Physics AUC, vol. 25, 75-96 (2015) PHYSICS AUC

No-go results on consistent cross-couplings between a massless tensor field with the mixed symmetry $(k, 1)$ and a spin-2 field

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Abstract

Here we approach all consistent and nontrivial couplings that can be introduced between a massless tensor field with the mixed symmetry $(k, 1)$ for $k \geq 4$ and a Pauli–Fierz field in the context of the antifield-BRST deformation method under some standard "selection rules" from Quantum Field Theory.

PACS: 11.10.Ef

1 Introduction

The class of tensor fields with mixed symmetries (neither completely antisymmetric nor fully symmetric), i.e. transforming according to the irreducible exotic representations of the group $GL(D, \mathbb{R})$, became of real interest in theoretical high energy physics due to its involvement in many important physical theories (superstrings, supergravity, supersymmetric high-spin theories). The enhancement of gauge field theory spectrum such as to include bosonic tensor fields with mixed symmetries (i.e. transforming in true exotic representations of the Lorentz group) made possible a successful approach to several important issues, such as the proof of the absence of nontrivial consistent interactions in the dual formulation of linearized gravity in $D = 5$ [1], the connection of such models to M-theory [2–4], or the development of consistent interactions between this class of gauge field theories and gravity on both Minkowski and anti-de Sitter backgrounds [5–7].

Here we focus on the class of massless real tensor fields transforming in irreducible exotic representations of the Lorentz group corresponding to the so-called "hook" Young diagrams: two-column, with $(k + 1)$ cells, and displaying $k \geq 4$ rows — also known as tensor fields with the mixed symmetry $(k, 1)$. For arbitrary values of k, such tensor fields (massless and massive) have initially been investigated more than two decades ago [8–12]. Their key feature follows from the fact their free action provides one of the dual formulations of linearized gravity in $D = k + 3$. Thus, the raised interest in constructing gravity-like dual theories unveiled the prominent role played in this context by various types of mixed-symmetry tensor fields, like for instance that of fundamental field in the "magnetic representation" [13].

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The main purpose of the present paper is to construct all consistent and nontrivial couplings between a massless tensor field with the mixed symmetry $(k, 1)$ ($D \geq k + 2$, $k \geq 4$) and a spin-2 gauge field. Regarding the spin-2 gauge field, we start from the Pauli–Fierz formulation [26, 27] based on a massless symmetric tensor field of order two endowed with the gauge algebra of linearized diffeomorphisms. We employ one of the main applications of the antifield-BRST formalism [14–17], namely the construction of consistent interactions in gauge field theories [18–21] by means of deforming the solution to the classical master equation. This method requires the computation of the local cohomology of the BRST differential in ghost number 0 and in maximum form degree [22–25]. In addition, we ask that the deformations agree with the usual selection rules of Quantum Field Theory: analyticity in the coupling constant, spacetime locality, Lorentz covariance, Poincaré invariance, and conservation of the differential order of the free field equations at the level of the coupled theories. The last hypothesis is strengthened by requiring that the interacting vertices preserve the maximum derivative order the free Lagrangian density at all orders of perturbation theory, namely two. The findings exposed here complement and extend various developments [28–47].

Under the specified working hypotheses, we will prove the following main results:

1. There are no consistent and nontrivial cross-coupling first-order deformations that check all the imposed selection rules. The main obstruction seems to originate in the maximum derivative order equal to two of the interacting Lagrangian density at all orders of perturbation theory. It is possible that the relaxation of this assumption leads to nontrivial cross-couplings;

2. The only eligible terms from the deformed solution to the master equation correspond to the graviton self-interactions, materialized in the Einstein–Hilbert action (possibly with a cosmological term), invariant under diffeomorphisms, and respectively to the self-interactions of a single tensor field with the mixed symmetry $(k, 1)$ (nontrivial only for $k = 2\bar{k}$ and in $D = 4\bar{k}$).

2 Free limit: Lagrangian formulation and antifield-BRST symmetry

The starting point is a free Lagrangian action describing a massless tensor field with the mixed symmetry $(k, 1)$ ($k \geq 4$) and a Pauli–Fierz field

$$S_0[t_{\mu_1\ldots\mu_k|\alpha}, h_{\mu|\alpha}] = S_0^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}] + S_0^{\mathrm{h}}[h_{\mu|\alpha}], \tag{1}$$

where

$$S_0^{\mathrm{t}} = -\tfrac{1}{2\cdot(k+1)!}\int \left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha} - (k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right]d^Dx, \tag{2}$$

$$\begin{aligned} S_0^{\mathrm{h}} = \int \Big[& -\tfrac{1}{2}(\partial^\rho h^{\mu|\alpha})(\partial_\rho h_{\mu|\alpha}) + (\partial_\mu h^{\mu|\alpha})(\partial^\nu h_{\nu|\alpha}) \\ & - (\partial^\alpha h)(\partial^\mu h_{\mu|\alpha}) + \tfrac{1}{2}(\partial^\rho h)(\partial_\rho h)\Big]d^Dx. \end{aligned} \tag{3}$$

We work on a Minkowski spacetime $\mathcal{M}$ of dimension $D \geq k+2 \geq 6$ with a mostly positive metric $\sigma_{\mu\nu} = \sigma^{\mu\nu} = (- + \ldots +)$ and define the Levi-Civita symbol in D dimensions $\varepsilon^{\mu_1\ldots\mu_D}$ by $\varepsilon^{01\ldots D-1} = -1$. The mixed symmetry $(k, 1)$ of the field $t_{\mu_1\ldots\mu_k|\alpha}$ means it is antisymmetric in its first k indices and satisfies the identities $t_{[\mu_1\ldots\mu_k|\alpha]} \equiv 0$. Its trace, $t_{\mu_1\ldots\mu_{k-1}} = t_{\mu_1\ldots\mu_k|\alpha}\sigma^{\mu_k\alpha}$, is a completely antisymmetric tensor of order $(k-1)$, while

$$F_{\mu_1\ldots\mu_{k+1}|\alpha} = \partial_{[\mu_1} t_{\mu_2\ldots\mu_{k+1}]|\alpha}, \tag{4}$$

$$F_{\mu_1\ldots\mu_k} \equiv F_{\mu_1\ldots\mu_{k+1}|\alpha}\sigma^{\mu_{k+1}\alpha} = \partial_{[\mu_1} t_{\mu_2\ldots\mu_k]} + (-)^k \partial^\alpha t_{\mu_1\ldots\mu_k|\alpha}, \tag{5}$$

so the tensor $F_{\mu_1\ldots\mu_{k+1}|\alpha}$ displays the mixed symmetry $(k+1,1)$ and its trace is completely antisymmetric. The field $h_{\mu|\alpha}$ is a symmetric two-tensor and h denotes its trace, $h \equiv h_{\mu|\alpha}\sigma^{\mu\alpha}$. Functional S_0^{h} is precisely the Pauli–Fierz action [26, 27] and coincides with the linearized Einstein–Hilbert action (without a cosmologic term). Everywhere in this paper the notation $[\mu\ldots\nu]$ signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The stationary surface of this free model is defined by the field equations

$$\frac{\delta S_0^{\mathrm{t}}}{\delta t_{\nu_1\ldots\nu_k|\alpha}} \equiv \tfrac{1}{k!} T^{\nu_1\ldots\nu_k|\alpha} \approx 0, \tag{6}$$

$$\frac{\delta S_0^{\mathrm{h}}}{\delta h_{\mu|\alpha}} \equiv H^{\mu|\alpha} \approx 0, \tag{7}$$

with

$$T^{\nu_1\ldots\nu_k|\alpha} = \partial_\mu F^{\mu\nu_1\ldots\nu_k|\alpha} - \sigma^{\alpha[\nu_1}\partial_\mu F^{\nu_2\ldots\nu_k\mu]}, \tag{8}$$

$$H^{\mu|\alpha} = \Box h^{\mu|\alpha} - \partial^{(\mu} h^{\alpha)|\beta,}{}_\beta + \partial^\mu\partial^\alpha h + \sigma^{\mu\alpha}(\partial_\nu\partial_\beta h^{\nu|\beta} - \Box h). \tag{9}$$

In (9) we employed the standard notation $f_{,\mu} \equiv \partial_\mu f$. Obviously, $T^{\nu_1\ldots\nu_k|\alpha}$ and $H_{\mu|\alpha}$ preserve the symmetry properties of the corresponding fields, namely $T^{\nu_1\ldots\nu_k|\alpha}$ exhibits the mixed symmetry $(k,1)$ and $H_{\mu|\alpha}$ is symmetric.

A generating set of (infinitesimal) gauge transformations of action (1) can be taken as

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} t_{\mu_1\ldots\mu_k|\alpha} = \partial_{[\mu_1} \overset{(1)}{\theta}{}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1} \overset{(1)}{\epsilon}{}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\partial_\alpha \overset{(1)}{\epsilon}{}_{\mu_1\ldots\mu_k}, \tag{10}$$

together with the linearized version of diffeomorphisms in the Pauli–Fierz sector

$$\delta_\xi h_{\mu|\alpha} = \partial_{(\mu}\xi_{\alpha)}. \tag{11}$$

The gauge parameters from the $(k,1)$ sector are some real, arbitrary tensors on the spacetime manifold $\mathcal{M}$ such that in addition $\overset{(1)}{\theta}{}_{\mu_1\ldots\mu_{k-1}|\alpha}$ possesses the mixed symmetry $(k-1,1)$ and $\overset{(1)}{\epsilon}{}_{\mu_1\ldots\mu_k}$ is completely antisymmetric. The bosonic gauge parameter ξ is an arbitrary vector field on $\mathcal{M}$. The most general gauge-invariant quantities of this free theory are functions of the gauge-invariant objects from both sectors. Related to the Pauli–Fierz model, these are precisely the components of the linearized Riemann tensor

$$K^{\mathrm{h}}_{\mu\nu|\alpha\beta} = \partial_{[\mu} h_{\nu]|[\beta,\alpha]} \equiv \partial_\mu\partial_\alpha h_{\nu|\beta} - \partial_\nu\partial_\alpha h_{\mu|\beta} + \partial_\nu\partial_\beta h_{\mu|\alpha} - \partial_\mu\partial_\beta h_{\nu|\alpha} \tag{12}$$

together with their spacetime derivatives. The linearized Riemann tensor is linear in the Pauli–Fierz field, of order two in its spacetime derivatives, and exhibits the mixed symmetry $(2,2)$. This means it is separately antisymmetric in $\{\mu,\nu\}$ and $\{\alpha,\beta\}$, symmetric under their exchange ($\{\mu,\nu\} \leftrightarrow \{\alpha,\beta\}$), and satisfies the first (algebraic) Bianchi identities

$$K^{\mathrm{h}}_{[\mu\nu|\alpha]\beta} = 0. \tag{13}$$

In addition, it verifies also the second (differential) Bianchi identities

$$\partial_{[\lambda} K^{\mathrm{h}}_{\mu\nu]|\alpha\beta} = 0. \tag{14}$$

The following property will be useful in what follows: if a (real) symmetric tensor $\bar{H}^{\mu|\alpha}$ is divergence-free, then there exists a (real) tensor with the mixed symmetry (2,2) (of the linearized Riemann tensor), such that $\bar{H}^{\mu|\alpha}$ is written (up to the metric tensor) like its double divergence

$$\partial_\mu \bar{H}^{\mu|\alpha} = 0 \Leftrightarrow \left(\bar{H}^{\mu|\alpha} = \partial_\nu \partial_\beta \bar{\Phi}^{\mu\nu|\alpha\beta} + c\sigma^{\mu\alpha},\ \ c \in \mathbb{R}\right). \tag{15}$$

