# Stability analysis and synchronization of Chua systems 

R. Constantinescu, C. Ionescu, I. Petrisor<br>Deptartment of Physics, University of Craiova, 13 A. I. Cuza Str., Craiova 200585, Romania


#### Abstract

The paper investigates two important phenomena which characterize the behavior of an important example of nonlinear dynamical system expressed by a simple electronic circuit, the Chua model.


## 1 Introduction

In the signal theory, the Chua circuits are specific representation of very simple nonlinear circuits with chaotic behavior. Important studies were devoted to this electronic circuit, and many results related to the stability of signals, to the analogy with other circuits, or to the possibility of its synchronization were published. In recent decades, synchronization of coupled chaotic systems has been exploated for secure communication, generating a new important field of communication sciences called anticontrol theory. The phenomenon of chaos synchronization was first revealed by Pecora and Carroll [1]. In recent years new appoaches for synchronization of chaotic systems has been proposed. In the scientific literature one can find mentions on: the complete synchronization [2], the phase synchronization [3], the generalized synchronization [4], the adaptive synchronization [5], the lag synchronization [6], or on the impulsive control [7].

The present paper will reffere to the stability analysis and the control of chaotic oscillations, which are of great interest because of their practical applications. We will apply the Rough-Hurvitz criteria and we will decide when exactly the analized systems is stable or not. We will continue, by studying the synchronization of two identical smooth Chua systems, using the classic approach of the master-slave system configuration. One of the two Chua systems will be considered as master, the second one as slave, and the system of errors will be investigated in terms of some uncertain parameters. Nonlinear filtering processes will be used in order to preserve the chaotic synchronization.

## 2 Chua circuit: stability analysis

Chua circuit (Figure 1) is a well known electronic circuit generating chaotic signals. It became an important toy model in many domains, starting from neurosciences (model of neural networks), till various engineering applications, as for example generator of electronic music.


Figure 1: Chua's electronic circuit


By writing down the Kirchoff 's law for the circuit, a set of three differential equations are generated. They have the form:

$$
\begin{gather*}
\dot{x}=\alpha[(y-x)+f(x)]  \tag{1}\\
\dot{y}=x-y+z  \tag{2}\\
\dot{z}=-\beta y \tag{3}
\end{gather*}
$$

The function $f(x)$ describe the characteristic $I=f(V)$ of the nonlinear element know as Chua diode. Many types of nonlinearities were considered. In this paper we shall choose:

$$
f(x)=\left\{\begin{array}{c}
\sin k x, x \in\left[-\frac{\pi}{2 k}, \frac{\pi}{2 k}\right]  \tag{4}\\
-1, x<-\frac{\pi}{2 k} \\
1, x>\frac{\pi}{2 k}
\end{array}\right.
$$

This choice corresponds to a $C^{1}$ function on $R$ whose graphic is presented in Figure 2.

We shall be interested in studying the chaotic behavior and the regular regime corresponding to this circuit. In order to do this study, we shall compute the equilibrium points. They are given by:

$$
\begin{array}{r}
(y-x)+f(x)=0 \\
x-y+z=0  \tag{5}\\
-\beta y=0
\end{array}
$$

or, equivalent

$$
\begin{align*}
x-f(x) & =0 \\
x+z & =0  \tag{6}\\
y & =0
\end{align*}
$$

Three important cases can be identified:
Case 1 - An unique equilibrium point:

$$
\begin{equation*}
k \leq 1 \rightarrow P_{1}=(0,0,0) \tag{7}
\end{equation*}
$$

Case 2 - Three equilibrium points:

$$
\begin{equation*}
\frac{\pi}{2}<k<1 \rightarrow P_{1}=(0,0,0) ; P_{2}=\left(x_{0}, 0,-x_{0}\right) ; P_{3}=\left(-x_{0}, 0, x_{0}\right) \tag{8}
\end{equation*}
$$

where $x_{0}$ is the positive solution for the following equation

$$
\begin{equation*}
x=\sin k x \tag{9}
\end{equation*}
$$

Case $\mathbf{3}$ - Three equilibrium points:

$$
\begin{equation*}
k>\frac{\pi}{2} \rightarrow P_{1}=(0,0,0) ; \bar{P}_{2}=(1,0,-1) ; \bar{P}_{3}=(-1,0,1) \tag{10}
\end{equation*}
$$

