43, 15, 20, 21]. Here we will use the one of them that is most convenient to our purposes.

Assume we know the solution $\xi_0^\pm(x,t,\lambda)$ of the RHP which have simple poles at the points $\lambda_j^\pm \in \mathbb{C}_\pm$, $j = 1,\dots,k-1$. Let us denote the corresponding FAS of L_0 by $\chi_0^\pm(x,t,\lambda)$.

Next we apply the dressing method to construct the new FAS $\xi^\pm(x,t,\lambda)$ which are related to $\xi_0^\pm(x,t,\lambda)$ by the dressing factor:

$$\xi^\pm(x,t,\lambda) = u(x,t,\lambda)\xi_0^\pm(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \tag{64}$$

which has simple poles also for $\lambda = \lambda_k^\pm \neq \lambda_j^\pm$. Obviously $u(x,t,\lambda) \in SO(2r+1)$. This can be ensured if we choose:

$$u(x,t,\lambda) = \exp(\ln(c_k(\lambda)(P_k - \bar{P}_k), \qquad c_k(\lambda) = \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-}, \qquad \bar{P}_k = S_0 P_k^T S_0. \tag{65}$$

It is easy to check that $P_k - \bar{P}_k \in so(2r+1)$, which means that with the choice in (65) $u(x,t,\lambda) \in SO(2r+1)$ for any choices of P_k. We choose P_k to be a rank 1 projector such that $P_k\bar{P}_k = 0$. Such choice allows us to evaluate the right hand side of (65) with the result:

$$
\begin{aligned}
u(x,t,\lambda) &= \mathbb{1} + (c_1(\lambda) - 1)P_k + \left(\frac{1}{c_1(\lambda)} - 1\right)\bar{P}_k, \\
u^{-1}(x,t,\lambda) &= \mathbb{1} + \left(\frac{1}{c_1(\lambda)} - 1\right)P_k + (c_1(\lambda) - 1)\bar{P}_k.
\end{aligned}
\tag{66}
$$

Following Zakharov-Shabat ideas we request that if $\chi_0^{\pm}(x,t,\lambda)$ is a FAS of L_0 with potential $Q_0(x,t)$, i.e.

$$
L_0\chi_0^{\pm}(x,t,\lambda) \equiv i\frac{\partial \chi_0}{\partial x} + (Q_0(x,t) - \lambda J)\chi_0^{\pm}(x,t,\lambda) = 0, \tag{67}
$$

then $\xi^{\pm}(x,t,\lambda)$ will be a FAS of the operator L with potential $Q(x,t)$:

$$
L\chi^{\pm}(x,t,\lambda) \equiv i\frac{\partial \chi}{\partial x} + (Q(x,t) - \lambda J)\chi^{\pm}(x,t,\lambda) = 0. \tag{68}
$$

Thus the dressing factor $u(x,t,\lambda)$ and its inverse must satisfy the equations

$$
\begin{aligned}
&i\frac{\partial u}{\partial x} + (Q(x,t) - \lambda J)u(x,t,\lambda) - u(x,t,\lambda)(Q_0(x,t) - \lambda J) = 0, \\
&i\frac{\partial \hat{u}}{\partial x} + (Q_0(x,t) - \lambda J)\hat{u}(x,t,\lambda) - \hat{u}(x,t,\lambda)(Q(x,t) - \lambda J) = 0
\end{aligned}
\tag{69}
$$

identically with respect to λ. Since $u(x,t,\lambda)$ and $\hat{u}(x,t,\lambda)$ have poles at $\lambda = \lambda_k^{\pm}$, then the residues of the left hand sides of the eqs. (69) must vanish. This leads to the following equations for the projectors P_k and $\bar{P}_k$

$$
\begin{aligned}
&i\frac{\partial P_k}{\partial x} + (Q(x,t) - \lambda_k^{-} J)P_k(x,t) - P_k(x,t)(Q_0(x,t) - \lambda_k^{-} J) = 0, \\
&i\frac{\partial P_k}{\partial x} + (Q_0(x,t) - \lambda_k^{+} J)P_k(x,t) - P_k(x,t)(Q(x,t) - \lambda_k^{+} J) = 0,
\end{aligned}
\tag{70}
$$

and similar equations for $\bar{P}_k = S_0 P_k^T S_0$. Taking the limit of eqs. (69) for $\lambda \to \infty$ we find the new potential $Q(x,t)$

$$
Q(x,t) - Q_0(x,t) = -(\lambda_k^{+} - \lambda_k^{-})[J, P_k - \bar{P}_k]. \tag{71}
$$

