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Consistent interactions between a massless tensor field with the mixed symmetry $(k, 1)$ and an Abelian vector field from local BRST cohomology

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Abstract

In this paper we analyze all consistent and nontrivial couplings that can be introduced between a massless tensor field with the mixed symmetry $(k, 1)$ for $k \geq 4$ and an Abelian vector field in the context of the antifield-BRST deformation method under some standard "selection rules" from Quantum Field Theory.

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1 Introduction

Tensor fields with mixed symmetry were brought into attention along with the raised interest in irreducible exotic representations of the group $GL(D, \mathbb{R})$. Among the many reasons for studying this class of fields it is worth mentioning that they are involved in several important physical theories such as superstrings, supergravity, or supersymmetric high-spin theories. The analysis of gauge theories containing bosonic tensor fields with mixed symmetries opened a variety of perspectives in both theoretical and mathematical physics, like for instance their correlation with high-spin gauge theories[1–5].

In this paper we target the class of massless real tensor fields transforming according to the irreducible representations of $GL(D, \mathbb{R})$ corresponding to some "spin-2" (two-column) Young diagrams with $(k + 1)$ cells and $k \geq 4$ rows (the so-called "hook diagrams"), also known as fields with the mixed symmetry $(k, 1)$. For arbitrary values of k, such tensor fields (massless and massive) have initially been investigated more than two decades ago [6–10] and more recently for instance in [11–13] due to their presence in the bosonic sector of Chern–Simons-like supergravity in odd dimensions following from the fact that their free action gives one of the dual formulations of linearized gravity in $D = k+3$ dimensions. The construction of gravity-like dual theories benefits of a raised interest in the context of some new results, like the extension to higher dimensions of a formulation of linearized gravity in 4 spacetime dimensions that is manifestly invariant under "duality rotations" in the space spanned by a graviton and its dual [14].

The main aim of this work is to generate all consistent and nontrivial couplings between a massless tensor field with the mixed symmetry $(k, 1)$ $(D \geq k + 2,\ k \geq 4)$ and a spin-1

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gauge field. Regarding the spin-1 gauge field, we start from the formulation based on a massless vector field (Maxwell theory in the absence of sources) with an Abelian $U(1)$ gauge symmetry. We rely on one of the main applications of the antifield-BRST formalism [15–18] regarding the construction of consistent interactions in gauge field theories [19–22] by means of deforming the solution to the classical master equation. This setting necessitates the computation of the local cohomology of the BRST differential in ghost number 0 and in maximum form degree [23–26]. The entire analysis is done in the presence of the following working hypotheses imposed on the deformations: analyticity in the coupling constant, spacetime locality, Lorentz covariance, Poincaré invariance, and conservation of the differential order of the free field equations at the level of the coupled theories. The last hypothesis is strengthened by asking that the interacting vertices display the maximum derivative order of the free Lagrangian density at any order in the coupling constant, namely two. The results reported here complement and extend various developments [27–46].

The main results deduced under the above mentioned working hypotheses may be synthesized into:

1. The nontrivial first-order deformation responsible by the cross-couplings between the spin-1 gauge vector field and the $(k, 1)$ tensor with $k \geq 4$ is nonvanishing only in antighost number 0 and 1 and is defined on a Minkowski spacetime of minimum dimension, $D = k + 2$, where the tensor $(k, 1)$ has no physical degrees of freedom. The component of antighost number equal to 1 is simultaneously linear in the undifferentiated antifield of the vector field and in the first-order derivatives of the antisymmetric ghost of pure ghost number equal to 1 from the $(k, 1)$ sector. These two quantities are assembled via the $(k + 2)$-dimensional Levi-Civita symbol. The piece of antighost number 0, identified as the Lagrangian density at order one of perturbation theory, is simultaneously linear in the (Abelian) field strength of the vector field and in some combinations of the first-order derivatives of the field with the mixed symmetry $(k, 1)$;

2. Due to the fact that in $D = k + 2$ with $k \geq 4$ there are no self-interactions of the tensor $(k, 1)$ and the gauge vector field displays no self-interactions that comply with the derivative order assumption, it follows that the overall first-order deformation reduces to the cross-coupling component;

3. The second-order deformation is nonvanishing and contains only terms of antighost number 0 that are quadratic in the first-order derivatives of the tensor $(k, 1)$ (but nontrivial in the local BRST cohomology), while all the higher-order deformations can be taken to vanish;

4. Consequently, the Lagrangian action of the interacting model includes only contributions of order one and respectively two in the coupling constant, expressed by "mixing-component" terms, which, moreover, break the PT invariance. These pieces may be organized, together with the original Lagrangian density of the gauge vector field, into a nonintegrated density that is quadratic into a deformed field strength of the of the 1 field. This deformed field strength coincides with the original, Abelian field strength of the vector field in the free limit and includes a neat contribution in order one of perturbation theory linear in the first-order derivatives of the tensor $(k, 1)$;

5. The generating set of infinitesimal gauge transformations of the interacting action gets deformed only at the level of the vector field, in order one of perturbation theory, by a term linear in the first-order derivatives of the antisymmetric gauge parameters from the $(k,1)$ sector. It is interesting to mention that the deformed field strength of the spin-1 field is invariant under the deformed gauge symmetries;

6. The other gauge features of the coupled model are not affected by the deformation procedure (the associated gauge algebra remains Abelian and the reducibility functions/relations coincide with the initial ones, from the free limit).

2 Starting free model: Lagrangian formulation and BRST symmetry

The starting point is given by the Lagrangian action describing a free massless tensor field with the mixed symmetry $(k,1)$ for $k \geq 4$ and an Abelian vector field

$$S_0[t_{\mu_1\ldots\mu_k|\alpha}, V_\mu] = S_0^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}] + S_0^{\mathrm{V}}[V_\mu]. \tag{1}$$

We work on a Minkowski spacetime $\mathcal{M}$ of dimension $D \geq k+2 \geq 6$ endowed with a mostly positive metric $\sigma_{\mu\nu} = \sigma^{\mu\nu} = (-+\ldots+)$ and define the Levi-Civita symbol in D dimensions $\varepsilon^{\mu_1\ldots\mu_D}$ by $\varepsilon^{01\ldots D-1} = -1$. The field $t_{\mu_1\ldots\mu_k|\alpha}$ is antisymmetric in its first k (Lorentz) indices and satisfies the identities $t_{[\mu_1\ldots\mu_k|\alpha]} \equiv 0$, while its trace, $t_{\mu_1\ldots\mu_{k-1}} = t_{\mu_1\ldots\mu_k|\alpha}\sigma^{\mu_k\alpha}$, is a completely antisymmetric tensor of order $(k-1)$. The components S_0^{t} and S_0^{V} read as

$$S_0^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}] = -\tfrac{1}{2\cdot(k+1)!}\int\left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha} - (k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right]d^Dx, \tag{2}$$

$$S_0^{\mathrm{V}}[V_\mu] = -\tfrac{1}{4}\int F_{\mu\nu}^{\mathrm{V}}F^{\mathrm{V}\mu\nu}d^Dx, \qquad F_{\mu\nu}^{\mathrm{V}} \equiv \partial_{[\mu}V_{\nu]}. \tag{3}$$

In the above

$$F_{\mu_1\ldots\mu_{k+1}|\alpha} = \partial_{[\mu_1}t_{\mu_2\ldots\mu_{k+1}]|\alpha}, \tag{4}$$

$$F_{\mu_1\ldots\mu_k} \equiv F_{\mu_1\ldots\mu_{k+1}|\alpha}\sigma^{\mu_{k+1}\alpha} = \partial_{[\mu_1}t_{\mu_2\ldots\mu_k]} + (-)^k\partial^\alpha t_{\mu_1\ldots\mu_k|\alpha}, \tag{5}$$

so the tensor $F_{\mu_1\ldots\mu_{k+1}|\alpha}$ displays the mixed symmetry $(k+1,1)$ and its trace, $F_{\mu_1\ldots\mu_k}$, is completely antisymmetric. Everywhere in this paper the notation $[\mu\ldots\nu]$ signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The stationary surface of this free model is defined by the field equations

$$\frac{\delta S_0^{\mathrm{t}}}{\delta t_{\nu_1\ldots\nu_k|\alpha}} \equiv \tfrac{1}{k!}T^{\nu_1\ldots\nu_k|\alpha} \approx 0, \tag{6}$$

$$\frac{\delta S_0^{\mathrm{V}}}{\delta V_\mu} \equiv \partial_\nu F^{\mathrm{V}\nu\mu} \approx 0, \tag{7}$$

where the tensor $T^{\nu_1\ldots\nu_k|\alpha}$ exhibits the mixed symmetry $(k,1)$ and reads as

$$T^{\nu_1\ldots\nu_k|\alpha} = \partial_\mu F^{\mu\nu_1\ldots\nu_k|\alpha} - \sigma^{\alpha[\nu_1}\partial_\mu F^{\nu_2\ldots\nu_k\mu]}. \tag{8}$$

A generating set of infinitesimal gauge transformations of action (1) is given by the set corresponding to the $(k,1)$ sector

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} t_{\mu_1\ldots\mu_k|\alpha} = \partial_{[\mu_1} \overset{(1)}{\theta}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1} \overset{(1)}{\epsilon}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\partial_\alpha \overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}, \tag{9}$$

supplemented by the local $U(1)$ symmetry for the vector field

$$\delta_\xi V_\mu = \partial_\mu \xi. \tag{10}$$

The gauge parameters from the $(k,1)$ sector are some real, arbitrary tensors on the spacetime manifold $\mathcal{M}$ up to the requirements that $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}$ possesses the mixed symmetry $(k-1,1)$ and $\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}$ is completely antisymmetric. The gauge parameter from the vector sector, ξ, is an arbitrary real scalar function on $\mathcal{M}$. The overall theory, governed by action (1), inherits all the features of the free massless tensor field with the mixed symmetry $(k,1)$: the gauge algebra is Abelian, the above generating set is off-shell reducible of order $(k-1)$, and the total Cauchy order of this linear gauge theory is equal to $(k+1)$. The vector field component is administered by the Lagrangian action (3), invariant under the $U(1)$ gauge symmetry, which is Abelian and irreducible, such that the "photonic" sector (by abuse, we maintain the standard terminology from electrodynamics also in $D>4$) is separately described by a linear gauge theory with the Cauchy order equal to 2. The general gauge-invariant quantities of this free model are polynomials in the curvature tensor from the $(k,1)$ sector

$$K_{\mu_1\ldots\mu_{k+1}|\alpha\beta} = \partial_\alpha F_{\mu_1\ldots\mu_{k+1}|\beta} - F_{\mu_1\ldots\mu_{k+1}|\alpha} \equiv \partial_{[\mu_1} t_{\mu_2\ldots\mu_{k+1}]|[\beta,\alpha]}, \tag{11}$$

in the Abelian field strength of the vector field introduced in the latter formula from (3), as well as in their spacetime derivatives.