Regarding the free $(k,1)$ model alone, the most general gauge-invariant quantities are represented by the "curvature tensor"

$$K_{\mu_1\ldots\mu_{k+1}|\alpha\beta} = \partial_\alpha F_{\mu_1\ldots\mu_{k+1}|\beta} - F_{\mu_1\ldots\mu_{k+1}|\alpha} \equiv \partial_{[\mu_1} t_{\mu_2\ldots\mu_{k+1}]|[\beta,\alpha]}, \tag{16}$$

and its spacetime derivatives. The tensor $K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}$ exhibits the mixed symmetry $(k+1,2)$, so it is separately antisymmetric in its first $k+1$ and respectively last two indices, satisfies the (first Bianchi) algebraic identities $K_{[\mu_1\ldots\mu_{k+1}|\alpha]\beta} \equiv 0$ and also the (second Bianchi) differential identities $\partial_{[\mu_1} K_{\mu_2\ldots\mu_{k+2}]|\alpha\beta} \equiv 0$ together with $K_{\mu_1\ldots\mu_{k+1}|[\alpha\beta,\gamma]} \equiv 0$. We notice that $K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}$ plays, in the context of the free theory $(k,1)$, exactly the same role like the linearized Riemann tensor in the framework of the Pauli–Fierz model. Moreover, if $\bar{T}^{\mu_1\ldots\mu_k|\alpha}$ is a (real) covariant tensor with the mixed symmetry $(k,1)$ whose both kinds of divergences are vanishing, then there exists a (real) tensor $\bar{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}$ with the mixed symmetry $(k+1,2)$ of the curvature tensor such that $\bar{T}^{\mu_1\ldots\mu_k|\alpha}$ is represented like its double divergence

$$\left(\partial_{\mu_1} \bar{T}^{\mu_1\ldots\mu_k|\alpha} = 0,\ \partial_\alpha \bar{T}^{\mu_1\ldots\mu_k|\alpha} = 0\right) \Leftrightarrow \bar{T}^{\mu_1\ldots\mu_k|\alpha} = \partial_{\mu_{k+1}} \partial_\beta \bar{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}. \tag{17}$$

The generating set of gauge transformations (11) for linearized gravity are irreducible and generate an Abelian algebra, whereas the Pauli–Fierz field equations were shown to be linear in the spin-2 field and of order two in its derivatives. Therefore the "graviton" sector is described by a linear gauge theory with the Cauchy order equal to 2. Altogether, action (1) inherits all the properties of the free massless tensor field with the mixed symmetry $(k,1)$: the gauge algebra remains Abelian, but the overall generating set of gauge transformations becomes reducible of order $(k-1)$, such that its Cauchy order is $(k+1) \geq 5$.

The BRST algebra of this free model is constructed starting from the generators corresponding to the $(k,1)$ sector

$$\Phi^A \equiv \left\{ t_{\mu_1\ldots\mu_k|\alpha},\ \left\{ \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha},\ \overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} \right\}_{m=\overline{1,k-1}},\ \overset{(k)}{\eta}_\mu \right\}, \tag{18}$$

$$\Phi^*_A \equiv \left\{ t^{*\mu_1\ldots\mu_k|\alpha},\ \left\{ \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha},\ \overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} \right\}_{m=\overline{1,k-1}},\ \overset{(k)}{\eta}{}^{*\mu} \right\}, \tag{19}$$

whose properties are detailed in [40, 46] (a synthetic view is given in Table 1 from [46]), to which we add the generators from Table 1 with the properties specified therein. We

BRST generator	pgh	agh	gh	ε
$h_{\mu\|\alpha}$	0	0	0	0
$\mathcal{C}_{\alpha}$	1	0	1	1
$h^{*\mu\|\alpha}$	0	1	-1	1
$\mathcal{C}^{*\alpha}$	0	2	-2	0

Table 1: Various gradings of the BRST generators from the Pauli–Fierz sector.

mention that the ghosts $\mathcal{C}_{\alpha}$ are due to the gauge parameters ξ_{α}. The antifields $h^{*\mu|\alpha}$ preserve the symmetry property of the Pauli–Fierz field components, $h^{*\mu|\alpha}=h^{*\alpha|\mu}$. In this case, the BRST differential simply decomposes like

$$s=\delta+\gamma,\quad s^{2}=0\Leftrightarrow(\delta^{2}=0,\,\gamma^{2}=0,\,\delta\gamma+\gamma\delta=0)\tag{20}$$

into the sum between the Koszul–Tate differential δ ($\mathbb{N}$-graded in terms of the antighost number agh, $\mathrm{agh}(\delta)=-1$) and the longitudinal exterior derivative γ (here a true differential that anticommutes with δ and is $\mathbb{N}$-graded along the pure ghost number pgh, $\mathrm{pgh}(\gamma)=1$). The BRST differential is $\mathbb{Z}$-graded according to the ghost number gh (defined in the standard fashion like pgh $-$ agh), such that $\mathrm{gh}(s)=\mathrm{gh}(\delta)=\mathrm{gh}(\gamma)=1$. The actions of δ and γ on the $(k,1)$ BRST generators can be found in [46] (formulas (15)–(23)), while on the "graviton" ones read as

$$\gamma h_{\mu|\alpha}=\partial_{(\mu}\mathcal{C}_{\alpha)},\qquad \gamma\mathcal{C}_{\alpha}=0,\qquad \gamma h^{*\mu|\alpha}=0=\gamma\mathcal{C}^{*\alpha},\tag{21}$$

$$\delta h_{\mu|\alpha}=0=\delta\mathcal{C}_{\alpha},\qquad \delta h^{*\mu|\alpha}=-H^{\mu|\alpha},\qquad \delta\mathcal{C}^{*\alpha}=-2\partial_{\mu}h^{*\mu|\alpha}.\tag{22}$$

The solution to the classical master equation is nothing but the sum between those associated with the $(k,1)$ sector and the Pauli–Fierz model

$$S=S^{\mathrm{t}}+S^{\mathrm{h}},\tag{23}$$

$$\begin{aligned}S^{\mathrm{t}}=&S_{0}^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}]+\int\Big\{t^{*\mu_1\ldots\mu_k|\alpha}\Big[\partial_{[\mu_1}\overset{(1)}{C}_{\mu_2\ldots\mu_k]|\alpha}+\partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_k\alpha]}\\&+(-)^{k+1}(k+1)\partial_{\alpha}\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}\Big]+\overset{(k-1)}{C}{}^{*\mu_1|\alpha}\partial_{(\mu_1}\overset{(k)}{\eta}_{\alpha)}\\&+\sum_{m=1}^{k-2}\overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha}\Big[\partial_{[\mu_1}\overset{(m+1)}{C}_{\mu_2\ldots\mu_{k-m}]|\alpha}+\partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m}\alpha]}\\&+(-)^{k-m+1}(k-m+1)\partial_{\alpha}\overset{(m+1)}{\eta}_{\mu_1\ldots\mu_{k-m}}\Big]\\&+\sum_{m=1}^{k-1}\tfrac{k-m}{k-m+2}\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}}\partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m+1}]}\Big\}d^{D}x,\end{aligned}\tag{24}$$

$$S^{\mathrm{h}}=S_{0}^{\mathrm{h}}[h_{\mu|\alpha}]+\int h^{*\mu|\alpha}\partial_{(\mu}\mathcal{C}_{\alpha)}d^{D}x.\tag{25}$$

It is useful to denote all the BRST generators in a condensed form like

$$\bar{\Phi}^{\bar{A}}=\{\Phi^{A},h_{\mu|\alpha},\mathcal{C}_{\alpha}\},\qquad \bar{\Phi}^{*}_{\bar{A}}=\{\Phi^{*}_{A},h^{*\mu|\alpha},\mathcal{C}^{*\alpha}\},\tag{26}$$

with Φ^{A} si Φ^{*}_{A} like in (18) and (19), respectively.

3 Main properties of the local BRST cohomology

From the perspective of analyzing the (nontrivial) consistent interactions that can be added to the free action (1), here we discuss the main properties of the local BRST cohomology in maximum form degree (D) for the free model under study computed in the algebra of local forms, $H^D(s|d)$. The last algebra is defined via the coefficients of the underlying forms, which are required to be elements of the BRST algebra of local functions $\bar{\mathcal{A}}$, defined by polynomials in ghosts, antifields, and their derivatives up to a finite order, 'smooth' functions in the undifferentiated fields t and h, and again polynomials in the field derivatives up to a finite order. All the BRST cohomological results reported in Refs. [46, 47] in relation to a single massless tensor field with the mixed symmetry $(k,1)$ still hold in the present broader context up to properly include the supplementary dependence on the Pauli–Fierz BRST generators. Related to the local BRST cohomology corresponding to the Pauli–Fierz model we adopt the line from Ref. [48] where we eliminate the graviton collection indices.