Skipping the details (see [10, 15, 12]) we obtain:

$$
P_k = \frac{|n_k\rangle\langle m_k|}{\langle m_k|n_k\rangle}, \quad |n_k\rangle = \chi_0^{+}(x,t,\lambda_k^{+})|n_{k0}\rangle, \quad \langle m_k| = \langle m_{k0}|\hat{\chi}_0^{-}(x,t,\lambda_k^{-}). \tag{72}
$$

For $Q_0 = 0$ we have $|n_k\rangle = e^{(z_k - i\phi_k)J}|n_{k0}\rangle$ where

$$
\lambda_k^{\pm} = \mu_k \pm i\nu_k, \qquad z_k = \nu_k(x + 2\mu_k t), \qquad \phi_k = \mu_k x + (\mu_k^2 - \nu_k^2)t
$$

, and $|n_{k0}\rangle$ and $\langle m_{k0}|$ are constant polarization vector satisfying $\langle m_{k0}|S_0|n_{k0}\rangle = 0$.

The soliton solutions of the generalized KS model are parametrized by the eigenvalues $\lambda_k^{\pm}$ and by the polarization vectors $|n_{k0}\rangle$ and $\langle m_{k0}|$, which due to the typical reductions are related by $\langle m_{k0}| = |n_{k0}\rangle^\dagger$. The vectors are constrained by the condition $\langle m_{k0}|S_0|n_{k0}\rangle = 0$. If we introduce

$$|n_{k0}\rangle = (n_{k0,1}, \vec{\nu}_{k0}, \bar{n}_{k0,1})^T, \quad \langle m_{k0}|S_0|n_{k0}\rangle = 2n_{k0,1}\bar{n}_{k0,1} - \vec{\nu}_{k0}^T s_0 \vec{\nu}_{k0} = 0. \tag{73}$$

For the special case when $\chi_0^\pm(x,t,\lambda)$ is the regular solution of the RHP, i.e. $\chi_0^\pm(x,t,\lambda) = \exp(-i(\lambda x + \lambda^2 t)J)$ and for $r = 3$ we have the 3-component KS model and the one-soliton solution $\vec{q}_{1s}$ takes the form:

$$\vec{q}_{1s}(x,t;z_1,\phi_1) = -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1)} \left(e^{-z_1} s_0|\vec{\nu}_{01}\rangle + e^{z_1}|\vec{\nu}_{01}^*\rangle\right)}{\cosh(2z_1) + (\vec{\nu}_{01}^\dagger, \vec{\nu}_{01})}. \tag{74}$$

We will need also the limits of the dressing factors for $x \to \pm\infty$. The results is:

Lemma 3 ([15]) *The asymptotics of $P_s(x,t)$ for $z_s \to \pm\infty$ are given by:*

$$\lim_{z_s\to\infty} P_k(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \lim_{z_s\to-\infty} P_k(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{75}$$

Therefore the asymptotics of $u_s(x,t,\lambda)$ for $z_s \to \pm\infty$ take the form:

$$\begin{aligned} \lim_{x\to\infty} u(x,t,\lambda) &= e^{J\ln c_k(\lambda)} = diag\,(c_k(\lambda), \mathbb{1}, c_k^{-1}(\lambda)), \\ \lim_{x\to-\infty} u(x,t,\lambda) &= e^{-J\ln c_k(\lambda)} = diag\,(c_k^{-1}(\lambda), \mathbb{1}, c_k(\lambda)). \end{aligned} \tag{76}$$

Proof 3 *The proof of the lemma is given in [15].*

Note that $u_k^\pm(\lambda)$ are x and t independent, and are given by diagonal matrices.