Next, we pass to the construction of the antifield-BRST symmetry for the theory under study. Regarding the $(k,1)$ sector, we maintain all the notations, conventions, formulas, and results from [39, 45, 45]. Consequently, the BRST differential algebra is constructed starting with the generators from the $(k,1)$ sector

$$\Phi^A \equiv \left\{ t_{\mu_1\ldots\mu_k|\alpha}, \left\{ \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} \right\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}_\mu \right\}, \tag{12}$$

$$\Phi^*_A \equiv \left\{ t^{*\mu_1\ldots\mu_k|\alpha}, \left\{ \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} \right\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}{}^{*\mu} \right\}, \tag{13}$$

whose properties are detailed in [39, 45] (a synthetic view is given in Table 1 from [45]), supplemented in the $U(1)$ sector by the vector field V_μ, the ghost $\mathcal{C}$ corresponding to the gauge parameter ξ, and their antifields. Their properties are listed in Table 1. The BRST differential simply decomposes like

$$s = \delta + \gamma, \quad s^2 = 0 \Leftrightarrow (\delta^2 = 0,\, \gamma^2 = 0,\, \delta\gamma + \gamma\delta = 0) \tag{14}$$

into the sum between the Koszul–Tate differential δ ($\mathbb{N}$-graded in terms of the antighost number agh, $\mathrm{agh}(\delta) = -1$) and the longitudinal exterior derivative γ (a true differential in this case, which anticommutes with δ and is $\mathbb{N}$-graded along the pure ghost number pgh, $\mathrm{pgh}(\gamma) = 1$). The BRST differential is $\mathbb{Z}$-graded in terms of the ghost number gh defined like $\mathrm{pgh} - \mathrm{agh}$, such that $\mathrm{gh}(s) = \mathrm{gh}(\delta) = \mathrm{gh}(\gamma) = 1$. The actions of the operators

BRST generator	pgh	agh	gh	ε
V_μ	0	0	0	0
$\mathcal{C}$	1	0	1	1
$V^{*\mu}$	0	1	-1	1
$\mathcal{C}^*$	0	2	-2	0

Table 1: Degrees of the BRST generators from the vector sector.

δ and γ on the BRST generators from the $(k,1)$ sector are given for instance in [45] (see formulas (15)–(23) therein), while on those from the $U(1)$ sector read as

$$\gamma V_\mu = \partial_\mu \mathcal{C}, \qquad \gamma \mathcal{C} = 0, \qquad \gamma V^{*\mu} = 0 = \gamma \mathcal{C}^*, \tag{15}$$

$$\delta V_\mu = 0 = \delta \mathcal{C}, \qquad \delta V^{*\mu} = -\partial_\nu F^{\mathrm{V}\nu\mu}, \qquad \delta \mathcal{C}^* = -\partial_\mu V^{*\mu}. \tag{16}$$

The solution to the classical master equation reduces to the sum between that associated with the $(k,1)$ sector and the one corresponding to the Abelian gauge field

$$S = S^{\mathrm{t}} + S^{\mathrm{V}}, \tag{17}$$

$$\begin{aligned} S^{\mathrm{t}} = & S_0^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}] + \int \Big\{ t^{*\mu_1\ldots\mu_k|\alpha} \Big[\partial_{[\mu_1} \overset{(1)}{C}{}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1} \overset{(1)}{\eta}{}_{\mu_2\ldots\mu_k\alpha]} \\ & + (-)^{k+1}(k+1)\partial_\alpha \overset{(1)}{\eta}{}_{\mu_1\ldots\mu_k} \Big] + \overset{(k-1)}{C}{}^{*\mu_1|\alpha} \partial_{(\mu_1} \overset{(k)}{\eta}{}_{\alpha)} \\ & + \sum_{m=1}^{k-2} \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha} \Big[\partial_{[\mu_1} \overset{(m+1)}{C}{}_{\mu_2\ldots\mu_{k-m}]|\alpha} + \partial_{[\mu_1} \overset{(m+1)}{\eta}{}_{\mu_2\ldots\mu_{k-m}\alpha]} \\ & + (-)^{k-m+1}(k-m+1)\partial_\alpha \overset{(m+1)}{\eta}{}_{\mu_1\ldots\mu_{k-m}} \Big] \\ & + \sum_{m=1}^{k-1} \tfrac{k-m}{k-m+2} \overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} \partial_{[\mu_1} \overset{(m+1)}{\eta}{}_{\mu_2\ldots\mu_{k-m+1}]} \Big\} d^D x, \end{aligned} \tag{18}$$

$$S^{\mathrm{V}} = S_0^{\mathrm{V}}[V_\mu] + \int (V^{*\mu}\partial_\mu \mathcal{C}) d^D x. \tag{19}$$

It is useful to write all the BRST generators in a compact manner via the notations

$$\bar{\Phi}^{\bar{A}} = \{\Phi^A, V_\mu, \mathcal{C}\}, \qquad \bar{\Phi}^*_{\bar{A}} = \{\Phi^*_A, V^{*\mu}, \mathcal{C}^*\}, \tag{20}$$

where Φ^A and Φ^*_A are like in (12) and respectively (13).

3 Antifield-BRST deformation method

The reformulation of the problem of constructing consistent interactions in gauge field theories within the antifield-BRST formalism [19–22] is based on the fact that if consistent couplings can be introduced, then the solution to the classical master equation of the initial gauge theory, S, may be deformed into a solution to the classical master equation for the interacting gauge theory

$$\bar{S} = S + \lambda S_1 + \lambda^2 S_2 + \lambda^3 S_3 + \cdots, \quad \tfrac{1}{2}(\bar{S},\bar{S}) = 0. \tag{21}$$

Related to the coupled theory, we maintain the field, ghost, and antifield spectra of the original gauge theory in order to preserve the number of physical degrees of freedom. Also, we do not deform either the antibracket or the general properties $\bar{S}$ compared to those of the starting theory, but only the canonical generator itself, so $\bar{S}$ remains a bosonic functional of fields, ghosts, and antifields with the ghost number equal to 0. The projection of equation $\frac{1}{2}(\bar{S},\bar{S})=0$ on the various powers in the coupling constant λ is equivalent to the tower of equations

$$\lambda^0:\tfrac{1}{2}(S,S)=0,\lambda^1:(S_1,S)=0,\lambda^2:(S_2,S)+\tfrac{1}{2}(S_1,S_1)=0,\lambda^3:(S_3,S)+(S_1,S_2)=0,\cdots$$

known as the equation of the antifield-BRST deformation method. In this context the functionals S_i, $i\geq 1$, are called deformations of order i of the solution to the master equation. The first equation is fulfilled by assumption, while the others may be written (due to the canonical action $s\cdot=(\cdot,S)$) as

$$\lambda^1:sS_1=0,\quad \lambda^2:sS_2+\tfrac{1}{2}(S_1,S_1)=0,\quad \lambda^3:sS_3+(S_1,S_2)=0,\quad \cdots \tag{22}$$

The solutions to the first-order deformation equation $sS_1=0$ always exist since they belong to the cohomology of the BRST differential s in ghost number 0 computed in the space of all functionals (local and nonlocal) of fields, ghosts, and antifields, $H^0(s)$, which is nonempty due to its isomorphism to the algebra of physical observables of the initial gauge theory. Moreover, trivial first-order deformations, defined as trivial elements of $H^0(s)$ (s-exact functionals), should be ruled out due to the fact that they provoke trivial interactions in the sense of field theory (can be eliminated by some possibly nonlinear field redefinitions). The existence of solutions to the remaining higher-order equations from (22) has been shown in [20] by means of the triviality of the antibracket map in the BRST cohomology $H(s)$ computed in the space of all functionals. In conclusion, if we impose no restrictions on the interactions (spacetime locality, etc.), then the antifield-BRST deformation procedure can be developed without obstructions.

Nevertheless, if we work with local functionals, then the procedure goes as follows. We make the notation

$$S_1=\int a_1 d^D x\equiv\int \overset{[D]}{a_1}, \tag{23}$$

where the nonintegrated density of the first-order deformation, a_1, is now an element of the BRST algebra of local "functions", namely, it is polynomial in the ghosts, antifields, and their derivatives, smooth in the original fields, and polynomial in their derivatives up to a finite order, with or respectively without an explicit dependence on the spacetime coordinates x^μ. The overscript between brackets represents the form degree deg. (If we require the Poincaré invariance of the deformed solution to the master equation, then we work without an explicit dependence on x^μ.) In form language, $\overset{[D]}{a_1}$ is an element of the algebra of local forms with or without an explicit dependence on x^μ. The general properties of S_1 are transferred to a_1 and $\overset{[D]}{a_1}$

$$\varepsilon(a_1)=0,\quad \mathrm{gh}(a_1)=0,\quad \deg(\overset{[D]}{a_1})=D,\quad \mathrm{gh}(\overset{[D]}{a_1})=0. \tag{24}$$

The equation satisfied by the first-order deformation (the first equation from (22)) takes the local form

$$s\overset{[D]}{a_1}+d\overset{[D-1]}{b_1}=0,\quad \deg(\overset{[D-1]}{b_1})=D-1,\quad \mathrm{gh}(\overset{[D-1]}{b_1})=1, \tag{25}$$

or, equivalently, in dual language

$$sa_1 + \partial_\mu b_1^\mu = 0, \quad \varepsilon(b_1^\mu) = 1, \quad \mathrm{gh}(b_1^\mu) = 1, \tag{26}$$

where the $(D-1)$ form $\overset{[D-1]}{b_1} = 0$ and the current b_1^μ should be local. In other words, the first-order deformation defines precisely a class from the local BRST cohomology in maximum form degree and in ghost number equal to zero, $H^{0,D}(s|d)$, computed in the algebra of local forms with or without an explicit dependence on x^μ, where d symbolizes the spacetime exterior differential. From now on, the procedure is model-dependent via the properties of $H^{0,D}(s|d)$. Supposing equation (25) (or (26)) possesses local solutions, the resulting first-order deformations are then filtered (if necessary) according to the "selection rules" associated with other working hypotheses than the spacetime locality (such as Lorentz covariance, PT invariance, maximum derivative order of the interaction vertices, etc.). Meanwhile, all purely trivial contributions from $H^{0,D}(s|d)$ computed in the selected algebra of local forms

$$\overset{[D]}{a_1^{\mathrm{triv}}} = s\overset{[D]}{c} + d\overset{[D-1]}{e}, \tag{27}$$

$$\deg\left(\overset{[D]}{c}\right) = D, \quad \deg\left(\overset{[D-1]}{e}\right) = D-1, \quad \mathrm{gh}\left(\overset{[D]}{c}\right) = -1, \quad \mathrm{gh}\left(\overset{[D-1]}{e}\right) = 0, \tag{28}$$

should be discarded since they generate only trivial interactions. By trivial first-order deformations in the context of equation (26) we understand any s-exact object modulo a divergence

$$a_1^{\mathrm{triv}} = sc + \partial_\mu e^\mu, \tag{29}$$

$$\varepsilon(c) = 1, \quad \varepsilon(e^\mu) = 0, \quad \mathrm{gh}(c) = -1, \quad \mathrm{gh}(e^\mu) = 0, \tag{30}$$

with both c and e^μ local.

4 Analysis of the local BRST cohomoology

The main aim of this paper is to construct all the nontrivial and consistent interactions that can be added to the free model (1) by means of the antifield-BRST deformation method synthesized in the previous section. We require that the deformation of the solution to the master equation, (21), is analytical in the coupling constant, local in spacetime, Lorentz covariant, Poincaré invariant, and conserves the differential order of the free field equations at the level of the coupled theories. The last hypothesis is strengthened by asking that the interacting vertices display the maximum derivative order of the free Lagrangian density at any order in the coupling constant, namely two in this case. Due to the locality hypothesis, we introduce notation (23) and obtain in dual language that the nonintegrated density of the first-order deformation, a_1, is solution to equation (26), and thus, as argued in the previous section, should be a nontrivial element of the local BRST cohomology $H^{0,D}(s|d)$. The last cohomology will be computed in the BRST algebra of local forms $\bar{\Lambda}$ whose coefficients are elements of the BRST algebra of local "functions" $\bar{\mathcal{A}}$, namely polynomials in the ghosts, antifields, and their spacetime derivatives up to a finite order, smooth in the undifferentiated fields t and respectively V, and again polynomials in the field derivatives up to a finite order, and without an explicit dependence on the spacetime coordinates (due to the Poincaré invariance). Most of the BRST cohomological results exposed in Refs. [45, 46] related to the case a single massless tensor field with the mixed symmetry $(k,1)$ remain valid here up to the proper inclusion of the $U(1)$ sector. Regarding the local BRST cohomology of the $U(1)$ gauge field, we employ the general results for Abelian p-forms from Refs. [47–49] adapted to $p=1$. In this context it is interesting to mention the Hamiltonian approach to the construction of Stückelberg-coupled p- and $(p+1)$-forms [50].

