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Symmetries of a generalization of Hénon-Heiles model

Afrodita Liliana Boldea
IFIN-HH, Bucharest, Romania

Abstract

This paper proposes a direct study for the Lie symmetries investigation in the case of a 2D Hamiltonian system arising from astrophysics: the 2D Hénon-Heiles mechanical model. General Lie operators are deduced firstly and, in the the next step, the associated Lie invariants are derived.

Keywords: Generalized Symmetries, Hénon-Heiles equation.
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1 Introduction

In the last years, nonlinear dynamical systems were intensively studied for their important role in the sphere of basic theoretical researches from mathematics and physics, and in the application of these related branches of science.

Chaos is characteristic for many nonlinear dynamical systems with finite or infinite degrees of freedom. In nature, chaotic behavior more is a rule than an exception. The investigation methods of chaotic dynamics are studied in [1]. To prove that some of the dynamical systems are chaotic (nonintegrable systems), a study of the periodical orbits of the system can be done, case in which this analysis is possible, through numerical methods.

Opposite to the chaotic dynamical systems are the integrable ones, which present a regular behavior. In a tight connection with integrability is the problem of isolating the constants of motion for a given physical system. The determination of the invariants for autonomous or non-autonomous Hamiltonian integrable systems can be done by using direct or indirect methods. In the case of a 2-dimensional autonomous system, where the Hamiltonian does not depend explicitly on time, a first constant quantity is immediately found: the Hamiltonian itself that represents the system's total energy. Therefore, the only problem to be solved in order to fulfill the integrability condition would be to find a second invariant.

The indirect methods for constructing the invariants consist in their determining from the symmetries found for the analyzed system. Both, the Lie symmetries which leave invariant the evolution equations of the system (leave invariant the differential equation with partial derivatives which described the physics' process) and the Noether symmetries which leave invariant the action of the system, are in the attention of researchers. In [2], are pointed out the Noether and Lie symmetries and the associated invariants too, for the non-autonomous Kepler system. Two main indirect approaches of integrability are usually used: the Painlevé analysis [3] and the recursion operator method [4]. In the recent years, were also investigated another type of symmetries for various models. A special type of nonlocal symmetries are investigated in [5], [6].

The way followed in this paper is a more direct one. It consists in the check of the symmetries of the system using the Lie group approach [4]. After this introduction, we apply the general algorithm for the computation of Lie symmetries and associated constants of motion for the generalized Hénon-Heiles system.

The Hénon-Heiles (HH) model:

$$\left\{ \begin{array}{lcl} \ddot{x} & = & -x_1 - 2x_1 y_1 \\ \ddot{y} & = & -y_1 - x_1^2 + y_1^2 \end{array} \right. , \tag{1}$$

with the Hamiltonian

$$H_1 = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{y}_1^2 + \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + x_1 y_1^2 - \frac{1}{3}x_1^3. \tag{2}$$

was created for describing stellar motion in a cylindrical symmetric galaxy with gravitational potential and is formed by a set of two coupled non-linear second order ordinary differential equations. The original form with fixed parameters was used by Hnon and Heiles in 1964 [7] to examine the regular and chaotic motion of a star in the galaxy. The general form of this model is

$$\left\{ \begin{array}{lcl} \ddot{x} & = & -Ax - 2Bxy \\ \ddot{y} & = & -Cx^2 - Dy + Ey^2 \end{array} \right. , \tag{3}$$

which is know to be Hamiltonian and integrable in the original case: for A=D and B=C=-E.

After this introduction, we apply the general algorithm for the computation of Lie symmetries for the HH system which depend only on the independent and dependent variables (the classical point symmetries). In order to capture the full invariance properties of the analyzed system, we search also for Lie generalized symmetries, in the 4D space $(x, y, \dot{x}, \dot{y})$. By this, more general Lie symmetries than the standard ones are obtained. Some remarks and conclusions will end the paper.