We end this section noting that for the generalized KS models, i.e. for $s > 1$ we may construct one-soliton solutions using projectors of rank higher than 1. For example generic projector of rank 2 has the form:

$$P_k = \sum_{a,b=1}^{2} |n_a\rangle \widehat{M}_{ab} \langle m_b|, \quad M_{bc} = \langle m_b|n_c\rangle, \quad \widehat{M} \equiv M^{-1}, \tag{77}$$

where

$$|n_a\rangle = \chi_0^+(x,t,\lambda_k^+)|n_{a0}\rangle, \quad \langle m_b| = \langle m_{b0}|\hat{\chi}_0^-(x,t,\lambda_k^-), \quad a,b = 1,2. \tag{78}$$

A convenient parametrization of the vectors is:

$$\begin{gathered} |n_{a0}\rangle = \begin{pmatrix} |n_{a0}^1\rangle \\ |n_{a0}^2\rangle \\ |n_{a0}^3\rangle \end{pmatrix}, \qquad \langle m_{b0}| = \left(\langle m_{b0}^1|, \langle m_{b0}^2|, \langle m_{b0}^3|\right), \qquad a,b = 1,2; \\ \langle m_{b0}|S_0|n_{a0}\rangle = 0, \quad \text{i.e.} \quad \langle m_{b0}^1|s_0|n_{a0}^3\rangle - \langle m_{b0}^2|s_0|n_{a0}^2\rangle + \langle m_{b0}^3|s_0|n_{a0}^1\rangle = 0. \end{gathered} \tag{79}$$

Then

$$\begin{gathered} M = \begin{pmatrix} \langle m_1|n_1\rangle & \langle m_1|n_2\rangle \\ \langle m_2|n_1\rangle & \langle m_2|n_2\rangle \end{pmatrix}, \qquad M^{-1} = \frac{1}{\det M} \begin{pmatrix} \langle m_2|n_2\rangle & -\langle m_1|n_2\rangle \\ -\langle m_2|n_1\rangle & \langle m_1|n_1\rangle \end{pmatrix}, \\ \det M = \langle m_1|n_1\rangle\langle m_2|n_2\rangle - \langle m_1|n_2\rangle\langle m_2|n_1\rangle, \end{gathered} \tag{80}$$

where

$$
\begin{aligned}
\langle m_a|n_b\rangle &= \eta_{0;ab}\cosh(2z_k+\xi_{0;ab})+\kappa_{0;ab},\\
\eta_{0;ab} &= \sqrt{\langle m^1_{0a}|n^1_{0b}\rangle\langle m^3_{0a}|n^3_{0b}\rangle}, \qquad \xi_{0;ab}=\frac{1}{2}\ln\frac{\langle m^1_{0a}|n^1_{0b}\rangle}{\langle m^3_{0a}|n^3_{0b}\rangle}, \qquad \kappa_{0;ab}=\langle m^2_{0a}|n^2_{0b}\rangle.
\end{aligned} \tag{81}
$$

The corresponding reflectionless potential will take the form:

$$
Q(x,t)-Q_0(x,t) = -(\lambda_k^+-\lambda_k^-)\left[J, \frac{(-1)^{a+b}M_{3-b,3-a}}{\det M}\left(|n_a\rangle\langle m_b| - S_0|m_b\rangle\langle n_a|S_0\right)\right]. \tag{82}
$$

Obviously the explicit expression in the form of rational functions of hyper-trigonometric functions will be much more complicated now. And this is just the one-soliton solution, whose internal structure is described by a rank 2 projector. If we need to calculate the soliton solution of the reduced system we will have to impose on the polarization vectors additional constraints which will make them compatible with imposed reduction.

6 The resolvent of the Lax operators

6.1 The completeness relation of FAS of L

Let us consider first the kernel of the resolvent of L in eq. (26) by:

$$
\begin{aligned}
R^\pm(x,y,\lambda) &= \frac{1}{i}\chi^\pm(x,\lambda)\Theta^\pm(x-y)\hat{\chi}^\pm(y,\lambda),\\
\Theta^+(x-y) &= \operatorname{diag}\left(-\theta(y-x),\theta(x-y)\mathbb{1},\theta(x-y)\right) = -\pi_+ + \theta(x-y)\mathbb{1},\\
\Theta^-(x-y) &= \operatorname{diag}\left(\theta(x-y),\theta(x-y)\mathbb{1},-\theta(y-x)\right) = -\pi_- + \theta(x-y)\mathbb{1},
\end{aligned} \tag{83}
$$

where $\pi_+ = E_{11}$ and $\pi_- = E_{2r+1,2r+1}$. Then

$$
R^+(x,y,\lambda) \; - \; R^-(x,y,\lambda) \;\; = \;\; -\frac{1}{i}\left(\chi^+(x,\lambda)\pi_+\hat{\chi}^+(y,\lambda) - \chi^-(x,\lambda)\pi_-\hat{\chi}^-(y,\lambda)\right). \tag{84}
$$

