# The convergence rate in the triangular Bézier finite element 

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#### Abstract

We give a new simplified proof of the convergence of the two dimensional triangular Bézier polynomials to continuous functions based on the abstract topological Stone-Weierstrass theorem. We estimate the convergence rate and discuss the approximation problem on general triangular mesh.


## 1 Introduction

For the numerical simulation of the wall touching kink modes (WTKM) in the thin wall approximation [1] one of the problem is to extend the existing code, designed for the smooth tokamak wall, using triangular finite elements, to the case when to the tokamak wall is attached a limiter. In order to reduce the modification in the previously elaborated programme and input data, in the generic cases we are forced to solve the problem of nonconforming finite elements. Despite the construction of the finite elements on the limiter alone does not pose complex problems, on the contact line on the tokamak wall special problems appears due to nonconforming position of the finite element triangles. Due to non-conformity, the interpolation in general does not preserve the continuity of the exact function.

In this work we give a new, simplest proof of the convergence of Bernstein polynomials toward a two variable continuos function, defined on conforming or nonconforming triangular finite element and estimate the speed of convergence.

## 2 Convergence proof

### 2.1 Statement of the problem

Consider a two dimensional simplex $T$, a triangle with vertices

$$
\begin{equation*}
\mathbf{x}_{i} \in \mathbf{T} \subset \mathbb{R}^{2}, 1 \leq i \leq 3 \tag{1}
\end{equation*}
$$

where $\mathbb{R}^{2}$ is considered with standard real vector space structure and scalar product " $\cdot$ ". For a given point $\mathbf{x} \in \mathbf{T}$ we denote by $s_{i}(\mathbf{x}), \overline{i=1,3}$ its barycentric coordinates:

$$
\begin{align*}
\mathbf{x} & =\sum_{i=1}^{3} s_{i}(\mathbf{x}) \mathbf{x}_{i}  \tag{2}\\
\sum_{i=1}^{3} s_{i}(\mathbf{x}) & =1 ; s_{i}(\mathbf{x}) \geq 0, i=\overline{1,3} \tag{3}
\end{align*}
$$

Consider only the case when the triangle is non degenerated. The correspondence defined in $\operatorname{Eqs}(2,3)$,

$$
\begin{equation*}
\mathbb{R}^{2} \ni \mathbf{x} \rightarrow\left(s_{1}(\mathbf{x}), s_{2}(\mathbf{x}), s_{3}(\mathbf{x})\right):=S(\mathbf{x}) \in \mathbb{R}^{3} \tag{4}
\end{equation*}
$$

transform the triangle $T$ in the standard two dimensional simplex from $\mathbb{R}^{3}$, that is the convex hull of the unit vectors from $\mathbb{R}^{3}$. The map $S$ is one to one; denote its inverse by $R$ :

$$
\begin{align*}
\mathbf{T}_{0} & \ni\left(t_{1}, t_{2}, t_{3}\right) \xrightarrow{R} \mathbf{x}=\sum_{i=1}^{3} t_{i} \mathbf{x}_{i}:=R\left(t_{1}, t_{2}, t_{3}\right) \in \mathbf{T}  \tag{5}\\
\sum_{i=1}^{3} t_{i} & =1 ; t_{i} \geq 0, i=\overline{1,3} \tag{6}
\end{align*}
$$

The one dimensional analogue of the Eq.(3) was the key starting point in the probabilistic proof of the classical Weierstrass theorem by S. Bernstein in the case of functions of one variable [2], [3].

In analogy to the single variable case of Bernstein polynomial attached to the 1-simplex $[0,1]$,

$$
B_{k}^{n}(s)=\frac{n!}{k!(n-k)!} s^{k}(1-s)^{n-k} ; 0 \leq k \leq n ; k, n \text { integer }
$$

the Bernstein-Bézier polynomials [4], [5], [6] attached to the two dimensional simplex, the triangle $\mathbf{T}$ from (1) are

$$
\begin{align*}
B_{i, j, k}^{n}(\mathbf{x}) & :=\frac{n!}{i!j!k!}\left[s_{1}(\mathbf{x})\right]^{i}\left[s_{2}(\mathbf{x})\right]^{j}\left[s_{3}(\mathbf{x})\right]^{k}  \tag{7}\\
i+j+k & =n ; i, j, k \in \mathbb{N} \tag{8}
\end{align*}
$$