Regarding the cohomologies $H(\gamma)$ and $H(\gamma|d)$, the results from Ref. [46] apply up the following observations. At the level of the algebra of invariant polynomials, namely the cohomology of γ in pgh $= 0$ computed in $\bar{\mathcal{A}}$ for the entire free gauge theory, the Pauli–Fierz sector brings in an additional, polynomial dependence on the linearized Riemann tensor K^{t} given in (12), on the antifields h^*, $\mathcal{C}^*$, as well as on their spacetime derivatives up to a finite order, such that

$$H^0(\gamma) \text{ in } \bar{\mathcal{A}} = \{\text{algebra of invariant polynomials}\} \equiv \left\{\bar{\alpha}\left(\left[\bar{\Phi}^*_{\bar{A}}\right], [K], [K^{\mathrm{h}}]\right)\right\}, \tag{27}$$

where we employ the condensed notation from (26) and by $f([y])$ we mean that f depends on y and its derivatives up to a finite order. In strictly positive pure ghost numbers, the presence of the "graviton" component is manifested by a supplementary dependence on the undifferentiated fermionic ghosts $\mathcal{C}^\alpha$ (of pgh $= 1$) and of their antisymmetric first-order derivatives

$$\mathcal{F}^{\mathrm{h}\mu\nu} \equiv \partial^{[\mu}\mathcal{C}^{\nu]}, \qquad \mathcal{F}^{\mathrm{h}\mu\nu} = -\mathcal{F}^{\mathrm{h}\nu\mu}, \qquad \varepsilon(\mathcal{F}^{\mathrm{h}\mu\nu}) = 1, \qquad \mathrm{pgh}(\mathcal{F}^{\mathrm{h}\mu\nu}) = 1. \tag{28}$$

The symmetric first-order derivatives of the ghosts $\mathcal{C}$ are γ-exact, in agreement with the first formula from (21), and similarly the derivatives of $\mathcal{F}^{\mathrm{h}}$ of various orders due to

$$\partial^\rho \mathcal{F}^{\mathrm{h}\mu\nu} = \gamma(\partial^{[\mu} h^{\nu]|\rho}). \tag{29}$$

As a consequence, Table 2 from Ref. [46] should be replaced with Table 2 below, where the object $\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}$ is specific to the $(k,1)$ sector

$$\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}} \equiv \partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_{k+1}]}, \quad \varepsilon\Big(\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\Big) = 1, \quad \mathrm{pgh}\Big(\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\Big) = 1. \tag{30}$$

The derivatives of $\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}$ of various orders are trivial in $H(\gamma)$ (just like those of $\mathcal{F}^{\mathrm{h}}$) on account of the relation

$$\partial_{\rho_1}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}} = \gamma\Big(\tfrac{(-)^{k+1}}{k} F_{\mu_1\ldots\mu_{k+1}|\rho_1}\Big). \tag{31}$$

By virtue of Table 2, the general nontrivial elements a of the cohomology $H(\gamma)$ computed

BRST generator	Nontrivial representatives	pgh
$[t_{\mu_1\ldots\mu_k\vert\alpha}]$	$[K_{\mu_1\ldots\mu_{k+1}\vert\alpha\beta}]$	0
$[h_{\mu\vert\alpha}]$	$[K^{\mathrm{h}}_{\mu\nu\vert\alpha\beta}]$	0
$[\bar{\Phi}^*_{\bar{A}}]$	$[\bar{\Phi}^*_{\bar{A}}]$	0
$[\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}], [\overset{(1)}{C}_{\mu_1\ldots\mu_{k-1}\vert\alpha}]$	$\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}$	1
$[\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}}], [\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha}]$	—	$m,\ m=\overline{2,k-1}$
$[\overset{(k)}{\eta}_{\alpha}]$	$\overset{(k)}{\eta}_{\alpha}$	k
$[\mathcal{C}_\alpha]$	$\mathcal{C}_\alpha, \mathcal{F}^{\mathrm{h}}_{\mu\nu}$	1

Table 2: Nontrivial representatives of the cohomology $H(\gamma)$ computed in the algebra $\bar{\mathcal{A}}$.

in $\bar{\mathcal{A}}$ that satisfy the properties $\mathrm{pgh}(a) = l \geq 0$ and $\mathrm{agh}(a) = j \geq 0$, are expressed by

$$a = \sum_J \bar{\alpha}_J\big([\bar{\Phi}^*_{\bar{A}}], [K], [K^{\mathrm{h}}]\big)\bar{e}^J\Big(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}, \mathcal{F}^{\mathrm{h}}\Big), \quad \mathrm{agh}(\bar{\alpha}_J) = j \geq 0,\ \mathrm{pgh}(\bar{e}^J) = l \geq 0, \tag{32}$$

where $\bar{e}^J$ denote the elements of $\mathrm{pgh} = l$ of a basis in the objects $\overset{(1)}{\mathcal{F}}$, $\overset{(k)}{\eta}$, $\mathcal{C}$, and $\mathcal{F}^{\mathrm{h}}$. An important result is that the analogue of Corollary 3 from Ref. [46] is still valid in the presence of the Pauli–Fierz model.

The local cohomologies $H(\delta|d)$ (in $\mathrm{pgh} = 0$) and $H^{\mathrm{inv}}(\delta|d)$ are still subdued to the main properties from Ref. [46], up to the following specifications. The statements of Corollary 5, Lemma 6, Theorem 7, and Corollary 8 therein are to be completed by the eligible dependence on the Pauli–Fierz antifields. Actually, the nontrivial representatives that span the spaces $\big(H^D_j(\delta|d)\big)_{j=\overline{3,k+1}}$ (in $\mathrm{pgh} = 0$) and $\big(H^{\mathrm{inv}D}_j(\delta|d)\big)_{j=\overline{3,k+1}}$ coincide with those of the $(k,1)$ model. Only in $\mathrm{agh} = 2$ there appears a supplementary dependence on the components of the undifferentiated antifield $\mathcal{C}^*$, such that Table 1 from [47] will be replaced by Table 3 below, where

$$\overset{(m)}{C'}{}^{*\mu_1\ldots\mu_{k-m}||\alpha} \equiv \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha} + \tfrac{1}{k-m+2}\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m}\alpha}, \qquad m = \overline{1,k-1}. \tag{33}$$

The antifields $\overset{(m)}{C'}{}^{*}$ span, at each $\mathrm{agh} = m+1 \in \overline{2,k}$, all the independent components of the antifield spectrum from the $(k,1)$ sector. The double bar signifies that they display no mixed-symmetry property, but are only antisymmetric in their first $(k-m)$ indices, where applicable. Actually, we can equally define some ghost combinations

$$\overset{(m)}{C'}_{\mu_1\ldots\mu_{k-m}||\alpha} \equiv \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha} + (k-m+2)\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m}\alpha}, \qquad m = \overline{1,k-1}, \tag{34}$$

that span, at each $\mathrm{pgh} = m \in \overline{1,k-1}$, all the independent components of the ghost spectrum from the $(k,1)$ sector and, moreover, are respectively conjugated with (33) in the antibracket.

Finally, the local BRST cohomology in maximum form degree for the free theory (1) computed in the algebra of local forms, $H^D(s|d)$, is still adequately governed by Proposition 10 from Ref. [47] (up to the appropriate inclusion of the "graviton" sector),

agh	complete set of nontrivial representatives
$k+1$	$\overset{(k)}{\eta}{}^{*\alpha}$
$j=\overline{3,k}$	$\overset{(j-1)}{C'}{}^{*\mu_1\ldots\mu_{k-j+1}\|\|\alpha}$
2	$\overset{(1)}{C'}{}^{*\mu_1\ldots\mu_{k-1}\|\|\alpha}$, $\mathcal{C}^{*\alpha}$

Table 3: Nontrivial representatives spanning $\left(H_j^D(\delta|d)\right)_{j=\overline{2,k+1}}$ and $\left(H_j^{\mathrm{inv}D}(\delta|d)\right)_{j=\overline{2,k+1}}$.

but the analysis following this proposition should be changed appropriately. For instance, relation (99) from Ref. [47] becomes

$$\overset{[D]}{a}{}_{k+1} = \sum_J \overset{[D]}{\bar{\alpha}}{}_J \bar{e}^J\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}, \mathcal{F}^{\mathrm{h}}\right), \quad \overset{[D]}{\bar{\alpha}}{}_J \in H_{k+1}^{\mathrm{inv}D}(\delta|d), \quad \mathrm{pgh}(\bar{e}^J) = k+1+g. \tag{35}$$

4 Deformed solution of the classical master equation. No-go results

4.1 Antifield-BRST deformation method in brief

The reformulation of the problem of constructing consistent interactions in gauge field theories within the antifield-BRST formalism [18–21] is based on the fact that if consistent couplings can be introduced, then the solution to the classical master equation of the initial gauge theory, S, may be deformed into a solution to the classical master equation for the interacting gauge theory

$$\bar{S} = S + \lambda S_1 + \lambda^2 S_2 + \lambda^3 S_3 + \cdots, \quad \tfrac{1}{2}(\bar{S}, \bar{S}) = 0. \tag{36}$$

Related to the coupled theory, we maintain the field, ghost, and antifield spectra of the original gauge theory in order to preserve the number of physical degrees of freedom. In the above $\bar{S}$ is a bosonic functional of fields, ghosts, and antifields with the ghost number equal to 0. The projection of equation $\frac{1}{2}(\bar{S}, \bar{S}) = 0$ on the various powers in the coupling constant λ is equivalent to the tower of equations

$$\lambda^1 : sS_1 = 0, \quad \lambda^2 : sS_2 + \tfrac{1}{2}(S_1, S_1) = 0, \quad \lambda^3 : sS_3 + (S_1, S_2) = 0, \quad \cdots \tag{37}$$

known as the equation of the antifield-BRST deformation method. In this context the functionals S_i, $i \geq 1$, are called deformations of order i of the solution to the master equation. The solutions to the first-order deformation equation $sS_1 = 0$ always exist since they belong to the cohomology of the BRST differential s in ghost number 0 computed in the space of all functionals (local and nonlocal) of fields, ghosts, and antifields, $H^0(s)$, which is nonempty due to its isomorphism to the algebra of physical observables of the initial gauge theory. Moreover, trivial first-order deformations, defined as trivial elements of $H^0(s)$ (s-exact functionals), should be ruled out due to the fact that they provoke trivial interactions in the sense of field theory (that can be eliminated by some possibly nonlinear field redefinitions). The existence of solutions to the remaining higher-order equations from (37) has been shown in [19] by means of the triviality of the antibracket map in the BRST cohomology $H(s)$ computed in the space of all functionals. In conclusion, if we impose no restrictions on the interactions (spacetime locality, etc.), then the antifield-BRST deformation procedure can be developed without obstructions.

Nevertheless, if we work with local functionals, then the procedure starts as follows. We make the notation

$$S_1 = \int a_1 d^D x, \tag{38}$$

where the nonintegrated density of the first-order deformation, a_1, is now an element of the BRST algebra of local "functions". The general properties of S_1 are transferred to a_1

$$\varepsilon(a_1) = 0, \quad \mathrm{gh}(a_1) = 0. \tag{39}$$

The equation satisfied by the first-order deformation (the first equation from (37)) takes in dual language the local form

$$sa_1 + \partial_\mu b_1^\mu = 0, \quad \varepsilon(b_1^\mu) = 1, \quad \mathrm{gh}(b_1^\mu) = 1, \tag{40}$$

where the current b_1^μ should be local. In other words, the first-order deformation defines precisely a class from the local BRST cohomology in maximum form degree and in ghost number equal to zero computed in the algebra of local forms, $H^{0,D}(s|d)$, where d symbolizes the spacetime exterior differential. Meanwhile, all purely trivial contributions from $H^{0,D}(s|d)$ computed in the algebra of local forms should be discarded since they generate only trivial interactions

$$a_1^{\mathrm{triv}} = sc + \partial_\mu e^\mu, \tag{41}$$

$$\varepsilon(c) = 1, \quad \varepsilon(e^\mu) = 0, \quad \mathrm{gh}(c) = -1, \quad \mathrm{gh}(e^\mu) = 0, \tag{42}$$

with both c and e^μ local. Thus, the first step of the deformation method in the presence of the locality assumption is represented by finding nontrivial solutions to equation (40) that should be further filtered by additional selection rules if applicable. If no such solutions are detected, then the deformation procedure is obstructed and one concludes that no true interactions may be constructed with respect to the starting gauge theory.