Lemma 4 *The kernel of the resolvent satisfies:*

$$
R^+(x,y,\lambda) - R^-(x,y,\lambda) = -\frac{1}{i}\Big(\chi^+(x,\lambda)\pi_+\hat{\chi}^+(y,\lambda) - \chi^-(x,\lambda)\pi_-\hat{\chi}^-(y,\lambda)\Big); \tag{85}
$$

In addition:

$$
\operatorname*{Res}_{\lambda=\lambda_k^+}\chi^+(x,\lambda)\hat{\chi}^+(y,\lambda)=0, \qquad \operatorname*{Res}_{\lambda=\lambda_k^-}\chi^-(x,\lambda)\hat{\chi}^-(y,\lambda)=0. \tag{86}
$$

Proof 4 *Eq. (85) is a consequence of the relations between FAS and the Jost solutions:*

$$
\chi^\pm(x,\lambda) = \phi(x,t,\lambda)S^\pm, \qquad \lambda\in\mathbb{R}. \tag{87}
$$

Indeed, from (83) we get:

$$
\begin{aligned}
R^+(x,y,\lambda) &= -\chi^+(x,\lambda)\pi_+\hat{\chi}^+(y,\lambda) + \theta(x-y)\chi^+(x,\lambda)\hat{\chi}^+(y,\lambda)\\
&= -\chi^+(x,\lambda)\pi_+\hat{\chi}^+(y,\lambda) + \theta(x-y)\phi(x,\lambda)\hat{\phi}(y,\lambda),\\
R^-(x,y,\lambda) &= \chi^-(x,\lambda)\pi_-\hat{\chi}^-(y,\lambda) - \theta(x-y)\chi^-(x,\lambda)\hat{\chi}^-(y,\lambda)\\
&= \chi^-(x,\lambda)\pi_-\hat{\chi}^-(y,\lambda) - \theta(x-y)\phi(x,\lambda)\hat{\phi}(y,\lambda),
\end{aligned} \tag{88}
$$

which completes the proof of eq. (85). In order to prove eq. (86) we first assume that the potential $Q(x)$ is on finite support: $Q(x) = 0$ for $|x| > L_0 > 0$. It is well known, that in this case the Jost solutions are entire functions of λ. Therefore the scattering matrix $T(\lambda)$ and also the FAS will be meromorphic functions of λ. In particular the relations (87) hold true for all complex λ. Of course the FAS $\chi^{\pm}(x,\lambda)$ and their inverse $\hat{\chi}^{-}(y,\lambda)$ develop first order pole singularities for $\lambda \to \lambda_k^{\pm}$. As a result one may expect that kernel of the resolvent $R^{\pm}(x,y,\lambda)$ may develop second order pole singularities at $\lambda \to \lambda_k^{\pm}$. However,

$$\operatorname*{Res}_{\lambda=\lambda_k^{\pm}} \chi^{\pm}(x,\lambda)\hat{\chi}^{\pm}(y,\lambda) = \operatorname*{Res}_{\lambda=\lambda_k^{\pm}} \phi(x,\lambda)\hat{\phi}(y,\lambda) = 0. \tag{89}$$

Indeed, since the Jost solutions are entire functions of λ then they can have no singularities for any λ. The proof for generic potentials is concluded by taking the limit $L_0 \to \infty$.

Let us now consider the more resolvent of the more general 9×9 Lax operator given by eq. (30). It again has the form as in (83) but now

$$\pi_+ = \sum_{s=1}^{3} E_{ss}, \qquad \pi_- = \sum_{s=7}^{9} E_{ss}, \tag{90}$$

Theorem 5 *Let the potential $Q(x,t)$ be such that the FAS $\chi^{\pm}(x,t,\lambda)$ have finite number of pole singularities at the points $\lambda_k^{\pm} \in \mathbb{C}_{\pm}$, $k = 1,\dots,N$ respectively. Then $\chi^{\pm}(x,t,\lambda)$ satisfy the following completeness relation:*

$$\begin{split}\Pi_0\delta(x-y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda\; \left(\chi^{+}(x,\lambda)\pi_+\hat{\chi}^{+}(y,\lambda) - \chi^{-}(x,\lambda)\pi_-\hat{\chi}^{-}(y,\lambda)\right) \\ - \sum_{k=1}^{N}(R_k^{+}(x,y) + R_k^{-}(x,y)),\end{split} \tag{91}$$