In the following, we will denote by $M_{n}$, whose elements are triplets of integers:

$$
\begin{equation*}
\mathbb{N}^{\times 3} \supset M_{n}:=\left\{(i, j, k) \mid(i, j, k) \in \mathbb{N}^{\times 3} ; i+j+k=n\right\} \tag{9}
\end{equation*}
$$

The set of Bernstein-Bézier polynomials of given order $n$, in the case of a real continuous function of two real variables defined on the triangle $\mathbf{T}$

$$
\mathbb{R}^{2} \supset \mathbf{T} \rightarrow f(\mathbf{x}) \in \mathbb{R}
$$

generate a polynomial $B_{f}^{n}(\mathbf{y})$ in two real variables $y=\left(y_{1}, y_{2}\right)$, that is, by anticipating, an uniform polynomial approximant of the function $f(x)$

$$
\begin{align*}
B_{f}^{n}(\mathbf{x}) & :=\sum_{(i, j, k) \in M_{n}} f\left[R\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)\right] B_{i, j, k}^{n}(\mathbf{x})=  \tag{10}\\
& =\sum_{(i, j, k) \in M_{n}} f\left(\mathbf{x}_{1} \frac{i}{n}+\mathbf{x}_{2} \frac{j}{n}+\mathbf{x}_{3} \frac{k}{n}\right) B_{i, j, k}^{n}(\mathbf{x}) \tag{11}
\end{align*}
$$

The set of points,

$$
\begin{equation*}
C_{n}:=\left\{\left.R\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) \right\rvert\,(i, j, k) \in M_{n}\right\} \subset \mathbf{T} \tag{12}
\end{equation*}
$$

are called the set of control points on the triangle $\mathbf{T}$. We have

$$
\begin{equation*}
C_{n}=R\left(M_{n} / n\right) \tag{13}
\end{equation*}
$$

Our goal is to give a new simple proof that $B_{f}^{n}(x)$ is really an approximant of $f(x)$, and to estimate the error. We stress that we are not intend to give a new proof of some extension of the Weierstrass theorem on simplexes, rather we are interested in better understanding the approximation mechanism in order to handle the more complex geometries that appears in the case of nonconforming Bézier finite element analysis.

### 2.2 Convergence proof and error estimate for exponential functions

We denote by $C(\mathbf{T})$ space of continuos functions on the triangle $\mathbf{T}$, with the uniform convergence norm and denote by $E(\mathbf{T})$ the subspace of $C(\mathbf{T})$ generated by all finite linear combinations of the functions

$$
\begin{equation*}
\exp (\mathbf{a} \cdot \mathbf{x}) ; \mathbf{a} \in \mathbb{R}^{2} \tag{14}
\end{equation*}
$$

We start with a proposition that simplifies the proof:
Proposition 1 The subspace $E(\mathbf{T})$ contains functions that distinguish [7] every pair of points $\mathbf{b}_{1}, \mathbf{b}_{2} \in \mathbf{T}$.

Proof. We have to prove that there exists a function $g \in E(\mathbf{T})$ such that $g\left(\mathbf{b}_{1}\right) \neq g\left(\mathbf{b}_{2}\right)$. Consider

$$
\mathbf{a}=\mathbf{b}_{2}-\mathbf{b}_{1}
$$

The function $g(x)$ defined by

$$
g(\mathbf{x}):=\exp \left[\left(\mathbf{b}_{2}-\mathbf{b}_{1}\right) \cdot\left(\mathbf{x}-\mathbf{b}_{1}\right)\right]
$$

is also contained in $E(\mathbf{T})$ and $g\left(\mathbf{b}_{2}\right)>g\left(\mathbf{b}_{1}\right)$ that completes the proof.
Starting from this result we can justify our main lemma:
Lemma 2 The subspace $E(\mathbf{T})$ is dense in $C(\mathbf{T})$ in the sup norm topology.
Proof. The subspace $E(\mathbf{T})$ is closed under multiplication, contains identity element so it is a Banach subalgebra of $C(\mathbf{T})$ with identity, and according to the previous Proposition 1, distinguishes all of the points on the compact set T. So, according to Stone-Weierstrass theorem [7], $E(\mathbf{T})$ is dense in $C(\mathbf{T})$.

