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A reduction method for solving nonlinear PDEs

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Abstract

In this paper we are dealing with solving methods for finding solutions of various type of nonlinear PDEs. The focus will be put on two methods, both proposed by our group. The first one considers the use of an auxiliary equation, with well known solutions, in terms of which will be expressed the solutions of more complicated PDEs. More precisely, we will use the procedure called "functional expansion". The second procedure we are using here allows to reduce the order of differentiability by using a so called "attached flow". The two methods will be exemplified on few well known models of NPDEs.

Key words: Auxiliary equation, first integral method, KdV, BBM

1 Introduction

There is not a general approach allowing to solve Nonlinear Partial Derivative Equations (NPDE), despite of their huge importance in Physics and in other fields of Sciences and Engineering. Finding as many as possible NPDEs solutions offers us a better understanding of the phenomena. Classes of solutions are given by specific techniques and approaches as for example: the symmetry method and similarity reduction, the inverse scattering method, Hirota bilinear approach, Lax pair operators, etc [1], [2], [3], [4].

In this paper we will discuss how to solve NPDEs by reducing them to NODEs (Nonlinear Ordinary Differential Equations). This reduction can be achieved with the help of the so-called "wave variable" [5], [6]. Let us consider a NPDE a NPDE of the form:

$$\Delta(u(t,x,y,...),u_t,u_{x,}u_y,...)=0. \tag{1}$$

We apply the wave transformation that is in fact a change of variables for the independent variables of the form::

$$\xi = f(t,x,y,...) \tag{2}$$

It leads to a transformation of the dependent variable $u(t,x,y,...) \rightarrow U(\xi)$ and to a transformation of (1) into the NODE:

$$\Delta(u)=0 \rightarrow F(U,U',U",...)=0. \tag{3}$$

Solving (3) allows us to find a specific class of NPDEs solutions, the "traveling waves". If, supplementary, these solutions are very stable during the propagation, they becomes "solitary waves" or "solitons" [7]. There is a large literature related to the theory of solitons and, despite that, at the first glance, getting them seems to be simpler because it supposes not to solve a NPDE but a NODE, in reality there is not a general algorithm on how to do it. Numerous solving methods, not always easily to be applied, have been developed for solving NODEs. We will focus here on two approaches asking for supplementary reductions.

The first approach is based on the use of an "auxiliary equation". This equation is a linear or even nonlinear ODE that can be solved:

$$\Theta[G(\xi),\ G'(\xi), ...] = 0 \tag{4}$$

The method supposes to look for solutions of (3) that can be expressed as expansions of the solutions $G(\xi)$ of (4). It means that we will be able to find only a specific class of solutions accepted by the investigated equation, namely those that can be expressed as combinations of the known solutions of the auxiliary equation: $U(\xi) = U[G(\xi)]$.

The second approach consists in trying to solve (3) by means of a reduction of its differentiability order. This reduction is achieved by imposing a supplementary constraint. More precisely, we will attach to (3) a "flow equation" given by:

$$U'(\xi) = V(U) \tag{5}$$

From (3) and (5) we will get a new ODE in the "flow" variable V. It has a reduced order of differentiability, so it should be simpler to solve. After the flow is determined, we can come back to (5) for finding $U(\xi)$, and then, by pull back, to the initial variable $U(\xi) \to u(t, x, y, ...)$, solution for (1).

We will exemplify these two approaches on few important models of nonlinear equations from Mathematical Physics, KdV and BBM equations. It is well known that the KdV equation is a basic model in nonlinear wave theory and has been regarded as the classical model for studying soliton phenomena. The BBM equation is quite similar with the Korteweg-de-Vries equation. The KdV and BBM are two typical examples associated to effects of dissipation and disspersion. The similarity between the two equations becomes more obvious when we implement the change of variables, looking for traveling wave solutions.

2 Solving NPDEs with the auxiliary equation technique

As we mentioned, we will be interested in this paper to discuss the technique of solving a NPDE with the help of an auxiliary equation of the type (4), with known solutions $G = G(\xi)$. To be specific, we will consider the equation (3) and we will look for its solutions as expansions of the form:

$$U(\xi) = H(G(\xi),\ G'(\xi),\ ...G^{(n)}(\xi)) \tag{6}$$

Usualy, (6) is taken as a power expansion and its specific form depends on the chosen auxiliary equation. In the simple case, when the auxiliary equation is a first order ODE, the first derivative G' is extracted from the auxiliary equation and (6) simply becomes:

$$U(\xi) = H(G(\xi)) \tag{7}$$

When the auxiliary equation is of second order, the second order derivative, G'' , can be expressed in terms of G and G', and, a general form of (6) is:

$$U(\xi) = \sum_{i=-m}^{m} P_i(G)\,(G')^i\,. \tag{8}$$

Here $P_i(G)$ are unknown functionals that have to be determined. After that, we can write down the solutions $u(\xi)$ in a final form. It is therefore essential to establish what auxiliary equations can be chosen and how the solutions (6) depend on this choice.