Related to the cohomology of the longitudinal exterior differential $H(\gamma)$ and of its local version, $H(\gamma|d)$, both computed in $\bar{\mathcal{A}}$, all the results from Ref. [45] still hold up to the following observations. First, the vector sector is manifested at the level of the cohomology of the longitudinal exterior differential in pure ghost number 0 computed in $\bar{\mathcal{A}}$, $H^0(\gamma)$, also known as the algebra of invariant polynomials, by the additional (compared to the purely $(k,1)$ case analyzed in Ref. [45]) polynomial dependence on the Abelian field strength F^{V}, on the antifields V^*, $\mathcal{C}^*$, as well as on their spacetime derivatives up to a finite order

$$H^0(\gamma) \text{ in } \bar{\mathcal{A}} = \{\text{algebra of invariant polynomials}\} \equiv \left\{\bar{\alpha}\left(\left[\bar{\Phi}^*_{\bar{A}}\right],[K],\left[F^{\mathrm{V}}\right]\right)\right\}, \tag{31}$$

where K represent the components (11) of the curvature tensor, $\bar{\Phi}^*_{\bar{A}}$ is a collective notation for all the antifields (see (20)), and by $f([y])$ we mean that f depends on y and its derivatives up to a finite order. A direct consequence of this result is that the elements of the cohomology $H^0(\gamma)$ computed in the algebra of local forms $\bar{\Lambda}$ are nothing but

$$H^0(\gamma) = \bigoplus_{p=0}^{D} H^{0,p}(\gamma), \quad H^{0,p}(\gamma) \ni \overset{[p]}{\bar{\alpha}} = \tfrac{1}{p!}\bar{\alpha}_{\mu_1\ldots\mu_p}\left(\left[\bar{\Phi}^*_{\bar{A}}\right],[K],\left[F^{\mathrm{V}}\right]\right)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}. \tag{32}$$

Second, in pgh >0 the contribution of the "Maxwell" sector resides in the undifferentiated fermionic ghost $\mathcal{C}$ associated with the $U(1)$ gauge symmetries (its derivatives are γ-exact, in agreement with the first formula from (15)), while that of the $(k,1)$ sector may be found in Table 2 from Ref. [45]. Obviously, the fermionic behaviour of the ghost $\mathcal{C}$ introduces a dependence that is at most linear ($\mathcal{C}^2=0$). Under these considerations, the nontrivial representatives of the cohomology $H(\gamma)$ computed in the algebra $\bar{\mathcal{A}}$ (without an explicit dependence on x^μ) are synthesized in Table 2, where

$$\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}} \equiv \partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_{k+1}]}, \quad \varepsilon\left(\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\right)=1, \quad \mathrm{pgh}\left(\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\right)=1. \tag{33}$$

BRST generator	Nontrivial representatives	pgh
$[t_{\mu_1\ldots\mu_k\vert\alpha}]$	$[K_{\mu_1\ldots\mu_{k+1}\vert\alpha\beta}]$	0
$[V_\mu]$	$\left[F^{\mathrm{V}}_{\mu\nu}\right]$	0
$\left[\bar{\Phi}^*_{\bar{A}}\right]$	$\left[\bar{\Phi}^*_{\bar{A}}\right]$	0
$\left[\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}\right], \left[\overset{(1)}{C}_{\mu_1\ldots\mu_{k-1}\vert\alpha}\right]$	$\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}$	1
$\left[\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}}\right], \left[\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha}\right]$	—	$m,\ m=\overline{2,k-1}$
$\left[\overset{(k)}{\eta}_{\alpha}\right]$	$\overset{(k)}{\eta}_{\alpha}$	k
$[\mathcal{C}]$	$\mathcal{C}$	1

Table 2: Nontrivial representatives of the cohomology $H(\gamma)$ computed in the algebra $\bar{\mathcal{A}}$.

With the help of the above table we identify the general, nontrivial elements of $H(\gamma)$ evaluated in $\bar{\mathcal{A}}$ with the properties

$$\gamma a = 0, \quad a \in \mathcal{A}, \quad \mathrm{pgh}(a) = l \geq 0, \quad \mathrm{agh}(a) = j \geq 0, \tag{34}$$

under the form

$$a = \sum_J \bar{\alpha}_J\big([\bar{\Phi}^*_{\bar{A}}], [K], [F^{\mathrm{V}}]\big)\bar{e}^J\Big(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}\Big), \quad \mathrm{agh}(\bar{\alpha}_J) = j \geq 0, \quad \mathrm{pgh}(\bar{e}^J) = l \geq 0, \tag{35}$$

and also the general, nontrivial elements of $H(\gamma)$ computed in $\bar{\Lambda}$ with the properties

$$\gamma\varpi = 0, \quad \varpi \in \Lambda, \quad \deg(\varpi) = p \leq D, \quad \mathrm{pgh}(\varpi) = l \geq 0, \quad \mathrm{agh}(\varpi) = j \geq 0, \tag{36}$$

like

$$\varpi = \sum_J \overset{[p]}{\bar{\alpha}}_J\big([\bar{\Phi}^*_{\bar{A}}], [K], [F^{\mathrm{V}}]\big)\bar{e}^J\Big(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}\Big), \tag{37}$$

$$\deg(\overset{[p]}{\bar{\alpha}}_J) = p \leq D, \quad \mathrm{agh}(\overset{[p]}{\bar{\alpha}}_J) = j \geq 0, \quad \mathrm{pgh}(\bar{e}^J) = l \geq 0. \tag{38}$$

We notice that the elements of the basis in the ghosts, $\bar{e}^J$, is modified with respect to that of the $(k,1)$ case alone by including a novel dependence (at most linear) of the undifferentiated $U(1)$ ghost $\mathcal{C}$. Third, the analogue of Corollary 3 from [45] remains valid in the case where one adds a free massless vector field to the free theory describing a massless tensor field with the mixed symmetry $(k,1)$ and replaces the algebra Λ by $\bar{\Lambda}$.

Regarding the local cohomologies of the Koszul–Tate differential in gh $= 0$, $H(\delta|d)$ and $H^{\mathrm{inv}}(\delta|d)$, the relevant results from Ref. [46], contained in Corollary 5, Lemma 6, Theorem 7, and Corollary 8, still hold up to including the presence of the supplementary free spin-1 field. The nontrivial representatives that span the invariant characteristic cohomology spaces with agh valued between 3 and $(k+1)$, $\big(H_j^{\mathrm{inv}D}(\delta|d)\big)_{j=\overline{3,k+1}}$, gain no new elements compared with those from Ref. [46], but in agh $= 2$ there appears an additional contribution due to the vector sector, represented by the undifferentiated antifield $\mathcal{C}^*$, such that Table 1 from [46] should be replaced with Table 3 below. It is understood that in agh $= 1$ the (nontrivial) representatives of the space $H_1^D(\delta|d)$ will include also (linear) contributions in the antifield of the vector field, $[V^*]$.

agh	complete set of nontrivial representatives
$k+1$	$\overset{(k)}{\eta}{}^{*\alpha}$
$j = \overline{3,k}$	$\overset{(j-1)}{C'}{}^{*\mu_1\ldots\mu_{k-j+1}\|\alpha}$
2	$\overset{(1)}{C'}{}^{*\mu_1\ldots\mu_{k-1}\|\alpha}$, $\mathcal{C}^*$

Table 3: Nontrivial representatives spanning $\big(H_j^D(\delta|d)\big)_{j=\overline{2,k+1}}$ and $\big(H_j^{\mathrm{inv}D}(\delta|d)\big)_{j=\overline{2,k+1}}$.

The local BRST cohomology in maximum form degree for the model under consideration evaluated in the algebra of local forms without an explicit dependence of the spacetime coordinates, $\bar{\Lambda}$, is still governed by the analogue of Proposition 10 from Ref. [46] (up to the appropriate inclusion of the vector sector), but the discussion following this proposition should be adapted accordingly. For instance, formula (99) from Ref. [46] should be replaced by

$$\overset{[D]}{a}_{k+1}=\sum_J \overset{[D]}{\bar{\alpha}}_J \bar{e}^J\Big(\overset{(1)}{\mathcal{F}},\overset{(k)}{\eta},\mathcal{C}\Big),\quad \overset{[D]}{\bar{\alpha}}_J\in H_{k+1}^{\mathrm{inv}D}(\delta|d),\quad \mathrm{pgh}(\bar{e}^J)=k+1+g. \tag{39}$$

5 Construction of the deformed solution to the classical master equation

In this section we construct the deformed solution to the classical master equation that is consistent, nontrivial, and agrees with all the working hypotheses introduced in the preamble of Section 4 with respect to a tensor field with the mixed symmetry $(k,1)$ and a single vector field. We recall that the maximum derivative order of the interacting Lagrangian density at all orders of perturbation theory should be equal to two. We apply the antifield-BRST procedure exposed in Section 3 starting from formula (21), where S reads now as in (17) and the remaining pieces signify the solutions to equations (22).

Due to the locality presumption and maintaining notations (23), it follows that the nonintegrated density of the first-order deformation, a_1, is subdued to equation (26) and therefore defines a nontrivial element of the local BRST cohomology in maximum form degree and in ghost number equal to 0. We decompose S_1 in a natural way as a sum between three local pieces

$$S_1 = S_1^{\mathrm{t}}+S_1^{\mathrm{V}}+S_1^{\mathrm{t-V}}, \tag{40}$$

$$S_1^{\mathrm{t}}=\int a_1^{\mathrm{t}}d^Dx,\quad S_1^{\mathrm{V}}=\int a_1^{\mathrm{V}}d^Dx,\quad S_1^{\mathrm{t-V}}=\int a_1^{\mathrm{t-V}}d^Dx, \tag{41}$$

where S_1^{t} describes the self-interactions of the tensor $(k,1)$, S_1^{V} those of the vector field, and $S_1^{\mathrm{t-V}}$ the cross-couplings between these two field sectors, so a_1 inherits a similar decomposition

$$a_1=a_1^{\mathrm{t}}+a_1^{\mathrm{V}}+a_1^{\mathrm{t-V}}. \tag{42}$$

Because a_1^{t} may depend only on the BRST generators from the $(k,1)$ sector and a_1^{V} only on those from the vector one, while each term from $a_1^{\mathrm{t-V}}$ should contain at least one generator from each sector, equation (26) becomes equivalent with three independent equations

$$sa_1^{\mathrm{t}}+\partial_\mu b_1^{\mathrm{t}\mu}=0,\quad sa_1^{\mathrm{V}}+\partial_\mu b_1^{\mathrm{V}\mu}=0,\quad sa_1^{\mathrm{t-V}}+\partial_\mu b_1^{\mathrm{t-V}\mu}=0. \tag{43}$$

The first equation has been analyzed in Ref. [44] in the context of the same deformation method and general assumptions employed here, where it has been shown that we can stop the first-order deformation a_1^{t} in antighost number 1 and the current $b_1^{\mathrm{t}\mu}$ in antighost number 0

$$a_1^{\mathrm{t}}=a_{1,1}^{\mathrm{t}}+a_{1,0}^{\mathrm{t}},\quad b_1^{\mathrm{t}\mu}=b_{1,0}^{\mathrm{t}\mu}, \tag{44}$$

where the components $a_{1,1}^{\mathrm{t}}$ and $a_{1,0}^{\mathrm{t}}$ are expressed by

$$a_{1,1}^{\mathrm{t}} = c\delta_{2\bar{k}}^{k}\delta_{4\bar{k}}^{D}\varepsilon_{\mu_1\ldots\mu_{4\bar{k}}}t^{*\mu_1\ldots\mu_{2\bar{k}-1}}\overset{(1)}{\mathcal{F}}{}^{\mu_{2\bar{k}}\ldots\mu_{4\bar{k}}}, \tag{45}$$

$$a_{1,0}^{\mathrm{t}} = -c\delta_{2\bar{k}}^{k}\delta_{4\bar{k}}^{D}\tfrac{(2\bar{k}-1)(2\bar{k}+1)}{(2\bar{k})!8\bar{k}^2}\varepsilon_{\mu_1\ldots\mu_{4\bar{k}}}F^{\mu_1\ldots\mu_{2\bar{k}}}F^{\mu_{2\bar{k}+1}\ldots\mu_{4\bar{k}}}, \tag{46}$$

such that the first-order deformation from the $(k,1)$ sector takes the form

$$S_1^{\rm t} = \int c\varepsilon_{\mu_1\ldots\mu_{4\bar{k}}}\left(t^{*\mu_1\ldots\mu_{2\bar{k}-1}}\overset{(1)^{\mu_{2\bar{k}}\ldots\mu_{4\bar{k}}}}{\mathcal{F}} - \tfrac{(2\bar{k}-1)(2\bar{k}+1)}{(2\bar{k})!8\bar{k}^2}F^{\mu_1\ldots\mu_{2\bar{k}}}F^{\mu_{2\bar{k}+1}\ldots\mu_{4\bar{k}}}\right)d^{4\bar{k}}x. \tag{47}$$

Starting with formula (44), the second lower index of the quantities involved in the various orders of perturbation theory signifies their antighost number. In the above c is an arbitrary real constant and the supplementary factors $\delta_{2\bar{k}}^{k}$ and $\delta_{4\bar{k}}^{D}$ were introduced in order to highlight that relations (45)–(47) are valid solely for even values of k, equal to $2\bar{k}$, and only in $D = 4\bar{k}$ spacetime dimensions.