2 Classical symmetries of generalized Henon-Héiles system

Consider the equation Hénon-Heiles

$$\left\{ \begin{array}{lcl} \ddot{x} & = & -Ax - 2Bxy \\ \ddot{y} & = & -Cx^2 - Dy + Ey^2 \end{array} \right. , \tag{4}$$

written in the form

$$\left\{ \begin{array}{lcl} \ddot{x} - K_1 & = & \ddot{x} + Ax + 2Bxy = 0 \\ \ddot{y} - K_2 & = & \ddot{y} + Cx^2 + Dy - Ey^2 = 0 \end{array} \right. . \tag{5}$$

Let take a generalized vector field:

$$U = \xi\frac{\partial}{\partial t} + \varphi_1\frac{\partial}{\partial x} + \varphi_2\frac{\partial}{\partial y} \tag{6}$$

with a second order extension:

$$U^{(2)} = \xi\frac{\partial}{\partial t} + \varphi_1\frac{\partial}{\partial x} + \varphi_2\frac{\partial}{\partial y} + \varphi_1^t\frac{\partial}{\partial \dot{x}} + \varphi_2^t\frac{\partial}{\partial \dot{y}} + \varphi_1^{tt}\frac{\partial}{\partial \ddot{x}} + \varphi_2^{tt}\frac{\partial}{\partial \ddot{y}} \tag{7}$$

where

$$\begin{cases} \varphi_1^t &= \frac{d}{dt}[\varphi_1 - \xi\cdot x_t] + \xi x_{tt} \\ \varphi_2^t &= \frac{d}{dt}[\varphi_2 - \xi\cdot y_t] + \xi y_{tt} \\ \varphi_1^{tt} &= \frac{d^2}{dt^2}[\varphi_1 - \xi\cdot x_t] + \xi x_{ttt} \\ \varphi_2^{tt} &= \frac{d^2}{dt^2}[\varphi_2 - \xi\cdot y_t] + \xi y_{ttt} \end{cases}, \tag{8}$$

The invariant conditions of the evolution equations (5) are:

$$\begin{cases} U^{(2)}[\ddot{x} - K_1] &= \varphi_1^{tt} + \varphi_1(A + 2By) + 2\varphi_2 Bx = 0 \\ U^{(2)}[\ddot{y} - K_2] &= \varphi_2^{tt} + 2\varphi_1 Cx + \varphi_2(D - 2Ey) = 0 \end{cases}. \tag{9}$$

Supposing now that the infinitesimal generators $\xi = \xi(t,x,y)$, $\varphi_1 = \varphi_1(t,x,y)$ and $\varphi_2 = \varphi_2(t,x,y)$ depend only on the independent and dependent variables (*the case of classical Lie symmetries*), after vanishing the coefficients of various monomials of the form $\dot{x}^a\dot{y}^b$, $a,b = 0,1,2...$, in (9), we obtain a system S of certain number of equations with the unknown functions ξ φ_1 and φ_2.

The first of the equations (9) split into the equations:

$$(\varphi_1)_{xx} - 2\xi_{tx} = 0, \tag{10}$$

$$(\varphi_1)_{yy} = 0, \tag{11}$$

$$2(\varphi_1)_{xy} - 2\xi_{ty} = 0, \tag{12}$$

$$-2(\xi)_{xy} = 0, \tag{13}$$

$$-(\xi)_{xx} = 0, \tag{14}$$

$$-(\xi)_{yy} = 0, \tag{15}$$

$$2(\varphi_1)_{tx} - \xi_{tt} + 3\xi_x(Ax + 2Bxy) + \xi_y(Cx^2 + Dy - Ey^2) = 0, \tag{16}$$

$$2(\varphi_1)_{ty} + 2\xi_y(Ax + 2Bxy) = 0, \tag{17}$$

$$\begin{aligned} A\varphi_1 + 2By\varphi_1 + 2Bx\varphi_2 + (\varphi_1)_{tt} - (\varphi_1)_x(Ax + 2Bxy) &- \\ -(\varphi_1)_y(Cx^2 + Dy - Ey^2) + 2\xi_t(Ax + 2Bxy) &= 0. \end{aligned} \tag{18}$$