## Remarks:

1) The most interesting case we have here is the case: $1<k<\frac{\pi}{2}$. In this case the position of the equilibrium points depends by $k$.
2) If $k \rightarrow \frac{\pi}{2}$ then $P_{2} \rightarrow \bar{P}_{2}$ and $P_{3} \rightarrow \bar{P}_{3}$
3) If $k \rightarrow 1$ then $P_{2} \rightarrow P_{1}$ and $P_{3} \rightarrow P_{1}$ so there is no pitchfork bifurcation for $k=1$.
4) The equilibrium points $P_{2}$ si $P_{3}$ are situated symmetric in respect with $O(0,0,0)$ (that is also a fixed point) because the system has geometric symmetry from origin:

$$
\begin{equation*}
X_{C} \circ S=S \circ X_{C} \tag{11}
\end{equation*}
$$

Here $X_{C}(x, y, z)=(\alpha[(y-x)+f(x)], x-y+z,-\beta y)$ and $S(x, y, z)=(-x,-y,-z)$. So $P_{2}$ and $P_{3}$ will always have the same properties. What happens around of $P_{2}$ also happens around $P_{3}$.
5) The Jacobian matrix attached to the system is given by:

$$
J X_{C}(x, y, z)=\left(\begin{array}{ccc}
-1+k \cos k x & 1 & 0  \tag{12}\\
1 & -1 & 1 \\
0 & -\beta & 0
\end{array}\right)
$$

Let us effectively apply the study for the three possible equilibrium points.
Equilibrium points O (0,0,0)
The Jacobian matrix takes in this case the form:

$$
J X_{C}(0,0,0)=\left(\begin{array}{ccc}
-1+k & 1 & 0  \tag{13}\\
1 & -1 & 1 \\
0 & -\beta & 0
\end{array}\right)
$$

The eigenvalues are given by the equation:

$$
\left|\begin{array}{ccc}
k-1-\lambda & 1 & 0  \tag{14}\\
1 & -1-\lambda & 1 \\
0 & -\beta & -\lambda
\end{array}\right|=0
$$

The previous condition is equivalent with the following equation:

$$
\begin{equation*}
\lambda^{3}+(1-k) \lambda^{2}+(\beta-k) \lambda+\beta(1-k)=0 \tag{15}
\end{equation*}
$$

Let us use the notation:

$$
\begin{align*}
& 1-k=a_{1}=-\left(\lambda_{1}+\lambda_{2}+\lambda_{3}\right) \\
& \beta-k=a_{2}=\lambda_{1} \lambda_{2}+\lambda_{2} \lambda_{3}+\lambda_{1} \lambda_{3}  \tag{16}\\
& \beta(1-k)=a_{3}=-\lambda_{1} \lambda_{2} \lambda_{3}
\end{align*}
$$

The Rough-Hurvitz criteria asks for stability that:

$$
\begin{align*}
a_{1} & >0  \tag{17}\\
a_{3} & >0 \\
a_{1} a_{2}-a_{3} & >0
\end{align*}
$$

It is not satisfied for $k>1$ because $\beta(1-k)=-\lambda_{1} \lambda_{2} \lambda_{3}<0 \Rightarrow \lambda_{1} \lambda_{2} \lambda_{3}>0$. So $O(0,0,0)$ cannot be attractor point until $\operatorname{Re} \lambda_{1}<0 ; \operatorname{Re} \lambda_{2}<0 ; \operatorname{Re} \lambda 3<0$. The point $O(0,0,0)$ is stable.

The equilibrium point $P_{2}=\left(x_{0}, 0,-x_{0}\right)$
The Jacobian matrix takes in this case the form:

$$
J X_{C}\left(x_{0}, 0,-x_{0}\right)=\left(\begin{array}{ccc}
-1+k \cos k x_{0} & 1 & 0  \tag{18}\\
1 & -1 & 1 \\
0 & -\beta & 0
\end{array}\right)
$$

This means that the eigenvalues of $J X_{C}\left(x_{0}, 0,-x_{0}\right)$ are solutions of the following equation:

$$
\begin{equation*}
\lambda^{3}+\left(1-k \cos k x_{0}\right) \lambda^{2}+\left(\beta-k \cos k x_{0}\right) \lambda+\beta\left(1-k \cos k x_{0}\right)=0 \tag{19}
\end{equation*}
$$