The second equation in (43) has been analyzed in Ref. [48], where it has been shown that the only consistent first-order deformation that describes the self-interactions of a single massless vector field does not modify the $U(1)$ gauge symmetry

$$a_1^{\rm V} = a_{1,0}^{\rm V}, \qquad \gamma a_{1,0}^{\rm V} + \partial_\mu b_{1,0}^{{\rm V}\mu} = 0 \tag{48}$$

and reduces to a sum among generalized Abelian Chern–Simons terms

$$a_1^{\rm V} = \sum_{i\geq 1}\tilde{c}(i)\varepsilon^{\mu_1\mu_2\ldots\mu_{2i-1}\mu_{2i}\mu_{2i+1}}F_{\mu_1\mu_2}^{\rm V}\cdots F_{\mu_{2i-1}\mu_{2i}}^{\rm V}V_{\mu_{2i+1}}, \tag{49}$$

where $\tilde{c}(i)$ denote some arbitrary real constants. For each fixed i the associated Lagrangian density displays the derivative order equal to i. The simultaneous conditions $D \geq k+2$ and $k \geq 4$ selects from the above sum only the pieces with $i \geq 3$

$$a_1^{\rm V} = \sum_{i\geq 3}\tilde{c}(i)\varepsilon^{\mu_1\mu_2\ldots\mu_{2i-1}\mu_{2i}\mu_{2i+1}}F_{\mu_1\mu_2}^{\rm V}\cdots F_{\mu_{2i-1}\mu_{2i}}^{\rm V}V_{\mu_{2i+1}}, \tag{50}$$

which contain at least three spacetime derivatives and therefore must be discarded by virtue of the maximum derivative order assumption

$$\tilde{c}(3) = \tilde{c}(4) = \cdots = 0 \Rightarrow a_1^{\rm V} = 0. \tag{51}$$

If we started from a collection of Abelian vector fields, V_μ^a, then we would obtain some nontrivial solutions, such as the Yang–Mills couplings [51], due to the presence of more than one fermionic ghosts with pgh $= 1$, $\mathcal{C}^a$, which allows the existence of nontrivial elements in $H(\gamma)$ at pure ghost number equal to two, of the type $\mathcal{C}^a\mathcal{C}^b$. Nevertheless, in the present context of a single vector field we conclude that there are no nontrivial self-interactions that satisfy the working hypotheses (in particular the derivative order assumption) and hence we have that

$$S_1^{\rm V} = 0. \tag{52}$$

Next, we approach the cross-coupling first-order deformation as solution to the last equation in (43). We act in the standard manner and develop $a_1^{\rm t-V}$ and $b_1^{{\rm t-V}\mu}$ along agh. According, the analogue of Proposition 10 from Ref. [46] ensures that we can safely start from

$$a_1^{\rm t-V} = \sum_{j=0}^{k+1} a_{1,j}^{\rm t-V}, \qquad b_1^{{\rm t-V}\mu} = \sum_{j=0}^{k} b_{1,j}^{{\rm t-V}\mu}, \tag{53}$$

such that equation $sa_1^{\rm t-V} + \partial_\mu b_1^{{\rm t-V}\mu} = 0$ becomes equivalent to the chain

$$\gamma a_{1,k+1}^{\rm t-V} = 0, \tag{54}$$

$$\delta a_{1,k+1}^{\mathrm{t-V}} + \gamma a_{1,k}^{\mathrm{t-V}} + \partial_\mu b_{1,k}^{\mathrm{t-V}\mu} = 0, \tag{55}$$
$$\delta a_{1,k}^{\mathrm{t-V}} + \gamma a_{1,k-1}^{\mathrm{t-V}} + \partial_\mu b_{1,k-1}^{\mathrm{t-V}\mu} = 0, \tag{56}$$
$$\vdots$$
$$\delta a_{1,1}^{\mathrm{t-V}} + \gamma a_{1,0}^{\mathrm{t-V}} + \partial_\mu b_{1,0}^{\mathrm{t-V}\mu} = 0. \tag{57}$$

Moreover, result (39) particularized to $g = 0$ leads to

$$a_{1,k+1}^{\mathrm{t-V}} = \bar{\alpha}_{k+1}\bar{e}^{k+1}\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}\right), \quad \bar{\alpha}_{k+1} \leftrightarrow H_{k+1}^{\mathrm{inv}D}, \quad \mathrm{pgh}(\bar{e}^{k+1}) = k+1. \tag{58}$$

Table 3 in $\mathrm{agh} = k+1$ provides the invariant polynomial $\bar{\alpha}_{k+1}$ as an object linear in the undifferentiated antifield $\overset{(k)}{\eta}{}^{*}$ from the $(k,1)$ sector, such that the requirement of cross-coupling in relation with formula (58) necessarily demands that we are compelled to select among the elements of the basis in $\mathrm{pgh} = k+1$, $\bar{e}^{k+1}$, only those quantities depending on the ghost $\mathcal{C}$ from the vector sector. On the one hand, this scalar ghost is fermionic, so it may intervene only linearly (its square is identically vanishing), and, on the other hand, its pure ghost number is equal to one. Consequently, the eligible elements $\bar{e}^{k+1}\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}\right)$ reduce to

$$\bar{e}^{k+1}_{\mathrm{eligible}}\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C}\right) = \bar{e}^{k+1}\left(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \underline{\mathcal{C}}\right) = \left\{e^k\left(\overset{(1)}{\mathcal{F}}\right)\mathcal{C}, \overset{(k)}{\eta}\mathcal{C}\right\}, \tag{59}$$

with $e^k\left(\overset{(1)}{\mathcal{F}}\right)$ of the form

$$e^j\left(\overset{(1)}{\mathcal{F}}\right) \equiv \left(\overset{(1)}{\mathcal{F}}\right)^j = \overset{(1)}{\mathcal{F}}{}^{\mu_1^{(1)}\cdots\mu_{k+1}^{(1)}} \overset{(1)}{\mathcal{F}}{}^{\mu_1^{(2)}\cdots\mu_{k+1}^{(2)}} \cdots \overset{(1)}{\mathcal{F}}{}^{\mu_1^{(j)}\cdots\mu_{k+1}^{(j)}}, \quad j \geq 1 \tag{60}$$

for $j = k$. The underlined ghost signifies that the elements of the basis explicitly depend on $\mathcal{C}$. The derivative order assumption forbids the presence of the first class of elements in (58) (if consistent, such terms would produce a Lagrangian density with precisely $k+1 \geq 5$ derivatives), so we are left with

$$a_{1,k+1}^{\mathrm{t-V}} : \left\{\overset{(k)}{\eta}{}^{*} \rightleftharpoons \overset{(k)}{\eta}\mathcal{C}\right\}, \tag{61}$$

or, equivalently,

$$a_{1,k+1}^{\mathrm{t-V}} = \Upsilon_{\mu\alpha}\overset{(k)}{\eta}{}^{*\mu}\overset{(k)}{\eta}{}^{\alpha}\mathcal{C}, \tag{62}$$

with Υ a constant, non-derivative real tensor. Lorentz covariance and Poincaré invariance arguments combined with $D \geq k+2 \geq 6$ produce the unique option

$$\Upsilon_{\mu\alpha} = c_1\sigma_{\mu\alpha}, \qquad c_1 \in \mathbb{R}, \tag{63}$$

which further yields the most general expression of the solution to equation (54) that fulfills all the imposed properties like

$$a_{1,k+1}^{\mathrm{t-V}} = c_1\overset{(k)}{\eta}{}^{*}_{\alpha}\overset{(k)}{\eta}{}^{\alpha}\mathcal{C}. \tag{64}$$

Using the actions of the operators δ and γ on the BRST generators, we find the solution to equation (55) (up to the solutions of the homogeneous equation in $\mathrm{agh} = k$) of the form

$$a_{1,k}^{\mathrm{t-V}} = 2c_1\overset{(k-1)}{C'}{}^{*}_{\mu_1||\alpha}\left(\tfrac{1}{2}\overset{(k-1)}{C'}{}^{\mu_1||\alpha}\mathcal{C} - V^{\mu_1}\overset{(k)}{\eta}{}^{\alpha}\right), \tag{65}$$

where the ghost $\overset{(k-1)}{C'}$ together with the antifield $\overset{(k-1)^*}{C'}$ are defined by

$$\overset{(m)}{C'}_{\mu_1\ldots\mu_{k-m}||\alpha} \equiv \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha} + (k-m+2)\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m}\alpha}, \tag{66}$$

$$\overset{(m)^*\mu_1\ldots\mu_{k-m}||\alpha}{C'} \equiv \overset{(m)^*\mu_1\ldots\mu_{k-m}|\alpha}{C} + \tfrac{1}{k-m+2}\overset{(m)^*\mu_1\ldots\mu_{k-m}\alpha}{\eta}, \tag{67}$$

with $m = k-1$. We mention that the range of m in the last relations is $m = \overline{1, k-1}$. In order to generate the component of agh $= k-1$ as solution to equation (56), we start from (65) on which we act with δ. In this manner, after some computation we infer

$$\begin{aligned}\delta a_{1,k}^{\mathrm{t-V}} = &-2c_1 \overset{(k-2)^*}{C'}_{\mu_1\mu_2||\alpha} F^{\mathrm{V}\mu_1\mu_2} \overset{(k)^\alpha}{\eta} - \partial_\mu b_{1,k-1}^{\mathrm{t-V}\mu} \\ &- \gamma\left[c_1 \overset{(k-2)^*}{C'}_{\mu_1\mu_2||\alpha}\left(\overset{(k-2)^{\mu_1\mu_2||\alpha}}{C'}\mathcal{C} - V^{[\mu_1}\overset{(k-1)^{\mu_2]||\alpha}}{C'}\right)\right],\end{aligned} \tag{68}$$

with $\overset{(k-2)}{C'}$ and $\overset{(k-2)^*}{C'}$ like in (66) and respectively (67) for $m = k-2$. Inserting result (68) into equation (56), we notice that it possesses solutions with respect to $a_{1,k-1}^{\mathrm{t-V}}$ if and only if the first term from the right-hand side of relation (68) is γ-exact modulo a full divergence

$$2c_1 \overset{(k-2)^*}{C'}_{\mu_1\mu_2||\alpha} F^{\mathrm{V}\mu_1\mu_2} \overset{(k)^\alpha}{\eta} = \gamma a'^{\mathrm{t-V}}_{1,k-1} + \partial_\mu b'^{\mathrm{t-V}\mu}_{1,k-1}. \tag{69}$$

In order to analyze the last (necessary and sufficient) condition, we take its left functional derivative with respect to the antifield $\overset{(k-2)^*}{C'}$, use the commutation of this operation with the action of γ, and take into account the fact that the Euler–Lagrange (EL) derivatives of the divergence vanish identically, which finally leads to

$$2c_1 F^{\mathrm{V}\mu_1\mu_2} \overset{(k)^\alpha}{\eta} = \gamma\left(\frac{\delta^{\mathrm{L}} a'^{\mathrm{t-V}}_{1,k-1}}{\delta \overset{(k-2)^*}{C'}_{\mu_1\mu_2||\alpha}}\right). \tag{70}$$

By means of formula (35), we observe that the left-hand side of the above equation is a nontrivial element of agh $= 0$ pertaining to the cohomology space $H^k(\gamma)$, such that requirement (70) takes place if and only if

$$c_1 = 0, \tag{71}$$

which annihilates (64)

$$a_{1,k+1}^{\mathrm{t-V}} = 0. \tag{72}$$

In conclusion, the first-order deformation that couples the $(k,1)$ sector to the vector one cannot stop in maximum antighost number, equal to $(k+1)$.