4.2 Cross-coupling first-order deformation

The scope of this work is to generate all nontrivial, consistent interactions that can be added to the free model (1) in the framework of the antifield-BRST deformation method briefly exposed in the previous subsection under the following standard assumptions from Quantum Field Theory. We require that the deformation of the solution to the master equation, (36), is analytical in the coupling constant, local in spacetime, Lorentz covariant, Poincaré invariant, and conserves the differential order of each free field equation at the level of the coupled theories. The last hypothesis is strengthened by asking that the interacting vertices display the maximum derivative order of the free Lagrangian density at any order in the coupling constant, namely two in this case. Due to the locality hypothesis, we adopt notation (38) and find that the nonintegrated density of the first-order deformation, a_1, is solution to equation (40), and thus, as argued in the previous subsection, should be a nontrivial element of the local BRST cohomology $H^{0,D}(s|d)$.

The presence of the two distinct sectors induces a natural decomposition of the first-order deformation as a sum among three (local) pieces

$$S_1 = S_1^{\mathrm{t}} + S_1^{\mathrm{h}} + S_1^{\mathrm{t-h}}, \tag{43}$$

$$S_1^{\mathrm{t}} = \int a_1^{\mathrm{t}} d^D x, \quad S_1^{\mathrm{h}} = \int a_1^{\mathrm{h}} d^D x, \quad S_1^{\mathrm{t-h}} = \int a_1^{\mathrm{t-h}} d^D x, \tag{44}$$

where S_1^{t} and S_1^{h} describe the self-interactions of the tensor $(k, 1)$ and respectively of the Pauli–Fierz field and $S_1^{\mathrm{t-h}}$ the cross-couplings between these two fields, such that

$$a_1 = a_1^{\mathrm{t}} + a_1^{\mathrm{h}} + a_1^{\mathrm{t-h}}. \tag{45}$$

These three pieces are functionally independent (the first involves only BRST generators from the $(k, 1)$ sector, the second solely from the Pauli–Fierz sector, and the third mandatorily mixes both kinds of generators), so equation (40) splits into three equivalent, independent equations

$$sa_1^{\mathrm{t}} + \partial_\mu b_1^{\mathrm{t}\mu} = 0, \qquad sa_1^{\mathrm{h}} + \partial_\mu b_1^{\mathrm{h}\mu} = 0, \qquad sa_1^{\mathrm{t-h}} + \partial_\mu b_1^{\mathrm{t-h}\mu} = 0. \tag{46}$$

The first equation has been considered in Ref. [45], where it has been proved that we can take a_1^{t} to stop in $\mathrm{agh} = 1$

$$a_1^{\mathrm{t}} = a_{1,1}^{\mathrm{t}} + a_{1,0}^{\mathrm{t}}, \tag{47}$$

with

$$a_{1,1}^{\mathrm{t}} = \delta_{2\bar{k}}^{k} \delta_{4\bar{k}}^{D} \varepsilon_{\mu_1 \ldots \mu_{4\bar{k}}} t^{*\mu_1 \ldots \mu_{2\bar{k}-1}} \overset{(1)}{\mathcal{F}}{}^{\mu_{2\bar{k}} \cdots \mu_{4\bar{k}}}, \tag{48}$$

$$a_{1,0}^{\mathrm{t}} = -\delta_{2\bar{k}}^{k} \delta_{4\bar{k}}^{D} \frac{(2\bar{k}-1)(2\bar{k}+1)}{(2\bar{k})!8\bar{k}^2} \varepsilon_{\mu_1 \ldots \mu_{4\bar{k}}} F^{\mu_1 \ldots \mu_{2\bar{k}}} F^{\mu_{2\bar{k}+1} \cdots \mu_{4\bar{k}}}. \tag{49}$$

From now on, the second lower index of the quantities involved in the various orders of perturbation theory signifies their antighost number (for instance, see expansion (47)). The supplementary factors $\delta_{2\bar{k}}^{k}$ and $\delta_{4\bar{k}}^{D}$ were introduced to mark that relations (48) and (49) hold only for even values of k $(2\bar{k})$ and in $D = 4\bar{k}$ spacetime dimensions. The second equation from (46) has been analyzed in detail in Ref. [48] in the context of a collection of Pauli–Fierz fields. Eliminating the collection indices, we infer that the nonintegrated density of the first-order deformation for the purely "graviton" sector may be taken to stop in $\mathrm{agh} = 2$

$$a_1^{\mathrm{h}} = a_{1,0}^{\mathrm{h}} + a_{1,1}^{\mathrm{h}} + a_{1,2}^{\mathrm{h}}, \tag{50}$$

where

$$a_{1,2}^{\mathrm{h}} = \mathcal{C}^{*\mu} \mathcal{C}^{\nu} \partial_\mu \mathcal{C}_\nu, \qquad a_{1,1}^{\mathrm{h}} = h^{*\mu|\nu} \mathcal{C}^{\rho} (\partial_\rho h_{\mu|\nu} - \partial_{(\mu} h_{\nu)|\rho}) \tag{51}$$

and the Lagrangian density in order one of perturbation theory, $a_{1,0}^{\mathrm{h}}$, reduces to the sum between the cubic vertex of the Einstein–Hilbert Lagrangian plus a cosmologic term (linear in the trace of the Pauli–Fierz field).

In the sequel we construct the remaining piece, namely the cross-coupling first-order deformation as solution to the last equation from (46), using the cohomological ingredients analyzed in the previous section. For this purpose, we need the following result.

Proposition 1 *The nontrivial solutions to the homogeneous cross-coupling equations in strictly positive values of the antighost number that depend of at least one undifferentiated Pauli–Fierz ghost generate inconsistencies at the level of the first-order deformation* $a_1^{\mathrm{t-h}}$.

In order to understand the consequences of the above proposition, we take a non-integrated density of the first-order cross-coupling deformation that ends at a maximum, strictly positive value j of the antighost number

$$j > 0 : \bar{a}_1^{\mathrm{t-h}}|_j = \sum_{\bar{j}=0}^{j} \bar{a}_{1,\bar{j}}^{\mathrm{t-h}}|_j, \quad \varepsilon(\bar{a}_{1,\bar{j}}^{\mathrm{t-h}}|_j) = 0, \ \mathrm{gh}(\bar{a}_{1,\bar{j}}^{\mathrm{t-h}}|_j) = 0, \ \mathrm{agh}(\bar{a}_{1,\bar{j}}^{\mathrm{t-h}}|_j) = \bar{j}, \tag{52}$$

$$s\bar{a}_1^{\text{t-h}}|_j + \partial_\mu \bar{b}_1^{\text{t-h}\mu}|_j = 0. \tag{53}$$

On the one hand, the similar to Proposition 10 from Ref. [47] grants that we can take $j \leq k+1$ without affecting the generality of our approach. On the other hand, in view of decomposition $s = \delta + \gamma$ and taking into account expansion (52), the analogue of Corollary 3 from [46] ensures that we can take the current $\bar{b}_1^{\text{t-h}\mu}|_j$ to stop in $\text{agh} = j-1$

$$j > 0 : \bar{b}_1^{\text{t-h}\mu}|_j = \sum_{\bar{\jmath}=0}^{j-1} \bar{b}_{1,\bar{\jmath}}^{\text{t-h}\mu}|_j, \quad \varepsilon(\bar{b}_{1,\bar{\jmath}}^{\text{t-h}\mu}|_j) = 1, \ \text{gh}(\bar{b}_{1,\bar{\jmath}}^{\text{t-h}\mu}|_j) = 1, \ \text{agh}(\bar{b}_{1,\bar{\jmath}}^{\text{t-h}\mu}|_j) = \bar{\jmath}, \tag{54}$$

such that equation (53) becomes equivalent to the chain

$$\gamma \bar{a}_{1,j}^{\text{t-h}}|_j = 0, \tag{55}$$

$$\delta \bar{a}_{1,\bar{\jmath}+1}^{\text{t-h}}|_j + \gamma \bar{a}_{1,\bar{\jmath}}^{\text{t-h}}|_j + \partial_\mu \bar{b}_{1,\bar{\jmath}}^{\text{t-h}\mu}|_j = 0, \qquad \bar{\jmath} = \overline{0, j-1}. \tag{56}$$

The piece of maximum antighost $j > 0$ can always be constructed as solution to the homogenous equation (55), i.e. via the (nontrivial) elements of the cohomology $H^j(\gamma)$ with the supplementary properties that follow from (52). In agreement with result (32) for $l = j > 0$, the nontrivial solutions to the homogeneous equation (55) read

$$\bar{a}_{1,j}^{\text{t-h}}|_j = \bar{\alpha}_j\left(\left[\bar{\Phi}_{\bar{A}}^*\right], [K], [K^{\text{h}}]\right) \bar{e}^j \left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}, \mathcal{F}^{\text{h}}\right), \ \text{agh}(\bar{\alpha}_j) = j > 0, \ \text{pgh}(\bar{e}^j) = j > 0. \tag{57}$$

Proposition 1 states thus that if we start from a cross-coupling first-order deformation whose piece of maximum antighost number $j > 0$ is expressed like in (57) and the elements of the basis $\bar{e}^j$ effectively depend on the components of the undifferentiated Pauli–Fierz ghost $\mathcal{C}^\alpha$, then (irrespective of the corresponding invariant polynomial $\bar{\alpha}_j$) there exists a value $\bar{\jmath} \in \overline{0, j-1}$ such that the corresponding equation in $\text{agh} = \bar{\jmath}$ from (56) possesses no solutions with respect to $\bar{a}_{1,\bar{\jmath}}^{\text{t-h}}|_j$

$$\bar{a}_{1,j}^{\text{t-h}}|_j = \bar{\alpha}_j\left(\left[\bar{\Phi}_{\bar{A}}^*\right], [K], [K^{\text{h}}]\right) \bar{e}^j \left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \underline{\mathcal{C}}, \mathcal{F}^{\text{h}}\right) \Rightarrow \text{inconsistencies for } \bar{a}_1^{\text{t-h}}|_j, \tag{58}$$

where by underlining one or more arguments we mean an explicit dependence on it (them).