2.1 What auxiliary equations can be considered?

The specific form of the functionals $P_i(G)$ strongly depends on the choice of the auxiliary equation (2). The most frequent auxiliary equation that has been considered in literature was Riccati equation, a first order ODE. The *tanh method* [8], for example, can be seen as a method using Riccati as auxiliary equation. Other first order ODEs which were considered as auxiliary equations, as for example:

$$G' = \frac{A}{G} + BG + CG^3 \tag{9}$$

$$G' = 1 + G + ... + G^n$$

$$G' = c_2G^2 + c_4G^4 + c_6G^6 \tag{10}$$

Sometimes, it is convenient to look for the solutions (6) in terms of higher order auxiliary equations, with a more complex class of solutions. The authors working in this topic considered various types of *second order auxiliary equations*. The simplest example is the linear second order auxiliary equations used in [9]:

$$G'' + AG' + BG = 0 \tag{11}$$

More generaly, we could consider an auxiliary equation of the form:

$$G'' + AG' + BG + C = 0 \tag{12}$$

A more complex auxiliary equation, richer in the accepted classes of solutions, that has been considered as auxiliary equation is:

$$AGG'' - B(G')^2 - CGG' - EG^2 = 0 \tag{13}$$

If we look for solutions of differential equations with order higher than two, auxiliary equations of higher order, could be considered. If, for example, we are dealing with equations of third order, the third order derivative can be in principle expressed in terms of the second and first orders, so it can be eliminated, and (6) stops at terms at maximum second order, G''.

2.2 Details on the balancing procedure

Let us consider now a second order auxiliary equation of the form:

$$A(U)U'' + B(U)(U')^2 + C(U)U'E(U) = 0 \tag{14}$$

As we mentioned before, it accepts solutions of the form (8). Defining $N_i \equiv |N(P_i)|$, we can choose the functionals P_i as a sum of monomials:

$$P_i(G) = \sum_{\kappa=0}^{N_i} \pi_{i\kappa} G^{-\kappa}, \tag{15}$$

An important step in geting their effective form is related to determining the maximum value m to be considered in the expansion. In almost all the solving methods this problem is solved following one balancing procedure between the highest order derivative and the highest nonlinear term. In our approach, with the functionals P_i given by (15), we have in fact to find the two summation limits, that is the values for the parameters m and N_i appearing in (8), respectively in (15). These tasks can be achieved following a combined balancing procedure, after G' and, respectivelly, after G.

2.3 The example of Korteweg-de Vries Equation

To see exactly as this method is functioning, we will apply it to a specific case of the equation (14), namely to the Korteweg-de Vries (KdV) equation. It is a nonlinear third order equation that in the $(2+1)$-space has the mathematical form:

$$u_t + uu_x + \delta u_{xxx} = 0 \tag{16}$$

By passing to the wave variable $\xi = x - Vt$Zhang ZY, Zhong J, Dou SS, Liu J, Peng D, Gao T. Abundant exact travelling wave solutions for the Klein-Gordon-Zakharov equations via the tanh-coth expansion method and Jacobi elliptic function expansion method. Rom J Phys. 2013;58(7-8):749–765. and by integrating once, we get the attached ODE:

$$\delta u''(\xi) + \frac{1}{2}u^2(\xi) - Vu(\xi) + k = 0 \tag{17}$$

Here $\delta,\ k,\ V$ are constants which will be used as parameters. This equation represents a particular case of (14).The balancing procedure between the terms $\delta u''(\xi)$ and $\frac{1}{2}u^2(\xi)$ leads to $m = 2$, so the solutions (8) will have the form of the following expansion:

$$u(\xi) = \sum_{i=-2}^{2} P_i(G)(G')^i \tag{18}$$

where we have supposed that the function $G(\xi)$ satisfy the auxiliary equation of the form:

$$G'' + \lambda G' + \mu G = 0. \tag{19}$$

This example was intensively studied in [9], where it was proved that the most general solution has the form:

$$u(\xi) = V - \delta\lambda^2 - \frac{24\delta\omega_2 G'}{2\omega_2 G + \omega_1} - 12\delta\frac{(G')^2}{\omega_2 G^2 + \omega_1 G + \omega_0} \tag{20}$$

It is one of the largest solution that has been pointed out for the KdV equation.

3 The attached flow method

3.1 Description of the method

The method we are discussing here is in fact a version of the first integral method [10]. We consider a nonlinear partial differential equations in its general form:

$$u_t = \Delta(u, u_x, ..., u_{mx}); u_{mx} = \frac{\partial^m x}{\partial x^m} \tag{21}$$

As previously, the given partial nonlinear equation (21) can be converted into an ordinary differential equation introducing the transformation $u \equiv U(\xi)$, where ξ is given by (2):

$$F(\xi, U, U', ..., U^{(m)}) = 0; U^{(m)} = \frac{\mathrm{d}^m U}{\mathrm{d}\xi^m} \tag{22}$$

The main idea of the method we are proposing consists in attaching to the "master" equation a supplementary, flow type equation, of the form:

$$U' = V(U). \tag{23}$$

The quantity $V(U)$ can be a polynomial or a function of $U(\xi)$. The method is a "reduction method", leading to an equation in $V(U)$ with a reduced order of differentiability. We will apply the method on the BBM equation [11].