Before analyzing the remaining possibilities related to the structure of the nonintegrated density of the cross-coupling first-order deformation, we argue that the first class of elements of the ghost basis with pgh $= k+1$ present in the right-hand side of formula (59) does not meet the maximum derivative order criterion. Indeed, in this situation we consider in $a_{1,k+1}^{\mathrm{t-V}}$ a particular solution to equation (54) of the form $a_{1,k+1}^{\mathrm{t-V}} \sim \overset{(k)^*}{\eta} e^k \left(\overset{(1)}{\mathcal{F}}\right)\mathcal{C}$, where e^k reads as in (60) with $j = k$, so it comprises precisely k spacetime derivatives. If consistent, this solution generates some components of the first-order deformation of lower but strictly positive values of the antighost number (as solutions to equations (55), (56), and so on, but not to equation (57)) that preserve the initial number of derivatives. Indeed, all the nonvanishing actions of δ on the full antifield spectrum excepting the agh $= 1$ part together with all the nonvanishing actions of γ on the entire field/ghost spectrum increase the derivative order by one, such that the supplementary derivative coming from the action of δ will be absorbed (up to some divergences) by γ. To put it otherwise, all the pieces $a_{1,j}^{\text{t-V}}$ with $j = \overline{1,k}$ will be homogeneous functions of order k with respect to the total number of spacetime derivatives. This symmetry is broken when we act with δ on $a_{1,1}^{\text{t-V}}$ in order to generate $a_{1,0}^{\text{t-V}}$ as solution to equation (57) since in this context there appear two additional spacetime derivatives (the actions of delta on each antifield of the original fields is proportional with the EL derivatives of the free Lagrangian density with respect to the corresponding field, so they comprise exactly two spacetime derivatives). Thus, up to some irrelevant full divergences, only one of these derivatives will be absorbed by γ and will lead to an $a_{1,0}^{\text{t-V}}$ with exactly $k+1 \geq 5$ derivatives, which disagrees with the derivative order assumption.

The cases where the cross-coupling first-order deformation stops in maximum values of the antighost number $m = \overline{3,k}$ may be treated compactly. The analogue of Corollary 3 from Ref. [45] ensures that for every fixed value of m within this range we have that

$$a_1^{\text{t-V}} = \sum_{j=0}^{m} a_{1,j}^{\text{t-V}}, \qquad b_1^{\text{t-V}\mu} = \sum_{j=0}^{m-1} b_{1,j}^{\text{t-V}\mu}, \tag{73}$$

where the various components of the above decompositions fulfill the equations

$$\gamma a_{1,m}^{\text{t-V}} = 0, \tag{74}$$

$$\delta a_{1,m}^{\text{t-V}} + \gamma a_{1,m-1}^{\text{t-V}} + \partial_\mu b_{1,m-1}^{\text{t-V}\mu} = 0, \tag{75}$$

$$\vdots$$

$$\delta a_{1,1}^{\text{t-V}} + \gamma a_{1,0}^{\text{t-V}} + \partial_\mu b_{1,0}^{\text{t-V}\mu} = 0. \tag{76}$$

Moreover, the similar of Proposition 10 from Ref. [46] adapted to the present free model provides the piece of maximum antighost number like

$$a_{1,m}^{\text{t-V}} = \bar{\alpha}_m \bar{e}^m \Big(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \mathcal{C} \Big), \quad \bar{\alpha}_m \leftrightarrow H_m^{\text{inv}D}, \quad \text{pgh}(\bar{e}^m) = m. \tag{77}$$

Combining the information from Table 3 for agh $= m$ corresponding to the range under consideration, which yields that the invariant polynomial $\bar{\alpha}_m$ is linear in the undifferentiated antifields $\overset{(m-1)^*}{C'}$ from the $(k,1)$ sector, with the cross-coupling requirement related to expression (77), we conclude that the eligible elements of the basis $\bar{e}^m$ are those linear in the $U(1)$ ghost $\mathcal{C}$. Since its pure ghost number is equal to one, this further prevents the ghost $\overset{(k)}{\eta}$ (with pgh $= k$) to enter $\bar{e}^m$ and leads to

$$\bar{e}^m \Big(\overset{(1)}{\mathcal{F}}, \overset{(k)}{\eta}, \underline{\mathcal{C}} \Big) = e^{m-1} \Big(\overset{(1)}{\mathcal{F}} \Big) \mathcal{C}, \tag{78}$$

with $e^{m-1}\Big(\overset{(1)}{\mathcal{F}}\Big)$ as in (60) for $j = m-1$. So far, we argued that $a_{1,m}^{\text{t-V}}$ with $m = \overline{3,k}$ contains a single class of terms

$$a_{1,m}^{\text{t-V}} : \Big\{ \overset{(m-1)^*}{C'} \rightleftharpoons e^{m-1} \Big(\overset{(1)}{\mathcal{F}} \Big) \mathcal{C} \Big\}. \tag{79}$$

Due to the fact that e^{m-1} contains $(m-1)$ spacetime derivatives, a reasoning similar to that developed in the previous paragraph implies that if consistent, such a term will produce a Lagrangian density with exactly $m \geq 3$ derivatives, which contradicts the derivative order hypothesis. Consequently, the cross-coupling first-order deformation cannot stop either in maximum values of the antighost number within the range $\overline{3,k}$.

We pass to the next possible value of the maximum antighost number, equal to 2, and apply the result of Corollary 3 from Ref. [45] adapted here, which stipulates that we can start from

$$a_1^{\mathrm{t-V}} = a_{1,0}^{\mathrm{t-V}} + a_{1,1}^{\mathrm{t-V}} + a_{1,2}^{\mathrm{t-V}}, \qquad b_1^{\mathrm{t-V}\mu} = b_{1,0}^{\mathrm{t-V}\mu} + b_{1,1}^{\mathrm{t-V}\mu}, \tag{80}$$

as solutions to the equations

$$\gamma a_{1,2}^{\mathrm{t-V}} = 0, \quad \delta a_{1,2}^{\mathrm{t-V}} + \gamma a_{1,1}^{\mathrm{t-V}} + \partial_\mu b_{1,1}^{\mathrm{t-V}\mu} = 0, \quad \delta a_{1,1}^{\mathrm{t-V}} + \gamma a_{1,0}^{\mathrm{t-V}} + \partial_\mu b_{1,0}^{\mathrm{t-V}\mu} = 0. \tag{81}$$

In agreement with the analogue of Proposition 10 from Ref. [46], the solution to the first equation from the above chain can be represented like

$$a_{1,2}^{\mathrm{t-V}} = \bar{\alpha}_2 \bar{e}^2\left(\overset{(1)}{\mathcal{F}}, \mathcal{C}\right), \quad \bar{\alpha}_2 \leftrightarrow H_2^{\mathrm{inv}D}, \quad \mathrm{pgh}(\bar{e}^2) = 2. \tag{82}$$

We investigate the elements of $H_2^{\mathrm{inv}D}$ with the help of Table 3 and remark that this is the first time when the vector sector contributes to the invariant characteristic cohomology, such that the elements of the basis $\bar{e}^2$ are no longer restricted to depend on $\mathcal{C}$. Since

$$\bar{e}^2\left(\overset{(1)}{\mathcal{F}}, \mathcal{C}\right) = \left\{e^2\left(\overset{(1)}{\mathcal{F}}\right), \overset{(1)}{\mathcal{F}}\mathcal{C}\right\}, \tag{83}$$

the cross-coupling requirement allows three classes of terms in (82)

$$a_{1,2}^{\mathrm{t-V}} : \left\{\overset{(1)}{C'}{}^{*} \rightleftharpoons \overset{(1)}{\mathcal{F}}\mathcal{C}, \mathcal{C}^* \rightleftharpoons e^2\left(\overset{(1)}{\mathcal{F}}\right), \mathcal{C}^* \rightleftharpoons \overset{(1)}{\mathcal{F}}\mathcal{C}\right\}. \tag{84}$$

The second class does not comply with the derivative order criterion (if consistent, such terms would produce a Lagrangian density with three derivatives), which leaves two alternatives

$$a_{1,2}^{\mathrm{t-V}} = \left(\Upsilon_{\mu_1\ldots\mu_{k-1}||\mu_k||\nu_1\ldots\nu_{k+1}} \overset{(1)}{C'}{}^{*\mu_1\ldots\mu_{k-1}||\mu_k} + \Upsilon_{\nu_1\ldots\nu_{k+1}}\mathcal{C}^*\right) \overset{(1)}{\mathcal{F}}{}^{\nu_1\ldots\nu_{k+1}} \mathcal{C}, \tag{85}$$

where the tensors Υ are imposed to be real, constant, and non-derivative. The double bar signifies full antisymmetry (where applicable) only with respect to the delimited subgroups of Lorentz indices, without other mixed symmetry property. Taking into account the Lorentz covariance together with the Poincaré invariance, as well as the complete antisymmetry of the ghost $\overset{(1)}{\mathcal{F}}$ plus the antisymmetry of the antifield $\overset{(1)}{C'}{}^{*}$ with respect to its first $(k-1)$ indices, we infer the next eligible solutions in $D \geq k+2 \geq 6$

$$\Upsilon_{\mu_1\ldots\mu_{k-1}||\mu_k||\nu_1\ldots\nu_{k+1}} = \delta_{2k+1}^D c_2 (k+1) \varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}} + \delta_{2k-1}^D c_3 \sigma_{\mu_{k-1}\mu_k} \varepsilon_{\mu_1\ldots\mu_{k-2}\nu_1\ldots\nu_{k+1}}, \tag{86}$$

$$\Upsilon_{\nu_1\ldots\nu_{k+1}} = 0, \tag{87}$$

where $c_{2,3}$ stand for two arbitrary real constants and the numeric factor $(k+1)$ was added for further convenience. We introduce the previous results in (85) and process the resulting terms by recalling the expression of $\overset{(1)}{C'}{}^{*}$ in terms of $\overset{(1)}{C}{}^{*}$ and $\overset{(1)}{\eta}{}^{*}$ following from relation (67) particularized to $m = 1$. By virtue of the mixed symmetry of the type $(k-1,1)$ of $\overset{(1)}{C}{}^{*}$ and employing formula

$$\overset{(m)}{C'}{}^{*\mu_1\ldots\mu_{k-m-1}} = \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m-1}} \tag{88}$$

for $m = 1$, we finally obtain

$$a_{1,2}^{\mathrm{t-V}} = \Big(\delta_{2k+1}^{D} c_2 \varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}} \overset{(1)}{\eta}{}^{*\mu_1\ldots\mu_k}$$
$$+ \delta_{2k-1}^{D} c_3 \varepsilon_{\mu_1\ldots\mu_{k-2}\nu_1\ldots\nu_{k+1}} \overset{(1)}{C}{}^{*\mu_1\ldots\mu_{k-2}} \Big) \overset{(1)}{\mathcal{F}}{}^{\nu_1\ldots\nu_{k+1}} \mathcal{C}. \tag{89}$$
$$\equiv a_{1,2}^{\mathrm{t-V}}(c_2) + a_{1,2}^{\mathrm{t-V}}(c_3). \tag{90}$$