Let us discuss now the implications of the hypothesis on the maximum derivative-order of the cross-coupling Lagrangian density to be equal to two. We underline that all the results envisaged in this context hold *independently of Proposition 1*, even if we maintain some notations or recall certain equations or general results mentioned in the previous paragraph. We assume an expansion of the cross-coupling first-order deformation $\bar{a}_1^{\text{t-h}}|_j$ that stops at a maximum value of the antighost number strictly greater than one and less or equal to $(k+1)$

$$2 \leq j \leq k+1 : \quad \bar{a}_1^{\text{t-h}}|_j = \sum_{\bar{\jmath}=0}^{j} \bar{a}_{1,\bar{\jmath}}^{\text{t-h}}|_j \tag{59}$$

and satisfies equation (53). Clearly, the components $\bar{a}_{1,\bar{\jmath}}^{\text{t-h}}|_j$ display the properties given in (52). By standard arguments we find that the expansion of the corresponding current can be taken to end in $\text{agh} = j-1$

$$2 \leq j \leq k+1 : \quad \bar{b}_1^{\text{t-h}\mu}|_j = \sum_{\bar{\jmath}=0}^{j-1} \bar{b}_{1,\bar{\jmath}}^{\text{t-h}\mu}|_j, \tag{60}$$

such that (53) becomes equivalent to the (finite) descent

$$\gamma\bar{a}_{1,j}^{\mathrm{t-h}}|_j = 0, \delta\bar{a}_{1,j}^{\mathrm{t-h}}|_j + \gamma\bar{a}_{1,j-1}^{\mathrm{t-h}}|_j + \partial_\mu\bar{b}_{1,j-1}^{\mathrm{t-h}\mu}|_j = 0, \cdots, \delta\bar{a}_{1,1}^{\mathrm{t-h}}|_j + \gamma\bar{a}_{1,0}^{\mathrm{t-h}}|_j + \partial_\mu\bar{b}_{1,0}^{\mathrm{t-h}\mu}|_j = 0, \quad (61)$$

where the properties of the currents $\bar{b}_{1,\bar{j}}^{\mathrm{t-h}\mu}|_j$ read as in (54). For the sake of simplicity we consider only solutions to the homogeneous equation in maximum agh (the first equation in (61)) of the form (57) where we take the invariant polynomials to be linear in the undifferentiated antifields of antighost number $2 \leq j \leq k+1$. Recalling notation (26), this means that

$$\bar{\alpha}_j \sim \bar{\Phi}_{\bar{A}}^*|_j, \qquad 2 \leq j \leq k+1, \quad (62)$$

where $\bar{\Phi}_{\bar{A}}^*|_j$ symbolizes only the antifields of agh $= j$ from (26). Consequently, we start from the class of solutions to the homogeneous equation $\gamma\bar{a}_{1,j}^{\mathrm{t-h}}|_j = 0$ written like

$$2 \leq j \leq k+1, \quad \bar{a}_{1,j}^{\mathrm{t-h}}|_j : \left\{\bar{\Phi}_{\bar{A}}^*|_j \rightleftharpoons \bar{e}^j\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}, \mathcal{F}^{\mathrm{h}}\right)\right\}, \qquad \mathrm{pgh}(\bar{e}^j) = j. \quad (63)$$

At this point we retain among the elements of the basis $\bar{e}^j$ only the polynomials of minimum order equal to two in the objects $\overset{(1)}{\mathcal{F}}$ and $\mathcal{F}^{\mathrm{h}}$ (both fermionic, of pgh $= 1$, and containing a single spacetime derivative, in agreement with definitions (30) and (28), respectively)

$$2 \leq j \leq k+1 : \bar{e}^j\left(\overset{(1)}{\mathcal{F}}, \mathcal{C}, \mathcal{F}^{\mathrm{h}}\right) = e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2)}(\mathcal{C}), \quad 2 \leq j_1 + j_2 \leq j, \quad (64)$$

$$\mathrm{pgh}\left(e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\right) = j_1, \quad \mathrm{pgh}\left(\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\right) = j_2, \quad \mathrm{pgh}(\bar{e}^{j-(j_1+j_2)}(\mathcal{C})) = j - (j_1 + j_2). \quad (65)$$

The elements of the basis $e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)$ are polynomials of order j_1 in $\overset{(1)}{\mathcal{F}}$, of the type

$$e^j\left(\overset{(1)}{\mathcal{F}}\right) \equiv \left(\overset{(1)}{\mathcal{F}}\right)^j = \overset{(1)}{\mathcal{F}}{}^{\mu_1^{(1)}\ldots\mu_{k+1}^{(1)}}\,\overset{(1)}{\mathcal{F}}{}^{\mu_1^{(2)}\ldots\mu_{k+1}^{(2)}}\cdots\overset{(1)}{\mathcal{F}}{}^{\mu_1^{(j)}\ldots\mu_{k+1}^{(j)}}, \quad j \geq 1 \quad (66)$$

with $j \to j_1$, whereas the elements $\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)$ are of the same form, but in terms of $\mathcal{F}^{\mathrm{h}}$

$$\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right) \equiv \left(\mathcal{F}^{\mathrm{h}}\right)^{j_2} = \mathcal{F}^{\mathrm{h}\mu_1^{(1)}\nu_1^{(1)}}\cdots\mathcal{F}^{\mathrm{h}\mu_1^{(j_2)}\nu_1^{(j_2)}}, \qquad j_2 \geq 1. \quad (67)$$

If we did not fix the upper bound of j to $(k+1)$, we could include in (64) also a dependence on $\overset{(k)}{\eta}$ via $\bar{e}^{j-(j_1+j_2)}$. Under the given conditions: $j \leq k+1$ and $j_1 + j_2 \geq 2$, we deduce that $j - (j_1 + j_2) \leq k - 1$, so the presence of the ghost $\overset{(k)}{\eta}$ (with pgh $= k$) is not allowed in $\bar{e}^{j-(j_1+j_2)}$ (there are no BRST generators of strictly negative values of the pure ghost number, in particular -1). Since the undifferentiated Pauli–Fierz ghost is of pgh $= 1$, the elements of the basis $\bar{e}^l(\mathcal{C})$ for any value $l \geq 1$ of the pure ghost number will be represented also like polynomials of order l in its components

$$\bar{e}^l(\mathcal{C}) \equiv (\mathcal{C})^l = \mathcal{C}^{\rho_1}\cdots\mathcal{C}^{\rho_l}, \qquad l \geq 1. \quad (68)$$

Until now we selected among the nontrivial solutions to the homogeneous equation in agh $= j$ with $2 \leq j \leq k+1$ (the first equation from (61)) only those simultaneously linear in the undifferentiated antifields and at least quadratic in the objects $\overset{(1)}{\mathcal{F}}$ and $\mathcal{F}^{\mathrm{h}}$

$$2 \leq j \leq k+1, \quad \bar{a}_{1,j}^{\mathrm{t-h}}|_j : \left\{\bar{\Phi}_{\bar{A}}^*|_j \rightleftharpoons e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2)}(\mathcal{C})\right\}, \quad 2 \leq j_1 + j_2 \leq j. \quad (69)$$

Combining the derivative order equal to one of the objects $\overset{(1)}{\mathcal{F}}$ and $\mathcal{F}^{\mathrm{h}}$ with the assumed independence of the spacetime derivatives of the antifields at the level of the considered class of invariant polynomials, we can state that the derivative order of $\bar{a}_{1,j}^{\mathrm{t-h}}|_j$ expressed by (69) is equal to j_1+j_2. Assume that (69) generates consistent cross-coupling first-order deformations, i.e. there exist (local) solutions to the remaining equations from the set (61). If we combine the fact that the action of the operator δ on all the antifields of agh > 1 from the $(k,1)$ sector contains a single derivative with the last relation from (22), then by applying δ on (69) one obtains a quantity proportional with the first-order derivatives of the antifields with agh $= j-1$ included within (26). Transferring the derivative to act on the elements of the ghost basis (by some integrations by parts), we infer three distinct classes of possible terms, all linear in the undifferentiated antifields of agh $= j-1$, where the derivative acts on the elements $e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)$, $\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)$, and $\bar{e}^{j-(j_1+j_2)}(\mathcal{C})$, respectively. From formula (66) with $j\rightarrow j_1$, (67) and (68) particularized to $l\rightarrow j-(j_1+j_2)$, we reach the conclusion that related to the first class the derivative will act on a single object $\overset{(1)}{\mathcal{F}}$ times the elements $e^{j_1-1}\left(\overset{(1)}{\mathcal{F}}\right)$, regarding the second class on a sole quantity $\mathcal{F}^{\mathrm{h}}$ multiplied with the elements $\bar{e}^{j_2-1}\left(\mathcal{F}^{\mathrm{h}}\right)$, whereas with respect to the third class on a single ghost $\mathcal{C}$ times the elements $\bar{e}^{j-(j_1+j_2+1)}(\mathcal{C})$ (of course, if either of j_1, j_2 or $j-(j_1+j_2)$ is vanishing, it is understood that the accompanying basis elements are absent, so no terms with the pure ghost number apparently equal to -1 emerge)

$$\delta\bar{a}_{1,j}^{\mathrm{t-h}}|_j:\left\{\bar{\Phi}_{\bar{A}}^{*}|_{j-1}\rightleftharpoons\left(\partial\overset{(1)}{\mathcal{F}}\right)e^{j_1-1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2)}(\mathcal{C}),\right.$$
$$\bar{\Phi}_{\bar{A}}^{*}|_{j-1}\rightleftharpoons\left(\partial\mathcal{F}^{\mathrm{h}}\right)e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2-1}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2)}(\mathcal{C}),$$
$$\left.\bar{\Phi}_{\bar{A}}^{*}|_{j-1}\rightleftharpoons(\partial\mathcal{C})e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2+1)}(\mathcal{C})\right\}. \tag{70}$$

The assumption of consistency of (59) in agh $= j-1$ together with results (31), (29), and the first relation from (21), to which we add the γ-invariance of $\overset{(1)}{\mathcal{F}}$, $\mathcal{F}^{\mathrm{h}}$, and $\mathcal{C}$, generates from (70) three classes of possible terms into the solution to the second equation from the chain (61)

$$\bar{a}_{1,j-1}^{\mathrm{t-h}}|_j:\left\{\bar{\Phi}_{\bar{A}}^{*}|_{j-1}\rightleftharpoons(\partial t)e^{j_1-1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2)}(\mathcal{C}),\right.$$
$$\bar{\Phi}_{\bar{A}}^{*}|_{j-1}\rightleftharpoons(\partial h)e^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2-1}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2)}(\mathcal{C}),$$
$$\left.\bar{\Phi}_{\bar{A}}^{*}|_{j-1}\rightleftharpoons he^{j_1}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^{j_2}\left(\mathcal{F}^{\mathrm{h}}\right)\bar{e}^{j-(j_1+j_2+1)}(\mathcal{C})\right\}. \tag{71}$$