where $\Pi_0 = \pi_+ - \pi_-$ and

$$\begin{aligned} R_k^{+}(x,y) &= \chi_k^{+}(x)\pi_+\dot{\hat{\chi}}^{+}(y) + \dot{\chi}_k^{+}(x)\pi_+\hat{\chi}^{+}(y), \\ R_k^{-}(x,y) &= \chi_k^{+}(x)\pi_-\dot{\hat{\chi}}^{+}(y) + \dot{\chi}_k^{+}(x)\pi_-\hat{\chi}^{+}(y), \end{aligned} \tag{92}$$

Proof 5 *Let us apply the contour integration method to the integral:*

$$\mathcal{J}(x,y) = \frac{1}{2\pi i}\oint_{C_+} d\lambda\; R^{*}(x,y,\lambda) - \frac{1}{2\pi i}\oint_{C_-} d\lambda\; R^{-}(x,y,\lambda), \tag{93}$$

where the contours $C_{\pm}$ are shown on Figure 1. The integral $\mathcal{J}(x,y)$ can be evaluated by two methods: i) using the Cauchy residue theorem and ii) directly integrating along the contours. Equating the two answers we get:

$$\begin{aligned} \mathcal{J}(x,y) &= \sum_{k=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_k^{+}} R^{+}(x,y,\lambda) + \operatorname*{Res}_{\lambda=\lambda_k^{-}} R^{-}(x,y,\lambda)\right) \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(R^{+}(x,y,\lambda) - R^{-}(x,y,\lambda)\right) - i\Pi_0\delta(x-y). \end{aligned} \tag{94}$$

The term $\Pi_0\delta(x-y)$ comes from the integrals over the infinite semi-arcs $C_{\pm,\infty}$.

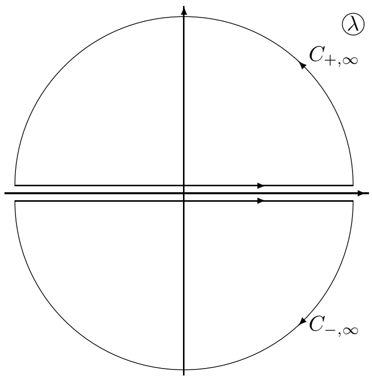

Figure 2: The contours of integrations $C_\pm = \mathbb{R} \cup C_{\pm,\infty}$.

6.2 Generating the M-operators of NLEE

Below we will show that the diagonal of the resolvent:

$$R(x,t,\lambda) = -iR(x,y,t,\lambda)|_{x=y}\,, \tag{95}$$

generates the hierarchy of M-operators for the hierarchy of NLEE related to L. This result generalizes the one of Gel'fand and Dickey for the KdV hierarchy.

First we note that the limit $y \to x$ in the kernel of the resolvent is singular. Doing it we have to regularize $R(x,y,t,\lambda)$. Skipping the details we use a regularization which comes to:

$$R(x,t,\lambda) = -\chi^+(x,t,\lambda)J\hat{\chi}(x,t,\lambda). \tag{96}$$

One of the important properties of the diagonals of the resolvent (95) is that they generate the hierarchy of M operators in the Lax representations. Indeed, it is easy to see that $R(x,t,\lambda)$ satisfy the equation:

$$i\frac{\partial R(x,t,\lambda)}{\partial x} + [Q(x,t) - \lambda J, R(x,t,\lambda)] = 0. \tag{97}$$

Since $\chi_\nu(x,t,\lambda)$ are analytic in λ then also $R(x,t,\lambda)$ will be analytic, and as a consequence we can use their asymptotic expansions over the inverse powers of λ:

$$R(x,t,\lambda) = -J + \sum_{s=1}^{\infty} \lambda^{-s} R_s(x,t)\,. \tag{98}$$

From (98) we find for s from 1 to infinity

$$i\frac{\partial R_s}{\partial x} + [Q, R_s] = [J, R_{s+1}] \tag{99}$$

while for $s=0$

$$[Q,J] + [J,R_1] = 0, \tag{100}$$

which coincide with the recursion relations for the M operator. Solving them we get