The second equation of (9) gives

$$(\varphi_2)_{yy} - 2\xi_{ty} = 0, \tag{19}$$

$$(\varphi_2)_{xx} = 0, \tag{20}$$

$$2(\varphi_2)_{xy} - 2\xi_{tx} = 0, \tag{21}$$

$$-2(\xi)_{yx} = 0, \tag{22}$$

$$-(\xi)_{yy} = 0, \tag{23}$$

$$-(\xi)_{xx} = 0, \tag{24}$$

$$2(\varphi_2)_{ty} - \xi_{tt} + \xi_x(Ax + 2Bxy) + 3\xi_y(Cx^2 + Dy - Ey^2) = 0, \tag{25}$$

$$2(\varphi_2)_{tx} + 2\xi_x(Cx^2 + Dy - Ey^2) = 0, \tag{26}$$

$$\begin{aligned} 2\varphi_1 Cx + D\varphi_2 - 2\varphi_2 Ey + (\varphi_2)_{tt} - (\varphi_2)_x(Ax + 2Bxy) \quad &- \\ -(\varphi_2)_y(Cx^2 + Dy - Ey^2) + 2\xi_t(Cx^2 + Dy - Ey^2) \quad &= \quad 0. \end{aligned} \tag{27}$$

From the equations (13), (14) and (15) we deduce that ξ is linear in x and y, and have the form

$$\xi = f(t)x + g(t)y + h(t).$$

From (11), we obtain that φ_1 is linear in y, and from (20), φ_2 must be linear in x.

Suppose that φ_1 and φ_2 have the form:

$$\varphi_1 = k_1 x + m_1 y + r_1 \tag{28}$$
$$\varphi_2 = k_2 x + m_2 y + r_2 \tag{29}$$

Using the first 18 equations (10)-(27), we obtain by reduction a new system:

$$2Br_2 = 0 \tag{30}$$
$$m_1 A + 2Br_1 - Dm_1 = 0 \tag{31}$$
$$2Bm_2 = 0 \tag{32}$$
$$2Bm_1 - Em_1 = 0 \tag{33}$$
$$2Bk_2 - Cm_1 = 0 \tag{34}$$
$$Em_2 = 0 \tag{35}$$
$$2C(k_1 - m_2) = 0 \tag{36}$$
$$2Cm_1 - 2Ek_2 - 2k_2 B = 0 \tag{37}$$
$$Er_2 = 0 \tag{38}$$
$$2Cr_1 + Dk_2 - Ak_2 = 0 \tag{39}$$
$$Dr_2 = 0 \tag{40}$$
$$Ar_1 = 0. \tag{41}$$

From these equations we can distingue some different cases:

- If B=0, E=0, C=0, $A = D \neq 0$ (*the trivial linear case*), we obtained the solutions

$$\xi = \mathrm{q}\ ,\ \varphi_1 = m_1 y + k_1 x\ ,\ \varphi_2 = m_2 x + k_2 y \tag{42}$$

where m_1, m_2, k_1, k_2 and q are constants. The infinitesimal generators of the symmetries are

$$U_1 = \frac{\partial}{\partial t}\ ,\ U_2 = x\frac{\partial}{\partial x}\ ,\ U_3 = y\frac{\partial}{\partial x}\ ,\ U_4 = x\frac{\partial}{\partial y}\ ,\ U_5 = y\frac{\partial}{\partial y};$$

- If B=0, E=0, C=0 and $A \neq D$ (*the second linear case*), we obtained the solutions

$$\xi = \mathrm{q}\ ,\ \varphi_1 = k_1 x\ ,\ \varphi_2 = 2k_1 y \tag{43}$$

where m_1, m_2, k_1, k_2 and q are constants.The infinitesimal generators of the symmetries are

$$U_1 = \frac{\partial}{\partial t}\ ,\ U_2 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y};$$