$P_{2}$ is stable if

$$
\left\{\begin{array}{c}
a_{1}=1-k \cos k x_{0}>0  \tag{20}\\
a_{3}=\beta\left(1-k \cos k x_{0}\right)>0 \\
a_{1} a_{2}-a_{3}=\left(1-k \cos k x_{0}\right)\left(\beta-k \cos k x_{0}\right)-\beta\left(1-k \cos k x_{0}\right)>0
\end{array}\right.
$$

That is equivalent with

$$
\begin{align*}
1-k \cos k x_{0} & >0  \tag{21}\\
\beta-k \cos k x_{0}-\beta & =-k \cos k x_{0}>0
\end{align*}
$$

it isn't possible because

$$
\begin{equation*}
x_{0} \in\left(0, \frac{\pi}{2 k}\right) \subset\left(0, \frac{\pi}{2}\right) \tag{22}
\end{equation*}
$$

so the Rough-Hurvitz criterion (sufficient condition) it is not satisfied and it can't be decided through this method if the points are stabile.

Nevertheless

$$
\begin{equation*}
\lambda_{1} \lambda_{2} \lambda_{3}=-\beta\left(1-k \cos k x_{0}\right) \tag{23}
\end{equation*}
$$

could take negative value as: $k \rightarrow \frac{\pi}{2}$.
Regarding what we said above we can conclude that no Hopf bifurcation appears (and no limit cycle forms because the ecuation (23) cannot have pure imaginary roots.

## 3 Syncronization of two Chua systems

Let us consider Chua system with nonlinearity of type th:

$$
\left\{\begin{array}{c}
\dot{x}=\alpha\left(y-x-\frac{e^{2 x}-1}{e^{2 x}+1}\right)  \tag{24}\\
\dot{y}=x-y+z \\
\dot{z}=-\beta y
\end{array}\right.
$$

The corresponding master-system will have the form:

$$
\left\{\begin{array}{c}
\dot{x_{1}}=\alpha\left(y_{1}-x_{1}-\frac{e^{2 x_{1}}-1}{e^{2 x_{1}}+1}\right)  \tag{25}\\
\dot{y}_{1}=x_{1}-y_{1}+z_{1} \\
\dot{z}_{1}=-\beta y_{1}
\end{array}\right.
$$

and the slave-system have the form:

$$
\left\{\begin{array}{c}
\dot{x_{2}}=\alpha\left(y_{2}-x_{2}-\frac{e^{2 x_{2}}-1}{e^{2 x_{2}}+1}+x_{2}^{2}\right)+u_{1}  \tag{26}\\
\dot{y}_{2}=x_{2}-y_{2}+z_{2}+u_{2} \\
\dot{z_{2}}=-\beta y_{2}+u_{3}
\end{array}\right.
$$

Let us introduce the errors:

$$
\begin{equation*}
e_{1}=x_{2}-x_{1}, e_{2}=y_{2}-y_{1}, e_{3}=z_{2}-z_{1} \tag{27}
\end{equation*}
$$

so that the errors-systems will have the form:

$$
\left\{\begin{array}{c}
\dot{e_{1}}=\alpha\left(e_{2}-e_{1}-\frac{e^{2 x_{2}-1}}{e^{2 x_{2}}+1}+\frac{e^{2 x_{1}-1}}{e^{2 x_{1}}+1}+x_{2}^{2}\right)+u_{1}  \tag{28}\\
\dot{e_{2}}=e_{1}-e_{2}+e_{3}+u_{2} \\
\dot{e_{3}}=-\beta e_{2}+u_{3}
\end{array}\right.
$$

We can choice the control parameters $u_{1}, u_{2}, u_{3}$ as:

$$
\begin{align*}
& u_{1}=\alpha\left(\frac{e^{2 x_{2}}-1}{e^{2 x_{2}}+1}-\frac{e^{2 x_{1}}-1}{e^{2 x_{1}}+1}-x_{2}^{2}\right)+V_{1}\left(e_{1}, e_{2}, e_{3}\right) \\
& u_{2}=V_{2}\left(e_{1}, e_{2}, e_{3}\right)  \tag{29}\\
& u_{3}=V_{3}\left(e_{1}, e_{2}, e_{3}\right)
\end{align*}
$$

and the error-system becames:

$$
\left\{\begin{array}{c}
\dot{e_{1}}=\alpha\left(e_{2}-e_{1}\right)+V_{1}\left(e_{1}, e_{2}, e_{3}\right)  \tag{30}\\
\dot{e}_{2}=e_{1}-e_{2}+e_{3}+V_{2}\left(e_{1}, e_{2}, e_{3}\right) \\
\dot{e_{3}}=-\beta e_{2}+V_{3}\left(e_{1}, e_{2}, e_{3}\right)
\end{array}\right.
$$