Since they evolve on Minkowski spacetime manifolds of different dimensions, $D = 2k+1$ and respectively $D = 2k-1$, the two terms from (90) should be independently consistent, i.e. the equations corresponding to agh $= 1$ and agh $= 0$ from (81) are to be split into two independent subsets

$$\delta a_{1,2}^{\mathrm{t-V}}(c_2) + \gamma a_{1,1}^{\mathrm{t-V}}(c_2) + \partial_\mu b_{1,1}^{\mathrm{t-V}\mu}(c_2) = 0,\ \delta a_{1,1}^{\mathrm{t-V}}(c_2) + \gamma a_{1,0}^{\mathrm{t-V}}(c_2) + \partial_\mu b_{1,0}^{\mathrm{t-V}\mu}(c_2) = 0, \tag{91}$$
$$\delta a_{1,2}^{\mathrm{t-V}}(c_3) + \gamma a_{1,1}^{\mathrm{t-V}}(c_3) + \partial_\mu b_{1,1}^{\mathrm{t-V}\mu}(c_3) = 0,\ \delta a_{1,1}^{\mathrm{t-V}}(c_3) + \gamma a_{1,0}^{\mathrm{t-V}}(c_3) + \partial_\mu b_{1,0}^{\mathrm{t-V}\mu}(c_3) = 0. \tag{92}$$

Regarding the first equation from (91), we start from the first term in the right-hand side of (89) and, after some computation, we find that

$$a_{1,1}^{\mathrm{t-V}}(c_2) = \delta_{2k+1}^{D} c_2 (k+1) \varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}} t^{*\mu_1\ldots\mu_k|\alpha} \left(\tfrac{1}{k} F^{\nu_1\ldots\nu_{k+1}|}{}_{\alpha} \mathcal{C} + (-)^k V_\alpha \overset{(1)}{\mathcal{F}}{}^{\nu_1\ldots\nu_{k+1}} \right). \tag{93}$$

Acting similarly with respect to the first equation from (92), we deduce

$$a_{1,1}^{\mathrm{t-V}}(c_3) = \delta_{2k-1}^{D} c_3 \tfrac{(-)^{k+1}k(k+1)}{k-1} \varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k} t^{*\mu_1\ldots\mu_{k-1}} \left(\tfrac{1}{k} F^{\nu_1\ldots\nu_k} \mathcal{C} + V_\mu \overset{(1)}{\mathcal{F}}{}^{\mu\nu_1\ldots\nu_k} \right), \tag{94}$$

where $t^{*\mu_1\ldots\mu_{k-1}}$ is the trace of the antifield $t^{*\mu_1\ldots\mu_k|\alpha}$.

In order to analyze the solutions to the last equation from (91), we act with δ on (93) and find

$$\delta a_{1,1}^{\mathrm{t-V}}(c_2) = \delta_{2k+1}^{D} c_2 \tfrac{k+1}{k!} \varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}} \Big[\tfrac{1}{k} \big(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha} + (-)^{k+1} \partial^\alpha F^{\mu_1\ldots\mu_k} \big) F^{\nu_1\ldots\nu_{k+1}|}{}_{\alpha} \mathcal{C}$$
$$+ (-)^k \big(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha} + (-)^{k+1} \partial^\alpha F^{\mu_1\ldots\mu_k} - \sigma^{\alpha[\mu_1} \partial_\rho F^{\mu_2\ldots\mu_k]\rho} \big) V_\alpha \overset{(1)}{\mathcal{F}}{}^{\nu_1\ldots\nu_{k+1}} \Big]. \tag{95}$$

Combining the result

$$\partial_{\rho_1} \overset{(1)}{\mathcal{F}}{}_{\mu_1\ldots\mu_{k+1}} = \gamma \Big(\tfrac{(-)^{k+1}}{k} F_{\mu_1\ldots\mu_{k+1}|\rho_1} \Big) \tag{96}$$

with the first definition in (15), we get two necessary conditions for the existence of the solutions $a_{1,0}^{\mathrm{t-V}}(c_2)$ to the last equation from (91)

$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}} \big(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha} + (-)^{k+1} \partial^\alpha F^{\mu_1\ldots\mu_k} \big) F^{\nu_1\ldots\nu_{k+1}|}{}_{\alpha} = \partial_\mu B^\mu, \tag{97}$$
$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}} \big(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha} + (-)^{k+1} \partial^\alpha F^{\mu_1\ldots\mu_k} - \sigma^{\alpha[\mu_1} \partial_\mu F^{\mu_2\ldots\mu_k]\mu} \big) V_\alpha$$

$$=\partial_\mu B^\mu{}_{\nu_1\ldots\nu_{k+1}}. \tag{98}$$

Let us analyze now (97) by means of relations (4) and (5). The second term in the left-hand side of (97) can be represented (up to a global phase factor $(-)^{k+1}$) under the form

$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(\partial^\alpha F^{\mu_1\ldots\mu_k}\right)F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha = \partial_\mu\left[\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(F^{\mu_1\ldots\mu_k}F^{\nu_1\ldots\nu_{k+1}|\mu}\right.\right.$$
$$\left.\left. - \sigma^{\mu[\mu_1}t^{\mu_2\ldots\mu_k]}\partial_\alpha F^{\nu_1\ldots\nu_{k+1}|\alpha}\right)\right] - (k+1)\varepsilon_{\mu\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\partial^\mu\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right), \tag{99}$$

such that condition (97) becomes equivalent to

$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha}\right)F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha$$
$$+(-)^k(k+1)\varepsilon_{\mu\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\partial^\mu\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right) = \partial_\mu\bar{B}^\mu, \tag{100}$$

where we made the notation

$$\bar{B}^\mu \equiv B^\mu + (-)^k\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(F^{\mu_1\ldots\mu_k}F^{\nu_1\ldots\nu_{k+1}|\mu} - \sigma^{\mu[\mu_1}t^{\mu_2\ldots\mu_k]}\partial_\alpha F^{\nu_1\ldots\nu_{k+1}|\alpha}\right). \tag{101}$$

It is easy to see that the following relations are valid in $D = 2k+1$

$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha}\right)F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha = \partial_\mu\left(\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}F^{\mu\mu_1\ldots\mu_k|\alpha}F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha\right)$$
$$(-)^{k+1}\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha}\right)F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha, \tag{102}$$

$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}F^{\mu\mu_1\ldots\mu_k|\alpha}F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha = (-)^k\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}F^{\mu\mu_1\ldots\mu_k|\alpha}F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha, \tag{103}$$

$$\varepsilon_{\mu\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\partial^\mu\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right) = \partial_\mu\left[\varepsilon^\mu{}_{\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right)\right]$$
$$+(-)^{k+1}\varepsilon_{\mu\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\partial^\mu\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right), \tag{104}$$

$$\varepsilon^\mu{}_{\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right) = (-)^k\varepsilon^\mu{}_{\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right). \tag{105}$$

Taking into account formulas (102)–(105) and the equivalence of the necessary condition (97) to (100), we remark that there appear two distinct situations:

1. for odd values, $k = 2\bar{k}+1$ ($\bar{k} \geq 2$), none of the two terms present in the left-hand side of formula (100) reduces to a divergence, such that in this case there are no solutions $a_{1,0}^{\text{t-V}}(c_2)$ to the second equation from (91). Consequently, inconsistency reasons force $c_2 = 0$, which automatically annihilates the component $a_{1,2}^{\text{t-V}}(c_2)$ from the piece of maximum antighost number equal to 2, (90), of the cross-coupling first-order deformation

$$a_{1,2}^{\text{t-V}}(c_2) = 0 \quad \text{for } k = 2\bar{k}+1, \quad \bar{k} \geq 2; \tag{106}$$

2. for even values, $k = 2\bar{k}$ ($\bar{k} \geq 2$), we have that $D = 4\bar{k}+1$, so (100) is indeed fulfilled

$$\varepsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{k+1}}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_k|\alpha}\right)F^{\nu_1\ldots\nu_{k+1}|}{}_\alpha$$
$$+(-)^k(k+1)\varepsilon_{\mu\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\left(\partial_\alpha t^{\mu_1\ldots\mu_k|\alpha}\right)\partial^\mu\left(\partial_\beta t^{\nu_1\ldots\nu_k|\beta}\right) \xrightarrow{k=2\bar{k},D=4\bar{k}+1}$$
$$\varepsilon_{\mu_1\ldots\mu_{2\bar{k}}\nu_1\ldots\nu_{2\bar{k}+1}}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_{2\bar{k}}|\alpha}\right)F^{\nu_1\ldots\nu_{2\bar{k}+1}|}{}_\alpha$$
$$+(2\bar{k}+1)\varepsilon_{\mu\mu_1\ldots\mu_{2\bar{k}}\nu_1\ldots\nu_{2\bar{k}}}\left(\partial_\alpha t^{\mu_1\ldots\mu_{2\bar{k}}|\alpha}\right)\partial^\mu\left(\partial_\beta t^{\nu_1\ldots\nu_{2\bar{k}}|\beta}\right) \equiv$$
$$\partial_\mu\left\{\tfrac{1}{2}\left[\varepsilon_{\mu_1\ldots\mu_{2\bar{k}}\nu_1\ldots\nu_{2\bar{k}+1}}F^{\mu\mu_1\ldots\mu_{2\bar{k}}|\alpha}F^{\nu_1\ldots\nu_{2\bar{k}+1}|}{}_\alpha\right.\right.$$

$$+ (2\bar{k}+1)\varepsilon^{\mu}{}_{\mu_1\ldots\mu_{2\bar{k}}\nu_1\ldots\nu_{2\bar{k}}}\left(\partial_\alpha t^{\mu_1\ldots\mu_{2\bar{k}}|\alpha}\right)\left(\partial_\beta t^{\nu_1\ldots\nu_{2\bar{k}}|\beta}\right)\Big]\Big\}, \tag{107}$$

so the first necessary condition for the existence of $a_{1,0}^{\mathrm{t-V}}(c_2)$, i.e. (97), indeed holds. Under these circumstances, it can be shown by direct computation that requirement (98) cannot take place, so no solutions $a_{1,0}^{\mathrm{t-V}}(c_2)$ to the second equation from (91) exist. The elimination of inconsistencies is enforced like in the previous case by taking $c_2 = 0$, which implements

$$a_{1,2}^{\mathrm{t-V}}(c_2) = 0 \quad \text{for } k = 2\bar{k}, \quad \bar{k} \geq 2 \tag{108}$$

into the piece of maximum antighost number equal to 2, (90), of the cross-coupling first-order deformation.