Each class is homogeneous with respect to the total number of derivatives, which is equal precisely to (j_1+j_2) and coincides with that of the piece of maximum antighost number ($\bar{a}_{1,j}^{\mathrm{t-h}}|_j$ of the form (69)). The same reasoning can be applied without modifications up to the component of antighost number 1, $\bar{a}_{1,1}^{\mathrm{t-h}}|_j$, which will also conserve the starting total number of derivatives — (j_1+j_2). The passing from $\bar{a}_{1,1}^{\mathrm{t-h}}|_j$ to $\bar{a}_{1,0}^{\mathrm{t-h}}|_j$ via the last equation from (61) produces a variation equal to two with respect to the total number of derivatives. Indeed, the fact that the action of δ on the antifield t^* is linear in the second-order derivatives of the field t and a similar behavior at the level of the Pauli–Fierz sector (see the second relation from (22) correlated with expression (9)) indicates that the action of δ on $\bar{a}_{1,1}^{\mathrm{t-h}}|_j$ leads also to homogeneous terms with respect to the total number of derivatives, which is now equal to (j_1+j_2+2), that are meanwhile monomials of order one in the quantities $\overset{(1)}{\mathcal{F}}$, $\mathcal{F}^{\mathrm{h}}$, and $\mathcal{C}$. Only one of these additional derivatives is absorbed in the form of γ-exact terms via formulas (31), (29), and the first relation from (21), such that the presumption of consistency in agh $=0$ provokes various types of terms in $\bar{a}_{1,0}^{\mathrm{t-h}}|_j$, all containing one more derivative than the starting component of agh $=j$, namely (j_1+j_2+1). In this manner we showed that the assumption $j_1+j_2\geq 2$ in (69) generated at order one of perturbation theory a Lagrangian density $\bar{a}_{1,0}^{\mathrm{t-h}}|_j$ containing at least three derivatives, which breaks the derivative-order hypothesis. This argument was conducted in the most favorable scenario, where the invariant polynomial that enters the component of maximum antighost number from the cross-coupling first-order deformation is derivative-free, so the above conclusion is automatically valid if one takes into consideration a more general dependence of the invariant polynomial on the derivatives of the eligible antifields and/or the curvature tensors and/or their derivatives. These considerations are synthesized by the next proposition.

Proposition 2 *Assuming they are consistent at order one of perturbation theory, the nontrivial solutions to the homogeneous cross-coupling equations that are at least quadratic in the quantities* $\overset{(1)}{\mathcal{F}}$ *and* $\mathcal{F}^{\mathrm{h}}$ *break the derivative-order assumption at the level of the corresponding Lagrangian densities.*

The systematic application of the last two propositions allows for a complete analysis of the cross-coupling first-order deformation.

In order to generate the solutions $a_1^{\mathrm{t-h}}$ to the last equation from (46), we invoke the similar of Proposition 10 from Ref. [47] and develop this nonintegrated density and the associated current up to the maximally allowed values of the antighost number, equal to $(k+1)$ and k, respectively,

$$a_1^{\mathrm{t-h}}=\sum_{j=0}^{k+1}a_{1,j}^{\mathrm{t-h}},\qquad b_1^{\mathrm{t-h}\mu}=\sum_{j=0}^{k}b_{1,j}^{\mathrm{t-h}\mu},\tag{72}$$

such that equation $sa_1^{\mathrm{t-h}}+\partial_\mu b_1^{\mathrm{t-h}\mu}=0$ becomes equivalent to the chain

$$\gamma a_{1,k+1}^{\mathrm{t-h}}=0,\tag{73}$$

$$\delta a_{1,k+1}^{\mathrm{t-h}}+\gamma a_{1,k}^{\mathrm{t-h}}+\partial_\mu b_{1,k}^{\mathrm{t-h}\mu}=0,\tag{74}$$

$$\delta a_{1,k}^{\mathrm{t-h}}+\gamma a_{1,k-1}^{\mathrm{t-h}}+\partial_\mu b_{1,k-1}^{\mathrm{t-h}\mu}=0,\tag{75}$$

$$\vdots$$

$$\delta a_{1,1}^{\mathrm{t-h}}+\gamma a_{1,0}^{\mathrm{t-h}}+\partial_\mu b_{1,0}^{\mathrm{t-h}\mu}=0.\tag{76}$$

By means of formula (35) for $g=0$ and Table 3 in agh $=k+1$ we obtain the generic form of $a_{1,k+1}^{\mathrm{t-h}}$ like

$$a_{1,k+1}^{\mathrm{t-h}}:\left\{\overset{(k)}{\eta}{}^{*}\rightleftharpoons\bar{e}^{k+1}\left(\overset{(1)}{\mathcal{F}},\overset{(k)}{\eta},\mathcal{C},\mathcal{F}^{\mathrm{h}}\right)\right\},\quad \mathrm{pgh}(\bar{e}^{k+1})=k+1.\tag{77}$$

Since the invariant polynomial may be constructed only in terms of BRST generators from the $(k,1)$ sector, the enforcement of cross-couplings requires that the elements of the basis $\bar{e}^{k+1}$ explicitly depend on the Pauli–Fierz ghost combinations $\mathcal{C}$ and $\mathcal{F}^{\mathrm{h}}$. Proposition 1 eliminates the dependence on $\mathcal{C}$, which leaves us with

$$a_{1,k+1}^{\mathrm{t-h}} : \left\{ \overset{(k)}{\eta}{}^{*} \rightleftharpoons \bar{e}^{k+1}\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \underline{\mathcal{F}^{\mathrm{h}}} \right) \right\}. \tag{78}$$

According to the discussion prior to the statement of Proposition 2, the dependence on $\mathcal{F}^{\mathrm{h}}$ becomes linear, which further discards the dependence on $\overset{(1)}{\mathcal{F}}$ (since otherwise Proposition 2 induces that the derivative-order criterion is not fulfilled). Due to the fact that the pure ghost number of $\overset{(k)}{\eta}$ is equal to k, we arrive at the unique possibility that the elements of the ghost basis are linear in both $\overset{(k)}{\eta}$ and $\mathcal{F}^{\mathrm{h}}$

$$a_{1,k+1}^{\mathrm{t-h}} = \Upsilon_{\alpha||\beta||\mu\nu} \overset{(k)}{\eta}{}^{*\alpha} \overset{(k)}{\eta}{}^{\beta} \mathcal{F}^{\mathrm{h}\mu\nu}, \tag{79}$$

with Υ a non-derivative, constant real tensor, antisymmetric in its last two indices. By arguments of Lorentz covariance, Poincaré invariance, and taking into account the restrictions $D \geq k+2$ and $k \geq 4$, we find a unique solution

$$\Upsilon_{\alpha||\beta||\mu\nu} = c_1 (\sigma_{\alpha\mu}\sigma_{\beta\nu} - \sigma_{\alpha\nu}\sigma_{\beta\mu}), \qquad c_1 \in \mathbb{R}. \tag{80}$$

Consequently, the most general nontrivial expression of the solution to equation (73) that agrees with all the imposed selection rules takes the simple form

$$a_{1,k+1}^{\mathrm{t-h}} = c_1 \overset{(k)}{\eta}{}^{*[\mu} \overset{(k)}{\eta}{}^{\nu]} \mathcal{F}_{\mu\nu}^{\mathrm{h}}. \tag{81}$$

After some computation, we deduce the solution to equation (74) (disregarding the solutions to the homogeneous equation in $\mathrm{agh} = k$)

$$a_{1,k}^{\mathrm{t-h}} = c_1 \overset{(k-1)}{C'}{}^{*\rho||[\mu} \left(\overset{(k-1)}{C'}{}^{\nu]}{}_{\rho||} \mathcal{F}_{\mu\nu}^{\mathrm{h}} - 2 \overset{(k)}{\eta}{}^{\nu]} \partial_{[\mu} h_{\nu]|\rho} \right), \tag{82}$$

where the antifield $\overset{(k-1)}{C'}{}^{*}$ and the ghost $\overset{(k-1)}{C'}$ follow from relations (33) and (34) with $m = k-1$. Regarding the component of $\mathrm{agh} = k-1$ as solution to equation (75), we act with δ on (82) and infer

$$\begin{aligned} \delta a_{1,k}^{\mathrm{t-h}} = & -2c_1 \overset{(k-2)}{C'}{}^{*\lambda\rho||[\mu} \overset{(k)}{\eta}{}^{\nu]} K_{\lambda\rho|\mu\nu}^{\mathrm{h}} - \partial_\mu b_{1,k-1}^{\mathrm{t-h}\mu} \\ & - \gamma \left[c_1 \overset{(k-2)}{C'}{}^{*\lambda\rho||[\mu} \left(\overset{(k-2)}{C'}{}^{\nu]}{}_{\lambda\rho||} \mathcal{F}_{\mu\nu}^{\mathrm{h}} - \partial_{[\mu} h_{\nu]|[\lambda} \overset{(k-1)}{C'}{}^{\nu]}{}_{\rho]||} \right) \right], \end{aligned} \tag{83}$$

where $\overset{(k-2)}{C'}{}^{*}$ and $\overset{(k-2)}{C'}$ are of the form (33) and (34) with $m = k-2$ and K^{h} stands for the linearized Riemann tensor defined in (12). Substituting (83) in equation (75), we remark that is possesses solutions with respect to $a_{1,k-1}^{\mathrm{t-h}}$ if and only if the first term from the right-hand side of relation (83) may be written in a γ-exact form modulo a full divergence

$$2c_1 \overset{(k-2)}{C'}{}^{*\lambda\rho||[\mu} \overset{(k)}{\eta}{}^{\nu]} K_{\lambda\rho|\mu\nu}^{\mathrm{h}} = \gamma a_{1,k-1}^{\prime\mathrm{t-h}} + \partial_\mu b_{1,k-1}^{\prime\mathrm{t-h}\mu}. \tag{84}$$

By means of expression (33) with $m = k-2$ and of the first Bianchi identities (13), the term from the left-hand side of equation (84) following by contracting the completely antisymmetric components of the antifield $\overset{(k-2)^{*}}{\eta}$ with those of K^{h} are identically vanishing, such that (84) takes the equivalent form

$$4c_1 \overset{(k-2)^{*\lambda\rho|\mu}}{C} \overset{(k)^{\nu}}{\eta} K^{\mathrm{h}}_{\lambda\rho|\mu\nu} = \gamma a'^{\mathrm{t-h}}_{1,k-1} + \partial_\mu b'^{\mathrm{t-h}\mu}_{1,k-1}. \tag{85}$$

The double bar from $\overset{(k-2)^{*}}{C'}$ has been replaced now by a single one at the level of the antifield $\overset{(k-2)^{*}}{C}$ since the last is a truly (2,1) mixed-symmetry tensor. In order to analyze the last condition we act in a standard fashion, namely, we take the left Euler–Lagrange derivative with respect to the antifield $\overset{(k-2)^{*}}{C}$ and use the property that this operation commutes with the action of the operator γ, which further implies

$$4c_1 K^{\mathrm{h}}_{\lambda\rho|\mu\nu} \overset{(k)^{\nu}}{\eta} = \gamma \left(\frac{\delta^{\mathrm{L}} a'^{\mathrm{t-h}}_{1,k-1}}{\delta \overset{(k-2)^{*\lambda\rho|\mu}}{C}} \right). \tag{86}$$

The left-hand side of the above equation is a nontrivial element of agh $= 0$ pertaining to the cohomology space $H^k(\gamma)$, such that (86) takes place if and only if $c_1 = 0$, which automatically annihilates the piece of maximum antighost number of the form (81). In this manner we argued that the first-order deformation that couples the $(k,1)$ tensor to the Pauli–Fierz field cannot stop in agh $= k+1$.