Now according with the previous Lemma, in order to prove that $B_{f}^{n}(x) \xrightarrow{n \rightarrow \infty} f(x)$, it is sufficient to prove the convergence on the generators $\exp (\mathbf{a} \cdot \mathbf{x})$ of the space $E(\mathbf{T})$. Denote

$$
\begin{equation*}
e_{\mathbf{a}, n}(\mathbf{x}):=B_{f}^{n}(\mathbf{x}) ; f(\mathbf{x}) \equiv \exp (\mathbf{a} \cdot \mathbf{x}) \tag{15}
\end{equation*}
$$

For the sake of clarity we decompose the proof in several steps. The main result, that explains the particular choice of $E(\mathbf{T})$ is the following algebraic relation:

Proposition 3 The Bézier polynomial for exponential function can be expressed as follows

$$
\begin{equation*}
e_{\mathbf{a}, n}(\mathbf{x})=\left[\sum_{j=1}^{3} s_{j}(\mathbf{x}) \exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right)\right]^{n} \tag{16}
\end{equation*}
$$

Proof. We use here a shorter notation: $s_{j}(x)$ denote simply as $s_{j}$. From Eqs.(11, 7, 14 8) results

$$
\begin{align*}
e_{\mathbf{a}, n}(\mathbf{x}) & =\sum_{(i, j, k) \in M_{n}} \exp \left[a \cdot\left(\mathbf{x}_{1} \frac{i}{n}+\mathbf{x}_{2} \frac{j}{n}+\mathbf{x}_{3} \frac{k}{n}\right)\right] B_{i, j, k}^{n}(\mathbf{x})  \tag{17}\\
& =\sum_{(i, j, k) \in M_{n}}\left[\exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{1}}{n}\right)\right]^{i}\left[\exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{2}}{n}\right)\right]^{j}\left[\exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{3}}{n}\right)\right] \frac{n!}{i!j!k!}\left[s_{1}\right]^{i}\left[s_{2}\right]^{j}\left[s_{3}\right]^{k} \\
& =\left[\sum_{j=1}^{3} s_{j} \exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right)\right]^{n}
\end{align*}
$$

where was used the definition (9) and the multinomial Newton formula.
By using Eqs.(16) we have the following
Proposition 4 In the limit of large $n$ we have the following convergence result of the Bézier polynomial $e_{\mathbf{a}, n}(x)$ on the triangle $\mathbf{T}$

$$
\begin{equation*}
\left|e_{\mathbf{a}, n}(\mathbf{x})-\exp (a \cdot \mathbf{x})\right| \leq\left[\frac{K}{n}+\mathcal{O}\left(\frac{1}{n^{2}}\right)\right] \exp (a \cdot \mathbf{x}) \tag{18}
\end{equation*}
$$

where $K$ is a constant, independent of $n$ and $x$.
Proof. By separating an $\mathcal{O}\left(1 / n^{2}\right)$ term

$$
\exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right)=1+\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}+\left[\exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right)-1-\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right]
$$

with the use of Eq.(3), the term in the right hand side from Eq.(16) we rewrite as follows

$$
\begin{align*}
\sum_{j=1}^{3} s_{j}(\mathbf{x}) \exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right) & =1+\sum_{j=1}^{3} s_{j}(\mathbf{x}) \frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}+r_{n}(\mathbf{x}) \\
& =1+\frac{\mathbf{a} \cdot \mathbf{x}}{n}+r_{n}(\mathbf{x}) \tag{19}
\end{align*}
$$

where in the last equality we used Eq.(2) and we denoted the residual $O\left(1 / n^{2}\right)$ term as

$$
\begin{equation*}
r_{n}(\mathbf{x}):=\sum_{j=1}^{3} s_{j}(\mathbf{x}) \exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right)-1-\frac{\mathbf{a} \cdot \mathbf{x}}{n} \tag{20}
\end{equation*}
$$