Next, we tackle the solutions to the last equation from (92). From the action of δ on the antifield of the field with the mixed symmetry $(k,1)$ where we take its contraction over the last two indices and particularize the emerging result to $D = 2k-1$, we find the action of the Koszul–Tate differential on the trace of the antifield t^*, which turns out to be proportional with the divergence of the trace of the tensor defined by (4)

$$\delta t^{*\mu_1\ldots\mu_{k-1}} = \tfrac{k-2}{k!}\partial_\mu F^{\mu\mu_1\ldots\mu_{k-1}}. \tag{109}$$

Now, we act with δ on (94) and slightly process the resulting expression, which further yields

$$\begin{aligned}\delta a_{1,1}^{\mathrm{t-V}}(c_3) =& \delta_{2k-1}^{D} c_3 \tfrac{(-)^k(k+1)(k-2)}{k!(k-1)}\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_{k-1}}\right)F^{\nu_1\ldots\nu_k}\mathcal{C}\\ &+ \delta_{2k-1}^{D} c_3 \tfrac{(k-2)}{(k-1)!}\varepsilon_{\mu_1\ldots\mu_{k-2}\nu_1\ldots\nu_{k+1}}V_\rho\left(\partial_\mu F^{\mu\rho\mu_1\ldots\mu_{k-2}}\right)\overset{(1)}{\mathcal{F}}{}^{\nu_1\ldots\nu_{k+1}}.\end{aligned} \tag{110}$$

Invoking again formula (96) together with the first definition from (15) we get that the existence of solutions $a_{1,0}^{\mathrm{t-V}}(c_3)$ to the latter equation from (92) will be governed in this case by the necessary conditions

$$\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_{k-1}}\right)F^{\nu_1\ldots\nu_k} = \partial_\mu M^\mu, \tag{111}$$

$$\varepsilon_{\mu_1\ldots\mu_{k-2}\nu_1\ldots\nu_{k+1}}V_\rho\left(\partial_\mu F^{\mu\rho\mu_1\ldots\mu_{k-2}}\right) = \partial_\mu M^{\mu}{}_{\nu_1\ldots\nu_{k+1}}, \tag{112}$$

which are nothing but the analogue of relations (97)–(98) from the previous situation. It is simple to check that the following identities hold in $D = 2k-1$

$$\begin{aligned}\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_{k-1}}\right)F^{\nu_1\ldots\nu_k} =& \partial_\mu\left(\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}F^{\mu\mu_1\ldots\mu_{k-1}}F^{\nu_1\ldots\nu_k}\right)\\ &+ (-)^k\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_{k-1}}\right)F^{\nu_1\ldots\nu_k},\end{aligned} \tag{113}$$

$$\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}F^{\mu\mu_1\ldots\mu_{k-1}}F^{\nu_1\ldots\nu_k} = (-)^{k+1}\varepsilon_{\mu_1\ldots\mu_{k-1}\nu_1\ldots\nu_k}F^{\mu\mu_1\ldots\mu_{k-1}}F^{\nu_1\ldots\nu_k}, \tag{114}$$

such that we find two complementary cases also in the present context:

1. for even values $k = 2\bar{k}$ ($\bar{k} \geq 2$) the left-hand side of (111) cannot be represented like a full divergence, so there are no solutions $a_{1,0}^{\mathrm{t-V}}(c_3)$ to the former equation from (92). Consequently, we must take $c_3 = 0$, which implies that the component $a_{1,2}^{\mathrm{t-V}}(c_3)$ from the piece of maximum antighost number equal to 2 present in the first-order cross-coupling deformation density vanishes

$$a_{1,2}^{\mathrm{t-V}}(c_3) = 0 \quad \text{for } k = 2\bar{k}, \quad \bar{k} \geq 2; \tag{115}$$

2. for odd values $k = 2\bar{k}+1$ ($\bar{k} \geq 2$), and hence $D = 4\bar{k}+1$, condition (111) is fulfilled

$$\begin{aligned}&\varepsilon_{\mu_1\ldots\mu_{2\bar{k}}\nu_1\ldots\nu_{2\bar{k}+1}}\left(\partial_\mu F^{\mu\mu_1\ldots\mu_{2\bar{k}}}\right)F^{\nu_1\ldots\nu_{2\bar{k}+1}}\\ =&\partial_\mu\left(\tfrac{1}{2}\varepsilon_{\mu_1\ldots\mu_{2\bar{k}}\nu_1\ldots\nu_{2\bar{k}+1}}F^{\mu\mu_1\ldots\mu_{2\bar{k}}}F^{\nu_1\ldots\nu_{2\bar{k}+1}}\right),\end{aligned} \tag{116}$$

but on the other hand this precise value of k implies that condition (112) cannot be fulfilled, so we must also take $c_3 = 0$, which annihilates one more time the corresponding piece of maximum antighost number equal to 2 from the nonintegrated density of the first-order deformation

$$a_{1,2}^{\text{t}-\text{V}}(c_3) = 0 \quad \text{for } k = 2\bar{k}+1, \quad \bar{k} \geq 2. \tag{117}$$

The main of conclusion of the arguments exposed so far is that the cross-coupling first-order deformation cannot stop in a nontrivial manner in maximum antighost number equal to 2.

In this way we reached the situation where the first-order deformation density $a_1^{\text{t}-\text{V}}$ stops in agh $= 1$. By applying the analogue of Corollary 3 from [45] adapted to the present model, we are able to stop the current $b_1^{\text{t}-\text{V}\mu}$ in agh $= 0$ without loss of nontrivial terms, so the starting point will be given by the following expansions and corresponding equations

$$a_1^{\text{t}-\text{V}} = a_{1,0}^{\text{t}-\text{V}} + a_{1,1}^{\text{t}-\text{V}}, \qquad b_1^{\text{t}-\text{V}\mu} = b_{1,0}^{\text{t}-\text{V}\mu}, \tag{118}$$

$$\gamma a_{1,1}^{\text{t}-\text{V}} = 0, \qquad \delta a_{1,1}^{\text{t}-\text{V}} + \gamma a_{1,0}^{\text{t}-\text{V}} + \partial_\mu b_{1,0}^{\text{t}-\text{V}\mu} = 0. \tag{119}$$

Result (35) for $j = 1 = l$ combined with the fact that the elements of the basis in pure ghost number 1, $\bar{e}^1$, are spanned by

$$\bar{e}^1\left(\overset{(1)}{\mathcal{F}},\mathcal{C}\right) = \left\{\overset{(1)}{\mathcal{F}},\mathcal{C}\right\}, \tag{120}$$

allow for a formal representation of $a_{1,1}^{\text{t}-\text{V}}$ as solution to the former equation in (119) as

$$a_{1,1}^{\text{t}-\text{V}} : \left\{\bar{\alpha}_1'\left([t^*],[V^*],[K],\left[F^{\text{V}}\right]\right) \rightleftharpoons \overset{(1)}{\mathcal{F}}, \quad \bar{\alpha}_1''\left([t^*],[V^*],[K],\left[F^{\text{V}}\right]\right) \rightleftharpoons \mathcal{C}\right\}, \tag{121}$$

where the invariant polynomials $\bar{\alpha}_1'$ and $\bar{\alpha}_1''$ display the antighost number 1 and hence are restricted to be monomials of order one in $[t^*]$ and $[V^*]$. One can always assume that these invariant polynomials are in fact some monomials of order one in the undifferentiated antifields t^* and respectively V^* (up to some irrelevant divergences, we can absolve the antifields from any derivatives acting on them and consequently eliminate the resulting terms containing the derivatives of the two types of ghosts due to their γ-exactness, in agreement with formula (96) and respectively the first definition in (15)). The derivative order assumption combined with the actions of the Koszul–Tate operator and of the longitudinal exterior derivative on the BRST generators imposes the next supplementary restrictions on $\bar{\alpha}_1'$ and $\bar{\alpha}_1''$:

– they cannot depend either on the components of the curvature tensor or on their derivatives;

– the terms linear in $\overset{(1)}{\mathcal{F}}$ cannot contain either the components of the Abelian field strength or their derivatives;

– the terms proportional with $\mathcal{C}$ may depend at most linearly on the Abelian field strength.

The above considerations completed by the cross-coupling selection rule narrow $a_{1,1}^{\mathrm{t-V}}$ to three classes of possible terms:

$$\begin{aligned} a_{1,1}^{\mathrm{t-V}} =& \Upsilon_{\mu_1\mu_2\ldots\mu_{k+2}} V^{*\mu_1} \overset{(1)}{\mathcal{F}}{}^{\mu_2\ldots\mu_{k+2}} \\ &+ \left(\Upsilon_{\mu_1\ldots\mu_k||\alpha||\rho_1\rho_2} t^{*\mu_1\ldots\mu_k|\alpha} F^{\mathrm{V}\rho_1\rho_2} + \Upsilon_{\mu_1\ldots\mu_k|\alpha} t^{*\mu_1\ldots\mu_k|\alpha}\right)\mathcal{C}, \end{aligned} \tag{122}$$

where the tensors denoted by Υ are real, constant, and non-derivative. Moreover, Lorentz covariance and Poincaré invariance arguments furnish the next nontrivial solutions in $D \geq k+2$ $(k \geq 4)$

$$\Upsilon_{\mu_1\mu_2\ldots\mu_{k+2}} = \delta^D_{k+2} c_4 \varepsilon_{\mu_1\mu_2\ldots\mu_{k+2}}, \quad \Upsilon_{\mu_1\ldots\mu_k||\alpha||\rho_1\rho_2} = \varepsilon_{\mu_1\ldots\mu_k\alpha\rho_1\rho_2}, \quad \Upsilon_{\mu_1\ldots\mu_k|\alpha} = 0, \tag{123}$$

with c_4 an arbitrary real constant. In addition, the identity $t^{*[\mu_1\ldots\mu_k|\alpha]} \equiv 0$ annihilates the second term from the right-hand side of (122), such that the general nontrivial solution to the former equation from (119) that agrees with all the enforced selection rules is expressed by

$$a_{1,1}^{\mathrm{t-V}} = \delta^D_{k+2} c_4 \varepsilon_{\mu_1\mu_2\ldots\mu_{k+2}} V^{*\mu_1} \overset{(1)}{\mathcal{F}}{}^{\mu_2\ldots\mu_{k+2}}. \tag{124}$$

From (124) we infer the solution to the latter equation in (119) (up to the solutions of the "homogeneous" equation in antighost number 0) like

$$a_{1,0}^{\mathrm{t-V}} = \delta^D_{k+2} c_4 \tfrac{k+1}{2k} \varepsilon_{\mu_1\ldots\mu_{k+2}} F^{\mathrm{V}\mu_1\mu_2} F^{\mu_3\ldots\mu_{k+2}}. \tag{125}$$

The final step in the analysis of the cross-couplings at order one of perturbation theory between the $(k,1)$ and the spin-1 fields is represented by the nontrivial solutions to the "homogeneous" equation

$$\gamma \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) + \partial_\mu \bar{b}_{1,0}^{\mathrm{t-V}\mu} = 0, \tag{126}$$

which does not modify the original generating set of gauge transformations corresponding to the free action (1). Since the analogue of Corollary 3 from [45] does not hold in agh $= 0$, we will analyze separately the solutions in the homogeneous case ($\bar{b}_{1,0}^{\mathrm{t-V}\mu} = 0$) from those present in the inhomogeneous one ($\bar{b}_{1,0}^{\mathrm{t-V}\mu} \neq 0$). In the homogeneous situation, the equation $\gamma \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) = 0$ is completely equivalent to the gauge invariance of $\bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) = 0$ under the original gauge transformations (9) and (10), $\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon},\xi} \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) = 0$. In other words, the associated Lagrangian densities are given by invariant polynomials with agh $= 0$ of the starting free model, so they follow from (31) in agh $= 0$, $\bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) \equiv \bar{a}_{1,0}^{\mathrm{t-V}}([K],[F^{\mathrm{V}}])$. The cross-coupling assumption forces such solutions to be at most linear in both the components of the curvature tensor K and in those of the Abelian field strength F^{V}, which amounts to vertices with three spacetime derivatives and breaks the derivative order hypothesis, so must be discarded. Regarding the solutions to the inhomogeneous equation, we act like in Ref. [44]. Using definition (14) and the first formula from (15), we get that equation (126) restricts the EL derivatives of $\bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])$ to fulfill the necessary conditions

$$\partial_{\mu_1}\left(\frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta t_{\mu_1\ldots\mu_k|\alpha}}\right) = 0, \quad \partial_{\alpha}\left(\frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta t_{\mu_1\ldots\mu_k|\alpha}}\right) = 0, \quad \partial_{\mu}\left(\frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta V_\mu}\right) = 0, \tag{127}$$

which allows us to represent them under the form

$$\frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta t_{\mu_1\ldots\mu_k|\alpha}} = \partial_{\mu_{k+1}}\partial_\beta \tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}([t],[V]), \quad \frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta V_\mu} = \partial_\nu \tilde{F}^{\nu\mu}([t],[V]), \tag{128}$$