- If $C = 0$, $E \neq 0$, and $A \neq 0$, we obtained the solutions

$$\xi = \mathrm{q}\ ,\ \varphi_1 = k_1 x\ ,\ \varphi_2 = 0 \tag{44}$$

where k_1 and q are constants. Here B and D can have any value. The infinitesimal generators of the symmetries are

$$U_1 = \frac{\partial}{\partial t}\ ,\ U_2 = x\frac{\partial}{\partial x}$$

- If $B \neq 0$, E=2B, C=0 and A=0, we obtained the solutions

$$\xi = \mathrm{q}\ ,\ \varphi_1 = m_1 y + k_1 x + \frac{D}{2B}m_1\ ,\ \varphi_2 = 0, \tag{45}$$

where m_1, k_1, and q are constants. The infinitesimal generators of the symmetries are

$$U_1 = \frac{\partial}{\partial t}\ ,\ U_2 = x\frac{\partial}{\partial x}\ ,\ U_3 = [y + \frac{D}{2B}]\frac{\partial}{\partial x}$$

- In any other cases the only classical symmetry admitted by the general Hénon-Heiles system is

$$\xi = \mathrm{q}\ ,\ \varphi_1 = 0\ ,\ \varphi_2 = 0, \tag{46}$$

where q is a constant. The infinitesimal generator of the symmetries is

$$U_1 = \frac{\partial}{\partial t}$$

The original Hénon- Heiles system is included in this category. Note that this symmetry correspond to a time-translation and is common to any autonomous system.

3 The generalized symmetries

The Lie generalized symmetries for a PDE system which describes a dynamical system are the solution of the Lie invariance condition of Olver (9) that involves the derivatives of the dependent variables x and y.

An alternate version of the Lie invariance condition can be obtained by introducing the couple $Q = (Q^1, Q^2)$, known as the characteristic of the symmetry operator (5):

$$Q^1 \equiv \varphi_1 - \xi \dot{x}\,,\; Q^2 \equiv \varphi_2 - \xi \dot{y} \tag{47}$$

the Lie invariance condition (9) become:

$$\left\{ \begin{array}{lcl} U^{(2)}[K_1 - \ddot{x}] & = & -D_t^2 Q^1 + Q^1 K_{1,x} + Q^2 K_{1,y} = 0 \\ U^{(2)}[K_2 - \ddot{y}] & = & -D_t^2 Q^2 + Q^1 K_{2,x} + Q^2 K_{2,y} = 0 \end{array} \right. . \tag{48}$$

where $K_1 = -Ax - 2Bxy$ and $K_2 = -Cx^2 - Dy + Ey^2$.

In this paper we restricted our-self to the case of linear dependence of the symmetry operators on the velocities $\dot{x}$ and $\dot{y}$, then the symmetries characteristics Q^1 and Q^2 are given by

$$\left\{ \begin{array}{lcl} Q^1 & = & Q_{11}(t,x,y)\dot{x} + Q_{12}(t,x,y)\dot{y} \\ Q^2 & = & Q_{21}(t,x,y)\dot{x} + Q_{22}(t,x,y)\dot{y} \end{array} \right. . \tag{49}$$

with Q_{ij} constants. In this case, the symmetry condition (48) rewrite as

$$\left\{ \begin{array}{lcl} (Q_{11}\dot{x} + Q_{12}\dot{y})K_{1,x} + (Q_{21}\dot{x} + Q_{22}\dot{y})K_{1,y} & = & Q_{11} D_t K_1 + Q_{12} D_t K_2 \\ (Q_{11}\dot{x} + Q_{12}\dot{y})K_{2,x} + (Q_{21}\dot{x} + Q_{22}\dot{y})K_{2,y} & = & Q_{21} D_t K_1 + Q_{22} D_t K_2 \end{array} \right. . \tag{50}$$