The cases where the cross-coupling first-order deformation stops in maximum values of the antighost number $m \in \overline{3,k}$ will be investigated simultaneously. The analogue of Corollary 3 from Ref. [46] induces that for every fixed value of m within the chosen range, we can start from the expansions

$$a^{\mathrm{t-h}}_1 = \sum_{j=0}^{m} a^{\mathrm{t-h}}_{1,j}, \qquad b^{\mathrm{t-h}\mu}_1 = \sum_{j=0}^{m-1} b^{\mathrm{t-h}\mu}_{1,j}, \tag{87}$$

where the above components are subject to the equations

$$\gamma a^{\mathrm{t-h}}_{1,m} = 0, \qquad \cdots, \qquad \delta a^{\mathrm{t-h}}_{1,1} + \gamma a^{\mathrm{t-h}}_{1,0} + \partial_\mu b^{\mathrm{t-h}\mu}_{1,0} = 0. \tag{88}$$

Moreover, the similar of Proposition 10 from Ref. [47] provides the piece of maximum antighost number in the form

$$a^{\mathrm{t-h}}_{1,m} = \bar{\alpha}_m \bar{e}^m \left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}, \mathcal{F}^{\mathrm{h}} \right), \quad \bar{\alpha}_m \leftrightarrow H^{\mathrm{inv}D}_m, \quad \mathrm{pgh}(\bar{e}^m) = m. \tag{89}$$

Table 3 in agh $= m \in \overline{3,k}$ restricts the invariant polynomial $\bar{\alpha}_m$ to be linear in the undifferentiated antifields $\overset{(m-1)^{*}}{C'}$ from the $(k,1)$ sector, such that the cross-coupling demand binds the elements $\bar{e}^m$ to effectively depend on the Pauli–Fierz objects $\mathcal{C}$ and/or $\mathcal{F}^{\mathrm{h}}$. Proposition 1 forbids $\mathcal{C}$, such that $\bar{e}^m$ remains to involve $\mathcal{F}^{\mathrm{h}}$. Since the pure ghost number of this quantity is equal to one, we have to exclude the ghost $\overset{(k)}{\eta}$ (with pgh $= k$) from the elements $\bar{e}^m$ with $m = \overline{3,k}$, which leaves us with

$$\bar{e}^m \left(\overset{(1)}{\mathcal{F}}, \underline{\mathcal{F}^{\mathrm{h}}} \right) = \sum_{n=1}^{m} e^{m-n} \left(\overset{(1)}{\mathcal{F}} \right) \bar{e}^n \left(\mathcal{F}^{\mathrm{h}} \right), \qquad m = \overline{3,k}, \tag{90}$$

with $e^{m-n}\left(\overset{(1)}{\mathcal{F}}\right)$ like in (66) for $j \to m-n$ and $\bar{e}^n$ of the type (67) for $j_2 \to n$. Consequently, the solution to the homogeneous equation in agh $= m$ (the first from chain (88)) will contain precisely m classes of possible terms

$$m = \overline{3,k}, \qquad a_{1,m}^{\mathrm{t-h}} : \left\{ \overset{(m-1)^*}{C'} \rightleftharpoons e^{m-n}\left(\overset{(1)}{\mathcal{F}}\right)\bar{e}^n\left(\mathcal{F}^{\mathrm{h}}\right)\right\}_{n=\overline{1,m}}. \tag{91}$$

Since the overall basis (90) is a polynomial of order $m \geq 3$ in the objects $\overset{(1)}{\mathcal{F}}$ and $\mathcal{F}^{\mathrm{h}}$, Proposition 2 leads to the conclusion that, if consistent, the pieces from (91) would generate nonintegrated Lagrangian densities that cannot comply with the derivative-order assumption, so they must be discarded. In view of this, we can state that the cross-coupling first-order deformation cannot stop either in maximum antighost numbers ranging between 3 and k.

Regarding the next possible maximum value of the antighost number, namely 2, we apply the analogue of Corollary 3 from Ref. [46] and begin with the expansions

$$a_1^{\mathrm{t-h}} = a_{1,0}^{\mathrm{t-h}} + a_{1,1}^{\mathrm{t-h}} + a_{1,2}^{\mathrm{t-h}}, \qquad b_1^{\mathrm{t-h}\mu} = b_{1,0}^{\mathrm{t-h}\mu} + b_{1,1}^{\mathrm{t-h}\mu}, \tag{92}$$

and the corresponding equations

$$\gamma a_{1,2}^{\mathrm{t-h}} = 0, \qquad \delta a_{1,2}^{\mathrm{t-h}} + \gamma a_{1,1}^{\mathrm{t-h}} + \partial_\mu b_{1,1}^{\mathrm{t-h}\mu} = 0, \qquad \delta a_{1,1}^{\mathrm{t-h}} + \gamma a_{1,0}^{\mathrm{t-h}} + \partial_\mu b_{1,0}^{\mathrm{t-h}\mu} = 0. \tag{93}$$

Proposition 10 from Ref. [47] adapted to the model under study provides the solution to the (homogenous) equation in agh $= 2$ like

$$a_{1,2}^{\mathrm{t-h}} = \bar{\alpha}_2 \bar{e}^2\left(\overset{(1)}{\mathcal{F}}, \mathcal{C}, \mathcal{F}^{\mathrm{h}}\right), \quad \bar{\alpha}_2 \leftrightarrow H_2^{\mathrm{inv}D}, \quad \mathrm{pgh}(\bar{e}^2) = 2, \tag{94}$$

and Proposition 1 eliminates the dependence of $\bar{e}^2$ on $\mathcal{C}$, such that

$$a_{1,2}^{\mathrm{t-h}} = \bar{\alpha}_2 \bar{e}^2\left(\overset{(1)}{\mathcal{F}}, \mathcal{F}^{\mathrm{h}}\right), \quad \bar{\alpha}_2 \leftrightarrow H_2^{\mathrm{inv}D}, \quad \mathrm{pgh}(\bar{e}^2) = 2. \tag{95}$$

By means of Table 3 in agh $= 2$, we remark this is the first case where the Pauli–Fierz sector may contribute to the invariant polynomials, such that the cross-coupling requirement furnishes four classes of eligible terms in (95)

$$a_{1,2}^{\mathrm{t-h}} : \left\{ \overset{(1)^*}{C'} \rightleftharpoons \overset{(1)}{\mathcal{F}}\mathcal{F}^{\mathrm{h}}, \overset{(1)^*}{C'} \rightleftharpoons \bar{e}^2\left(\mathcal{F}^{\mathrm{h}}\right), \mathcal{C}^* \rightleftharpoons e^2\left(\overset{(1)}{\mathcal{F}}\right), \mathcal{C}^* \rightleftharpoons \overset{(1)}{\mathcal{F}}\mathcal{F}^{\mathrm{h}}\right\}. \tag{96}$$

With the help of formulas (66) for $j = 2$ and (67) for $j_2 = 2$, we notice that all the above elements are polynomials of order two in $\overset{(1)}{\mathcal{F}}$ and $\mathcal{F}^{\mathrm{h}}$, such that we can safely eliminate them by Proposition 2. In this way we obtain that the cross-coupling first-order deformation cannot stop either in antighost number 2.

Next, we analyze the situation where the maximum value of agh is equal to 1, where the similar of Corollary 3 from Ref. [46] still applies. Therefore, the starting point is given here by the decompositions and accompanying equations

$$a_1^{\mathrm{t-h}} = a_{1,0}^{\mathrm{t-h}} + a_{1,1}^{\mathrm{t-h}}, \qquad b_1^{\mathrm{t-h}\mu} = b_{1,0}^{\mathrm{t-h}\mu}, \tag{97}$$

$$\gamma a_{1,1}^{\mathrm{t-h}} = 0, \qquad \delta a_{1,1}^{\mathrm{t-h}} + \gamma a_{1,0}^{\mathrm{t-h}} + \partial_\mu b_{1,0}^{\mathrm{t-h}\mu} = 0. \tag{98}$$

The solutions to the homogeneous equation follow from formula (32) for $j = 1 = l$

$$a_{1,1}^{\mathrm{t-h}} = \bar{\alpha}_1([t^*],[h^*],[K],[K^{\mathrm{h}}])\bar{e}^1\left(\overset{(1)}{\mathcal{F}},\mathcal{C},\mathcal{F}^{\mathrm{h}}\right), \qquad \mathrm{agh}(\bar{\alpha}_1) = 1, \qquad \mathrm{pgh}(\bar{e}^1) = 1, \tag{99}$$

where the invariant polynomials $\bar{\alpha}_1$ are of order one in $[t^*]$ and $[h^*]$. By virtue of Proposition 1 we can safely eliminate the dependence on $\mathcal{C}$ and work with

$$\bar{e}^1\left(\overset{(1)}{\mathcal{F}},\mathcal{C},\mathcal{F}^{\mathrm{h}}\right) \to \bar{e}^1\left(\overset{(1)}{\mathcal{F}},\mathcal{F}^{\mathrm{h}}\right) = \left\{\overset{(1)}{\mathcal{F}},\mathcal{F}^{\mathrm{h}}\right\}. \tag{100}$$

Since both elements of the basis $\bar{e}^1$ already contain a spacetime derivative, in agreement with the discussion from the paragraph preceding Proposition 2 the derivative-order assumption requires that the invariant polynomials $\bar{\alpha}_1$ are linear in the undifferentiated eligible antifields and do not depend on the components of either the curvature tensor, the linearized Riemann tensor, or their spacetime derivatives. The cross-coupling hypothesis finally selects only two possible classes of terms in $a_{1,1}^{\mathrm{t-h}}$

$$a_{1,1}^{\mathrm{t-h}} : \left\{t^* \rightleftharpoons \mathcal{F}^{\mathrm{h}}, h^* \rightleftharpoons \overset{(1)}{\mathcal{F}}\right\}. \tag{101}$$