We want to stabilize this system in $e_{1}, e_{2}, e_{3}$ in origin $O(0,0,0)$. There are many choises to control parameters $V_{1}, V_{2}, V_{3}$ :

$$
\left(\begin{array}{l}
V_{1}  \tag{31}\\
V_{2} \\
V_{3}
\end{array}\right)=\left(\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)\left(\begin{array}{l}
e_{1} \\
e_{2} \\
e_{3}
\end{array}\right)
$$

where $A=\left(\begin{array}{ccc}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{array}\right)$ is $3 x 3$ matrix. For the system to be stable, they need the $A$ 's own values must have the real negative side:

$$
\left(\begin{array}{l}
\dot{e}_{1}  \tag{32}\\
\dot{e}_{2} \\
\dot{e}_{3}
\end{array}\right)=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right)\left(\begin{array}{l}
e_{1} \\
e_{2} \\
e_{3}
\end{array}\right)
$$

For that, we need to choose

$$
\left(\begin{array}{l}
V_{1}  \tag{33}\\
V_{2} \\
V_{3}
\end{array}\right)=\left(\begin{array}{ccc}
\alpha-1 & -\alpha & 0 \\
-1 & 0 & -1 \\
0 & \beta & -1
\end{array}\right)\left(\begin{array}{l}
e_{1} \\
e_{2} \\
e_{3}
\end{array}\right)
$$

The characteristic polynomial is

$$
P(\lambda)=\left|\begin{array}{ccc}
\alpha-1 & -\alpha & 0  \tag{34}\\
-1 & 0 & -1 \\
0 & \beta & -1
\end{array}\right|
$$

and the characteristic equation has the form:

$$
\begin{equation*}
\lambda^{3}+(2-\alpha) \lambda^{2}+(\beta+1-2 \alpha) \lambda+\beta(1-\alpha)+\alpha=0 \tag{35}
\end{equation*}
$$

We want all three roots of the equation to have the real-side negative. This will imply some conditions on the $\alpha$ and $\beta$ coefficients. Let us use the notation:

$$
\left.\begin{array}{rl}
2-\alpha & =a_{1}
\end{array}=-\left(\lambda_{1}+\lambda_{2}+\lambda_{3}\right), ~=\lambda_{1}=\lambda_{1} \lambda_{2}+\lambda_{2} \lambda_{3}+\lambda_{1} \lambda_{3}\right)
$$

The Rough-Hurvitz criteria asks for stability that:

$$
\begin{align*}
a_{1} & >0 \\
a_{3} & >0  \tag{37}\\
a_{1} a_{2}-a_{3} & >0
\end{align*}
$$



Figure 2: $\alpha=2, \beta=1$

For our model this means

$$
\begin{align*}
2-\alpha & >0 \\
\beta(1-\alpha)+\alpha & >0  \tag{38}\\
(2-\alpha)(\beta+1-2 \alpha)-\beta(1-\alpha)+\alpha & >0
\end{align*}
$$

From first inequation we obtain

$$
\begin{equation*}
\alpha<2 \tag{39}
\end{equation*}
$$

and from the last one

$$
\begin{equation*}
\beta>-2(\alpha-1)^{2} \tag{40}
\end{equation*}
$$

The two above inequations leads to

$$
\begin{equation*}
\alpha-\beta<2\left[1+(\alpha-1)^{2}\right] \tag{41}
\end{equation*}
$$

This means

$$
\begin{gather*}
-\lambda_{1} \lambda_{2} \lambda_{3}=\beta(1-\alpha)+\alpha<\alpha-\beta<2\left[1+(\alpha-1)^{2}\right]<4  \tag{42}\\
\lambda_{1} \lambda_{2} \lambda_{3}>-4 \tag{43}
\end{gather*}
$$

This means that $O(0,0,0)$ can be attractor point because $\operatorname{Re} \lambda_{1}<0 ; \operatorname{Re} \lambda_{2}<0 ; \operatorname{Re} \lambda_{3}<0$ (for $-4<\lambda_{1} \lambda_{2} \lambda_{3}<0$ ) but may also be a stable point for $\lambda_{1} \lambda_{2} \lambda_{3}>0$ (and the systems of errors will synchronize).

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Figure 3: $\alpha=2, \beta=2.1$
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