But $D_t K_i = K_{i,x}\dot{x} + K_{i,y}\dot{y}$ for $i = 1, 2$, then, equaling the coefficients of $\dot{x}$, $\dot{y}$ in (50) one obtain the system

$$\left\{ \begin{array}{rcl} Q_{12}K_{2,x} & = & Q_{21}K_{1,y} \\ Q_{12}(K_{1,x} - K_{2,y}) & = & K_{1,y}(Q_{11} - Q_{22}) \\ Q_{21}(K_{1,x} - K_{2,y}) & = & K_{2,x}(Q_{11} - Q_{22}) \end{array} \right. . \tag{51}$$

One observe that the system (51) is obviously verified if $Q_{21} = Q_{12} = 0$ and $Q_{11} = Q_{22}$, then the system (4) admit in any situation the symmetry operator

$$U_0 = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}. \tag{52}$$

The main result of this section can be formulated as

Proposition 1 *Consider the generalized Hénon -Heiles system*

$$\left\{ \begin{array}{lcl} \ddot{x} & = & -Ax - 2Bxy \\ \ddot{y} & = & -Cx^2 - Dy + Ey^2 \end{array} \right. . \tag{53}$$

Then we have the following assertions:

a. *The system (53) always admit the symmetry operator (52)*

$$U_0 = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}.$$

b. *The operator*

$$U_{21} = \alpha\dot{y}\frac{\partial}{\partial x} + \beta\dot{x}\frac{\partial}{\partial y} \tag{54}$$

is symmetry for the system (53) if and only if

$$\begin{cases} A &= D \\ B &= -E \\ \alpha C &= \beta B \end{cases} . \tag{55}$$

c. *The operator*

$$U_{22} = (\alpha\dot{y} - \frac{\gamma}{2x}\dot{x})\frac{\partial}{\partial x} + (\beta\dot{x} + \frac{\gamma}{2x}\dot{y})\frac{\partial}{\partial y}. \tag{56}$$

is symmetry for the system (53) if and only if

$$\begin{cases} B &= -E \\ 2\gamma B &= \alpha(A-D) \\ 2\gamma C &= \beta(D-A) \end{cases} . \tag{57}$$

d. *Their are no other generalized symmetries that are linear in the velocities $\dot{x}$ and $\dot{v}$ for the generalized Hénon-Heiles system.*

Proof. By insertion of the expressions $K_1 = -Ax - 2Bxy$ and $K_2 = -Cx^2 - Dy + Ey^2$ into the system (51) and the re-notation $\alpha = Q_{12}$, $\beta = Q_{21}$ and $\gamma = Q_{22} - Q_{11}$, one obtains the solutions of the resulting system in the form described below.

Remark 1 *The classical Hénon-Heiles system verifies the condition (b.) from the Proposition 1.*

The cases of generalized symmetries that are not linear in velocities will be approached in a future paper.

4 Concluding remarks

We investigated the problem of the existence of classical and generalized symmetries of the generalized Hénon-Heiles equation

$$\begin{cases} \ddot{x} &= -Ax - 2Bxy \\ \ddot{y} &= -Cx^2 - Dy + Ey^2 \end{cases} , \tag{58}$$

using the Lie approach The main results we obtained could be synthesized as follows:

(i) The group of classical Lie symmetries for the equation Hénon-Heiles generalized is generated by one, two, three or four operators (the 42-46 formulas), depending of some conditions imposed to the coefficients A,B,C,D;

(ii) The generalized symmetries, linear in velocities $\dot{x}$ and $\dot{v}$, for this equation are spanned by (52) and (54) or (56), depending on some restriction of the coefficients A,B,C,D. Note that the existence of more complex symmetries was proved for two particular cases of the HH equation in [3].

The methodological approach exposed here do not require the investigated system to be Hamiltonian and can be easy adapted and applied in the case of other mechanical models of field theories, as, for examples, the case of Yang-Mills equation.

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