3.2 The example of the Benjamin-Bona-Mahony (BBM) Model

The BBM equation describes the uni-directional propagation of small-amplitude long waves on the surface of the water in a channel and also for hydromagnetic and acoustic waves. For a quantity $u(x,t)$ described in a 2-dimensional space, its mathematical form is:

$$u_t - u_{xxt} + u_x(1 + u^n) = 0 \tag{24}$$

For $n = 1$ the equation represents the BBM equation itself. We note that for this case the attached ODEs is similar with (17) attached to KdV. For $n = 2$ the equation represents the modified BBM equation. As far as our approach, we will reduce (24) to an ordinary differential equation by using the wave transformation (2). With the notation $u(x,t) = U(\xi)$, the equation (24) becomes:

$$\lambda U''' - U'(\lambda - 1 - U^n) = 0 \tag{25}$$

We will apply the attached flow method for $\{n = 1, 2\}$.

In the case $n = 1$, by integrating the ODE once with respect to ξ, will result the next equation:

$$\lambda U'' + U(1 - \lambda + \frac{1}{2}U^2) = 0 \tag{26}$$

We will try to find traveling wave solutions of (25) , considering the supplementary requirement $U' = V(U)$

$$\lambda V(U)\frac{\mathrm{d}V}{\mathrm{d}U} + U(1 - \lambda + \frac{1}{2}U^2) = 0 \tag{27}$$

The above equation can be solved, assuming the integration constant 0:

$$V(U) = \pm \frac{U}{\lambda\sqrt{3}}\sqrt{\lambda(3\lambda - 3 - U)} \tag{28}$$

$$\xi = \frac{2\lambda\sqrt{3}\tanh^{-1}(\frac{\sqrt{3\lambda(\lambda-1)-\lambda U}}{\sqrt{3\lambda(\lambda-1)}})}{\sqrt{3\lambda(\lambda-1)}} \tag{29}$$

Finnaly we get the solution of BBM equation:

$$u(x,t) = \frac{3(\lambda - 1)}{\cosh^2(\frac{1}{2\lambda}(x - \lambda t)\sqrt{\lambda(\lambda - 1)})} \tag{30}$$

For $n = 2$, by integrating (25) once with respect to ξ, will result the next equation:

$$\lambda U'' + U(1 - \lambda + \frac{1}{3}U^3) = 0 \tag{31}$$

$$\lambda V(U)\frac{\mathrm{d}V}{\mathrm{d}U} + U(1 - \lambda + \frac{1}{3}U^3) = 0 \tag{32}$$

$$V(U) = \pm \frac{1}{\lambda\sqrt{6}}\sqrt{-\lambda U(-6\lambda + 6 + U^2)} \tag{33}$$

$$\zeta = \frac{\lambda}{\sqrt{\lambda(\lambda - 1)}} \ln\left(\frac{12\lambda(\lambda - 1) + 2\sqrt{6\lambda(\lambda - 1)}\sqrt{6\lambda(\lambda - 1) - \lambda U^2}}{U}\right) \tag{34}$$

$$u(x,t) = \frac{24\lambda(\lambda - 1)\,(\cosh A + \sinh A)}{\cosh 2A + \sinh 2A + 24\lambda^2(\lambda - 1)} \tag{35}$$

where $A = \frac{(x-\lambda t)\sqrt{\lambda-1}}{\sqrt{\lambda}}$. We conclude that in both cases, for $n = 1$ and for $n = 2$, the BBM equation can be solved using the attached flow method.

4 Conclusions

It is well known that, depending on the initial or on the border conditions, the NPDEs can accept a large variety of solutions. There is not a general solution as for the linear equations and there is not an unique and well-defined solving procedure. Various solving methods [12], [13], [14], could work or not and could give different classses of solutions. In this paper we investigated two specific procedures allowing to find traveling wave solutions of various NPDEs: the *auxiliary equation* and the *attached flow* methods. As starting step for both approaches, the NPDE has to be transformed in a NODE by using the wave variable (2). Another common feature of the two approaches is that both appeal to reduction procedures: the auxiliary equation method reduces the class of solutions that can be determined to those of the form (8), related with the solutions $G(\xi)$ of the considered auxiliary equation; the second considered method, the attached flow, also reduce the class of obtenaible solutions to those satisfying the supplementary flow equation (23). The two methods were illustrated on two important models of nonlinear dynamics: the KdV and BBM models. Both models belong to a very large class of models, describing a lot of natural phenomena from hydrodynamics, optics or plasma physics and reunited in the commun form of the equation (14). Soliton-like solutions were pointed out for the two investigated models.

An important issue that will be investigated in future works [15] is related to the identification of the models for which the two methods are suitable. Rules of the form "go-no go" could be formulated for various specific polynomials $A(U)$, $B(U)$, $C(U)$ and $E(U)$ from (14). It will be possible to make a choice of the solving methods for various dynamical systems.

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