For large $n$, by using the remainder formula for Taylor series we have the inequality

$$
\begin{align*}
\left|r_{n}(\mathbf{x})\right| & \leq \frac{K_{1, n}}{n^{2}}  \tag{21}\\
K_{1, n} & =\frac{1}{2} \sum_{j=1}^{3}\left(\mathbf{a} \cdot \mathbf{x}_{j}\right)^{2} \exp \left(\frac{\mathbf{a} \cdot \mathbf{x}_{j}}{n}\right)<C \tag{22}
\end{align*}
$$

where the constant $C$ can be optimized by suitable coordinate change. From Eqs. $(16,19)$ results

$$
\begin{equation*}
e_{\mathbf{a}, n}(\mathbf{x})=\left[1+\frac{\mathbf{a} \cdot \mathbf{x}}{n}+r_{n}(\mathbf{x})\right]^{n} \tag{23}
\end{equation*}
$$

In order to find the speed of convergence by using Eqs. $(21,22)$ we compute

$$
\begin{align*}
\left|\log \frac{e_{\mathbf{a}, n}(\mathbf{x})}{\exp (\mathbf{a} \cdot \mathbf{x})}\right| & <\frac{K}{n}+\mathcal{O}\left(\frac{1}{n^{2}}\right)  \tag{24}\\
K & =C+\frac{1}{2} \max _{\mathbf{x} \in \mathbf{T}}(\mathbf{a} \cdot \mathbf{x}) \tag{25}
\end{align*}
$$

From Eq.(24) we obtain the relative error bound

$$
\begin{equation*}
\left|\frac{e_{\mathbf{a}, n}(\mathbf{x})-\exp (\mathbf{a} \cdot \mathbf{x})}{\exp (\mathbf{a} \cdot \mathbf{x})}\right|<\frac{K}{n}+\mathcal{O}\left(\frac{1}{n^{2}}\right) \tag{26}
\end{equation*}
$$

that completes the proof.

### 2.3 Convergence proof, and error estimate, general case.

Let $f(x) \in C^{0}(\mathbf{T})$ and $\varepsilon>0$. By Proposition 4 there exists $N(\varepsilon)$, vectors and coefficients $\mathbf{a}_{i}^{\varepsilon}, c_{i}^{\varepsilon}$ with $1 \leq i \leq N(\varepsilon)$ such that

$$
\begin{equation*}
\sup _{\mathbf{x} \in \mathbf{T}}\left|f(\mathbf{x})-\sum_{i=1}^{N(\varepsilon)} c_{i}^{\varepsilon} \exp \left(\mathbf{a}_{i}^{\varepsilon} \cdot \mathbf{x}\right)\right| \leq \frac{\varepsilon}{2} \tag{27}
\end{equation*}
$$

By Proposition 4 there exists an $M(\varepsilon)$ such that

$$
\begin{equation*}
\sup _{\mathbf{x} \in \mathbf{T}}\left|\sum_{i=1}^{N(\varepsilon)} c_{i}^{\varepsilon} e_{a_{i}^{\varepsilon}, M(\varepsilon)}(\mathbf{x})-\sum_{i=1}^{N(\varepsilon)} c_{i}^{\varepsilon} \exp \left(\mathbf{a}_{i}^{\varepsilon} \cdot \mathbf{x}\right)\right| \leq \frac{\varepsilon}{2} \tag{28}
\end{equation*}
$$

From Eqs. $(27,28)$ results

$$
\sup _{\mathbf{x} \in \mathbf{T}}\left|\sum_{i=1}^{N(\varepsilon)} c e_{a_{i}^{\varepsilon}, M(\varepsilon)}(\mathbf{x})-f(\mathbf{x})\right|
$$

Due to the fact that $e_{a_{\tilde{\varepsilon}}^{\varepsilon}, M(\varepsilon)}(\mathbf{x})$ can be expressed by Bézier polynomials, we proved that the function $f(x) \in C^{0}(\mathbf{T})$ can be approximated in unifom norm by Bézier polynomials, of fixed order $M(\varepsilon)$.

## 3 Conclusions

We presented a new proof of the convergence of the Bézier polynomials associated to conforming or non conforming triangular finite element. We proved that the relative error is dominated by an asymptotic term of the form $O(d k / n)$ where $n$ is the order of the Bézier polynomial, $d$ is the largest side of the triangle and $k$ is them largest wave number that appear in the Fourier expansion of the function to be approximated.

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