Equivalently, we can write

$$a_{1,1}^{\mathrm{t-h}} = \Upsilon_{\mu_1\ldots\mu_k||\alpha||\nu\rho}t^{*\mu_1\ldots\mu_k|\alpha}\mathcal{F}^{\mathrm{h}\nu\rho} + \Upsilon_{\mu||\nu||\rho_1\ldots\rho_{k+1}}h^{*\mu|\nu}\overset{(1)}{\mathcal{F}}{}^{\rho_1\ldots\rho_{k+1}}, \tag{102}$$

where the objects denoted by Υ represent some non-derivative, constant real tensors. Arguments of Lorentz covariance and Poincaré invariance on the spacetime manifold $\mathcal{M}$ of dimension $D \geq k+2$, with $k \geq 4$, lead to the solutions

$$\Upsilon_{\mu_1\ldots\mu_k||\alpha||\nu\rho} = c_2\varepsilon_{\mu_1\ldots\mu_k\alpha\nu\rho}, \qquad \Upsilon_{\mu||\nu||\mu_1\ldots\mu_{k+1}} = c_3\varepsilon_{\mu\nu\rho_1\ldots\rho_{k+1}}, \qquad c_2, c_3 \in \mathbb{R}, \tag{103}$$

which replaced in (102) annihilate the solution $a_{1,1}^{\mathrm{t-h}}$ respectively due to the mixed symmetry $(k,1)$ of t^*, $t^{*[\mu_1\ldots\mu_k|\alpha]} \equiv 0$ and to the symmetry of h^*, $h^{*\mu|\nu} = h^{*\nu|\mu}$

$$a_{1,1}^{\mathrm{t-h}} = 0. \tag{104}$$

As $a_{1,1}^{\mathrm{t-h}}$ contains all the information on the deformation of the generating set of gauge transformations, result (104) can be reformulated by the statement that in dimensions $D \geq k+2$ with $k \geq 4$ there are no consistent and nontrivial cross-couplings between a massless tensor field with the mixed symmetry $(k,1)$ and a Pauli–Fierz field in order one of perturbation theory that deform the gauge symmetries from the free limit.

We are now left with a single possibility, namely that the cross-coupling first-order deformation reduces to its component of antighost number 0, i.e., to the Lagrangian density at order one of perturbation theory

$$a_1^{\mathrm{t-h}} = a_{1,0}^{\mathrm{t-h}}([t],[h]), \qquad b_1^{\mathrm{t-h}\mu} = b_{1,0}^{\mathrm{t-h}\mu}. \tag{105}$$

The analogue of Corollary 3 from Ref. [46] is no longer valid, such that (105) is subject to the non-homogeneous equation

$$\gamma a_{1,0}^{\mathrm{t-h}}([t],[h]) + \partial_\mu b_{1,0}^{\mathrm{t-h}\mu} = 0. \tag{106}$$

We investigate separately the general solutions to the homogeneous equation ($b_{1,0}^{\mathrm{t-h}\mu} = 0$ in (106))

$$\gamma \bar{a}_{1,0}^{\mathrm{t-h}}([t],[h]) = 0, \tag{107}$$

which are given by invariant polynomials of agh $= 0$, so they follow from (27) where we discard the dependence on antifields and their derivatives

$$\bar{a}_{1,0}^{\mathrm{t-h}}([t],[h]) \equiv \bar{a}_{1,0}^{\mathrm{t-h}}([K],[K^{\mathrm{h}}]). \tag{108}$$

The cross-coupling requirement asks that $\bar{a}_{1,0}^{\mathrm{t-h}}$ is at least linear in both the components of the curvature tensor K and of the linearized Riemann tensor K^{h}, which is not acceptable since would lead to interaction vertices with at least four derivatives. In other words, we argued that there is no gauge-invariant cross-coupling Lagrangian density that agrees with all the imposed selection rules. The solutions to the non-homogeneous equation (106) ($b_{1,0}^{\mathrm{t-h}\mu} \neq 0$) are approached along a line similar to that exposed in Ref. [45] in relation with the couplings between a tensor with the mixed symmetry $(k,1)$ and a field with the mixed symmetry of the Riemann tensor. With the help of definition (20) and of the first formula in (21) we can show that equation (106) involves the following necessary conditions on the Euler–Lagrange derivatives of $a_{1,0}^{\mathrm{t-h}}$

$$\partial_{\mu_1}\left(\frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta t_{\mu_1\ldots\mu_k|\alpha}}\right) = 0, \quad \partial_{\alpha}\left(\frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta t_{\mu_1\ldots\mu_k|\alpha}}\right) = 0, \quad \partial_{\mu}\left(\frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta h_{\mu|\alpha}}\right) = 0. \tag{109}$$

Results (17) and (15) enable us to represent the solutions to the above equations like

$$\frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta t_{\mu_1\ldots\mu_k|\alpha}} = \partial_{\mu_{k+1}}\partial_\beta \tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}([t],[h]), \quad \frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta h_{\mu|\alpha}} = \partial_\nu\partial_\beta\bar{\Phi}^{\mu\nu|\alpha\beta}([t],[h]), \tag{110}$$

where the tensors $\tilde{\Phi}$ and $\bar{\Phi}$ exhibit the mixed symmetries $(k+1,2)$ and (2,2), respectively. (We set $c = 0$ in (15) since otherwise we would obtain a term linear in the trace of the Pauli–Fierz field, which obviously promotes no cross-coupling.) The derivative-order assumption forbids the dependence of both $\tilde{\Phi}$ and $\bar{\Phi}$ on any field derivative, while the cross-coupling requirement imposes that $\tilde{\Phi}$ effectively involves the Pauli–Fierz field and $\bar{\Phi}$ the mixed-symmetry $(k,1)$ tensor field. The last considerations are translated into

$$\frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta t_{\mu_1\ldots\mu_k|\alpha}} = \partial_{\mu_{k+1}}\partial_\beta \tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(t,\underline{h}), \quad \frac{\delta a_{1,0}^{\mathrm{t-h}}([t],[h])}{\delta h_{\mu|\alpha}} = \partial_\nu\partial_\beta\bar{\Phi}^{\mu\nu|\alpha\beta}(\underline{t},h). \tag{111}$$

From (111) we construct the Lagrangian density by the homotopy formula (where we omit the prospect divergences)

$$a_{1,0}^{\mathrm{t-h}}([t],[h]) = \int_0^1 d\tau \Big[\big(\partial_{\mu_{k+1}}\partial_\beta\tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(\tau t,\tau\underline{h})\big)t_{\mu_1\ldots\mu_k|\alpha} + \big(\partial_\nu\partial_\beta\bar{\Phi}^{\mu\nu|\alpha\beta}(\tau\underline{t},\tau h)\big)h_{\mu|\alpha}\Big]. \tag{112}$$

Integrating twice by parts in (112) and neglecting the resulting divergences, we have that

$$a_{1,0}^{\mathrm{t-h}}([t],[h]) = \int_0^1 d\tau \Big[\tfrac{(-)^{k+1}}{2(k+1)}\tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(\tau t,\tau\underline{h})K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}$$

$$+\tfrac{1}{4}\bar{\Phi}^{\mu\nu|\alpha\beta}(\tau\underline{t},\tau h)K^{\mathrm{t}}_{\mu\nu|\alpha\beta}\Big], \tag{113}$$

where K and K^t symbolize the curvature tensor and the linearized Riemann tensor, respectively. Applying again the operator γ on (113), after several steps we infer that the only nontrivial possibilities in $D = k + 2$ (with $k \geq 4$) are

$$\tilde{\Phi}^{\mu_1\cdots\mu_{k+1}|}{}_{\alpha\beta}(\tau t,\tau\underline{h}) = c_4\tau\big(\delta^{[\mu_1}_{\alpha}\delta^{\mu_2}_{\beta}t^{\mu_3\cdots\mu_{k+1}\rho]|}{}_{\rho}\big), \tag{114}$$

$$\bar{\Phi}^{\mu\nu|}{}_{\alpha\beta}(\tau\underline{t},\tau h) = c_5\tau\big(h^{[\mu|}{}_{\alpha}\delta^{\nu]}_{\beta} + h^{[\nu|}{}_{\beta}\delta^{\mu]}_{\alpha} - \delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}h\big), \tag{115}$$

with c_4 and c_5 two arbitrary real constants. Unfortunately, none of these variants satisfies the cross-coupling requirement (the right-hand side of (114) should explicitly depend on h and the right-hand side of (115) on t) so they must be annihilated by setting $c_4 = 0 = c_5$. Actually, the above solutions generate in (113) a linear combination of the two free Lagrangians, which provokes anyway a trivial element from $H^{0,D}(s|d)$ that may be cancelled. In conclusion, under the given working hypotheses there are no nontrivial cross-couplings at order one of perturbation theory between the fields t and h with the property of preserving the gauge symmetries from the free limit.

Assembling all the partial conclusions and results exposed so far we can state that one cannot construct any nontrivial and consistent first-order deformation responsible for the cross-couplings between a massless tensor field with the mixed symmetry $(k, 1)$ and a spin-2 field described in the free limit by the Pauli–Fierz action that agrees with the imposed selection rules. To put it otherwise, we are bound to implement $S_1^{\mathrm{t-h}} = 0$ in (43), which then yields

$$\bar{S} = \bar{S}^{\mathrm{t}} + \bar{S}^{\mathrm{h}}. \tag{116}$$

The Lagrangian formulation of (116) enables us to formulate the final conclusion of this section: the only terms that can be added to the free action (1) from the perspective of constructing consistent interactions are represented by the self-interactions of the Pauli–Fierz field, manifested through the Einstein–Hilbert action with a cosmologic term, invariant under diffeomorphisms, and the self-interactions of a single tensor field with the mixed symmetry $(k, 1)$, analyzed in Ref. [45].

5 Conclusion

The main result of this paper can be synthesized into the conclusion that there are no consistent and nontrivial couplings that can be added between a tensor field with the mixed symmetry $(k, 1)$ and a spin-2 field described in the free limit by the Pauli–Fierz model in the context of some standard selection rules from Quantum Field Theory completed by the requirement that the interactions vertices may contain at most two spacetime derivatives. It is possible that the relaxation of the derivative-order assumption in a rigorous, well-defined context may lead to different conclusions regarding the interactions among a collection of massless tensor fields with the mixed symmetry $(k, 1)$ and various graviton gauge theories, such as Einstein–Hilbert or Weyl.

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