$$R_1 = R_1^{\mathrm{f}} = Q(x,t), \tag{101}$$

and

$$i\frac{\partial R_s}{\partial x} + [Q, R_s] = [J, R_{s+1}]. \tag{102}$$

Note that the operator ad_J that appeared in the right hand side of (102) has a kernel. Therefore we do the splitting into block-diagonal and block-off-diagonal parts:

$$R_s = R_s^{\mathrm{f}} + R_s^{\mathrm{d}}, \tag{103}$$

For the block-diagonal part we obtain:

$$i\frac{\partial R_s^{\mathrm{d}}}{\partial x} + [Q, R_s^{\mathrm{f}}] = 0, \quad \text{i.e.} \quad R_s^{\mathrm{d}} = i\partial_x^{-1}[Q, R_s^{\mathrm{f}}]. \tag{104}$$

For the block-off-diagonal parts of the eq. (102) we get:

$$R_{s+1}^{\mathrm{f}} = \Lambda R_s^{\mathrm{f}}, \qquad \Lambda Z = \mathrm{ad}_J^{-1}\left(i\frac{\partial Z}{\partial x} + i\left[Q, \partial_x^{-1}[Q, R_s^{\mathrm{f}}]\right]\right). \tag{105}$$

where Λ is the recursion operator above.

Thus the M operators for NLEE with dispersion law $f(\lambda) = \lambda^p J$ will have a potential:

$$V(x,t,\lambda) - \lambda^p J = (R(x,t,\lambda))_+ = -\lambda^p J + \sum_{s=1}^{p} \lambda^{p-s} R_s. \tag{106}$$

6.3 Generating the integrals of motion of NLEE

As for the conservation laws we make use of the third type of Wronskian relations, see [43, 5, 8, 9]. Here we treat only the main series of integrals which have local densities. The result is:

$$\left\langle J, \hat{D}^+ \frac{\partial D^+}{\partial \lambda} \right\rangle = -i \int_{-\infty}^{\infty} dx \, \left(\langle J, R(x,\lambda)\rangle - \langle J, J\rangle \right). \tag{107}$$

The left hand side of (107) is analytic function of λ and can be expanded over the inverse powers of λ:

$$\left\langle J, \hat{D}^+ \frac{\partial D^+}{\partial \lambda} \right\rangle = \sum_{s=1}^{\infty} d_s^+ \lambda^{-s-1}. \tag{108}$$

In order to express the integrals of motion in terms of the recursion operators it remains to use expansion (98) for the right hand side of (107).

7 Discussion and conclusion

We outlined the derivation of the completeness relation for the FAS of the class of Lax operators L related to the BD.I symmetric spaces. The difficulty, as compare to the generic case [7, 9], is that the λ-dependent term is degenerate. As a result the FAS provide complete set of functions only on a subspace $\Pi_0\mathcal{M}$ of all vector-valued functions $\mathcal{M}$. On the complementary subspace $(\mathbb{1} - \Pi_0)\mathcal{M}$ the term λJ vanishes and the Lax operator acts as differential constraints.

We derived the completeness relation for the special subclass of BD.I symmetric spaces $SO(5)/SO(3)\otimes SO(2)$. However, it is not difficult to extend these results for any symmetric space $SO(2n+1)/SO(2n-2k+1)\otimes SO(2k)$. In these cases $J=\sum_{s=1}^{k} H_k = \pi_+ - \pi_-$ and the completeness relation (91) will hold provided we change Π_0 by $\pi_+ - \pi_-$, or more generally, by $\sum_{s=1}^{k}(E_{ss} - E_{2n-2s+1,2n-2s+1})$.

Our results can be generalized also to the classes of Lax operators with non-local reductions used, e.g. in [22, 25, 26, 27]. We also plan to apply the method to the class of Lax operators depending polynomially in λ [13]. These results will be published elsewhere. Our last remark here concerns another rather effective method for constructing integrable NLEE. It was discovered and developed by Shabat, Mikhailov, Sokolov et. al., see [35, 39, 40, 4] and the numerous references therein. The idea is to classify the NLEE that have higher or master symmetries. The presence of such symmetry ensures that the relevant equations have infinite number of conservation laws and must be integrable. For some of them Lax representations are known, but for many equations with master symmetries Lax representations are still unknown. This may be due to different factors, such as nontrivial dependence of the L and M operators on λ, combined with additional reductions of Mikhailov type. The spectral analysis for such Lax pairs could be related to RHP formulated on a complex contours, see e.g. [18]. This is one more argument in favor of analysis like the one in the present paper.

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