where $\tilde{\Phi}$ displays the mixed symmetry $(k+1,2)$ and $\tilde{F}$ is antisymmetric. The derivative order assumption on the one hand forbids the dependence of $\tilde{\Phi}$ on all field derivatives and on the other hand restricts the maximum derivative order of the tensor $\tilde{F}$ to one. Meanwhile, the cross-coupling condition enforces that $\tilde{\Phi}$ effectively involves the gauge vector field whereas $\tilde{F}$ truly depends on the $(k,1)$ field. This additional requirements are symbolized by

$$\frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta t_{\mu_1\ldots\mu_k|\alpha}} = \partial_{\mu_{k+1}}\partial_\beta \tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(t,\underline{V}), \quad \frac{\delta \bar{a}_{1,0}^{\mathrm{t-V}}([t],[V])}{\delta V_\mu} = \partial_\nu \tilde{F}_1^{\nu\mu}([\underline{t}],[V]), \tag{129}$$

where the lower index "1" of $\tilde{F}$ signifies the eligible maximum derivative order. We reconstruct the Lagrangian density $\bar{a}_{1,0}^{\mathrm{t-V}}$ from the EL derivatives by means of the homotopy formula (up to some irrelevant full divergences, which can be eliminated since they are trivial in the local BRST cohomology)

$$\begin{aligned}\bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) = \int_0^1 d\tau \Big[&\big(\partial_{\mu_{k+1}}\partial_\beta \tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(\tau t,\tau\underline{V})\big) t_{\mu_1\ldots\mu_k|\alpha} \\ &+ \big(\partial_\nu \tilde{F}_1^{\nu\mu}([\tau\underline{t}],[\tau V])\big) V_\mu \Big]\end{aligned} \tag{130}$$

and transfer all the derivatives to act on t and respectively on V by neglecting the prospect divergences, which finally produces

$$\bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) = \int_0^1 d\tau \Big[\tfrac{(-)^{k+1}}{2(k+1)} \tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(\tau t,\tau\underline{V}) K_{\mu_1\ldots\mu_{k+1}|\alpha\beta} - \tfrac{1}{2}\tilde{F}_1^{\mu\nu}([\tau\underline{t}],[\tau V]) F_{\mu\nu}^{\mathrm{V}} \Big], \tag{131}$$

where K and F^V are nothing but the gauge-invariant quantities with a minimum number of derivatives of the starting free model. Acting now with γ on (131), after some tedious computation that will be not reproduced here we deduce the necessary conditions that must be fulfilled by $\tilde{\Phi}$ and $\tilde{F}$ in order to ensure the existence of solutions to the inhomogeneous equation (126)

$$\begin{aligned}\tilde{\Phi}^{\mu_1\ldots\mu_{k+1}|\alpha\beta}(\tau t,\tau\underline{V}) &= \Upsilon^{\mu_1\ldots\mu_{k+1}||\alpha\beta||\mu}\tau V_\mu, \\ \tilde{F}_1^{\mu\nu}([\tau\underline{t}],[\tau V]) &= \Upsilon^{\mu\nu\rho||\mu_1\ldots\mu_k||\alpha}\tau\partial_\rho t_{\mu_1\ldots\mu_k|\alpha}.\end{aligned} \tag{132}$$

All the tensors denoted by Υ are required to be real, constant, non-derivative and antisymmetric (where appropriate) with respect to each index group delimited by double bars, while the right-hand sides of the previous relations are now linear in τ due to their linearity in V and respectively in the first-order derivatives ∂t. Substituting (132) in (131) and performing the integration over τ, we infer

$$\bar{a}_{1,0}^{\mathrm{t-V}}([t],[V]) = \tfrac{(-)^{k+1}}{4(k+1)} \Upsilon^{\mu_1\ldots\mu_{k+1}||\alpha\beta||\mu} K_{\mu_1\ldots\mu_{k+1}|\alpha\beta} V_\mu - \tfrac{1}{4}\Upsilon^{\mu\nu\rho||\mu_1\ldots\mu_k||\alpha} F_{\mu\nu}^{\mathrm{V}} \partial_\rho t_{\mu_1\ldots\mu_k|\alpha}. \tag{133}$$

The antisymmetry of $\Upsilon^{\mu\nu\rho||\mu_1\ldots\mu_k||\alpha}$ with respect to $\{\mu,\nu,\rho\}$ combined with the differential identity satisfied by F^{V} ($\partial_{[\rho}F^{\mathrm{V}}_{\mu\nu]}\equiv 0$) allow us to represent the last term from the right-hand side of (133) in a divergence-like form

$$-\tfrac{1}{4}\Upsilon^{\mu\nu\rho||\mu_1\ldots\mu_k||\alpha}F^{\mathrm{V}}_{\mu\nu}\partial_\rho t_{\mu_1\ldots\mu_k|\alpha}=\partial_\rho\big(-\tfrac{1}{4}\Upsilon^{\mu\nu\rho||\mu_1\ldots\mu_k||\alpha}F^{\mathrm{V}}_{\mu\nu}t_{\mu_1\ldots\mu_k|\alpha}\big), \tag{134}$$

so it can be safely eliminated from $\bar{a}^{\mathrm{t-V}}_{1,0}([t],[V])$, which leaves us with

$$\bar{a}^{\mathrm{t-V}}_{1,0}([t],[V])=\tfrac{(-)^{k+1}}{4(k+1)}\Upsilon^{\mu_1\ldots\mu_{k+1}||\alpha\beta||\mu}K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}V_\mu. \tag{135}$$

Applying once more the operator γ on (135), we entail the necessary and sufficient condition for the existence of solutions to equation (126)

$$\Upsilon^{\mu_1\ldots\mu_{k+1}||\alpha\beta||\mu}\partial_\mu K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}=0. \tag{136}$$

Because the only vanishing first-order differential combinations among the components of the curvature tensor are represented by the second Bianchi identities

$$\partial_{[\mu_1}K_{\mu_2\ldots\mu_{k+2}]|\alpha\beta}\equiv 0,\quad K_{\mu_1\ldots\mu_{k+1}|[\alpha\beta,\gamma]}\equiv 0 \tag{137}$$

together with their successive contractions and in addition we work in $D\geq k{+}2\geq 6$, it can be explicitly checked that all the solutions $\Upsilon^{\mu_1\ldots\mu_{k+1}||\alpha\beta||\mu}$ satisfying the above properties generate in $\bar{a}^{\mathrm{t-V}}_{1,0}([t],[V])$ some terms that vanish identically on behalf of the first Bianchi identities $K_{[\mu_1\ldots\mu_{k+1}|\alpha]\beta}\equiv 0$ and of their successive contractions. This observation together with the similar one from the purely homogenous case allows us to conclude that there are no nontrivial solutions to the "homogeneous" equation (126) in $\mathrm{agh}=0$

$$\bar{a}^{\mathrm{t-V}}_{1,0}([t],[V])=0. \tag{138}$$

Putting together all the results obtained so far related to the piece $a^{\mathrm{t-V}}_1$, we conclude that the nontrivial expression of the first-order deformation that couples a massless tensor field with the mixed symmetry $(k,1)$ to a gauge vector field and fulfills all the imposed selection rules evolves on a (Minkowski) spacetime of dimension $D=k+2$ and reads as

$$S^{\mathrm{t-V}}_1=c_4\int\varepsilon_{\mu_1\mu_2\ldots\mu_{k+2}}\left(V^{*\mu_1}\overset{(1)}{\mathcal{F}}{}^{\mu_2\ldots\mu_{k+2}}+\tfrac{k+1}{2k}F^{\mathrm{V}\mu_1\mu_2}F^{\mu_3\ldots\mu_{k+2}}\right)d^{k+2}x. \tag{139}$$

Since it possesses nonvanishing components only of antighost number equal to 1 and respectively 0, $S^{\mathrm{t-V}}_1$ will generate a nontrivial cross-coupling Lagrangian density and will also deform some gauge transformations in order one of perturbation theory, but will not affect either the original gauge algebra or the initial reducibility functions/relations. Inspecting decomposition (40), we remark that in fact we completed the overall first-order deformation of the solution to the classical master equation, where S^{t}_1 is given in (47), S^{V}_1 is vanishing in agreement with relation (51), and $S^{\mathrm{t-V}}_1$ takes the form (139). We notice that S^{t}_1 is nonvanishing only for even values $k=2\bar{k}$ ($\bar{k}\geq 2$) and in spacetime dimensions $D^{\mathrm{t}}=4\bar{k}$, while $S^{\mathrm{t-V}}_1$ is nonzero for any arbitrary value $k\geq 4$ investigated here, but for each such value solely in the corresponding minimum allowed spacetime dimension, $D^{\mathrm{t-V}}=k+2$. We try to make the two pieces simultaneously compatible at the level of both k and D, which means to restrict k to be even, $k=2\bar{k}$, and find the solutions $\bar{k}$ to the equation $D^{\mathrm{t}}=D^{\mathrm{t-V}}$, equivalent to $4\bar{k}=2\bar{k}+2$, within the set of natural numbers greater or equal to 2. Since there are no such solutions within the above set (the only solution to the previous equation is $\bar{k} = 1$, but it does not belong to the required set), it follows that the deformations $S_1^{\rm t}$ and $S_1^{\rm t-V}$ *cannot coexist*. We maintain the cross-couplings between the spin-1 field and the tensor $(k, 1)$ (as this is after all the purpose of the entire cohomological approach developed in this section) and give up the self-interactions of the last, such that the first-order deformation of the solution to the classical master equation will coincide with the cross-coupling piece (where in addition we normalize the constant c_4 to unit)

$$S_1 = S_1^{\rm t-V} \equiv \int \varepsilon_{\mu_1\mu_2\ldots\mu_{k+2}} \left(V^{*\mu_1} \overset{(1)}{\mathcal{F}}{}^{\mu_2\ldots\mu_{k+2}} + \tfrac{k+1}{2k} F^{{\rm V}\mu_1\mu_2} F^{\mu_3\ldots\mu_{k+2}} \right) d^{k+2}x. \tag{140}$$

The last step of the deformation procedure adopted here requires the construction of the higher-order deformations. Related to the second-order deformation, we start from the second equation in (22)

$$sS_2 + \tfrac{1}{2}(S_1, S_1) = 0, \tag{141}$$

and, by means of formula (140), we compute the antibracket (S_1, S_1)

$$\tfrac{1}{2}(S_1, S_1) = s\left[-\left(\tfrac{k+1}{k}\right)^2 \tfrac{k!}{2} \int F_{\mu_1\ldots\mu_k} F^{\mu_1\ldots\mu_k} d^{k+2}x \right], \tag{142}$$

such that (141) becomes equivalent to

$$s\left[S_2 - \left(\tfrac{k+1}{k}\right)^2 \tfrac{k!}{2} \int F_{\mu_1\ldots\mu_k} F^{\mu_1\ldots\mu_k} d^{k+2}x \right] = 0. \tag{143}$$

The solution to the above equation is unique up to the general solution to the homogeneous equation $s\bar{S}_2 = 0$, which has already been analyzed in the previous step, so we can take, without affecting the generality of our approach

$$S_2 = \left(\tfrac{k+1}{k}\right)^2 \tfrac{k!}{2} \int F_{\mu_1\ldots\mu_k} F^{\mu_1\ldots\mu_k} d^{k+2}x. \tag{144}$$

We remark that the nonintegrated density of the second-order deformation reduces to its component of antighost number 0

$$a_2 \equiv a_{2,0} = \left(\tfrac{k+1}{k}\right)^2 \tfrac{k!}{2} F_{\mu_1\ldots\mu_k} F^{\mu_1\ldots\mu_k} \tag{145}$$

and hence the second order of perturbation theory will strictly contribute to the Lagrangian density. We pass to the third-order deformation, governed by the third equation from (22)

$$sS_3 + (S_1, S_2) = 0 \tag{146}$$

and, employing relations (140) and (144), we deduce that the antibracket (S_1, S_2) is vanishing, so (146) reduces to

$$sS_3 = 0. \tag{147}$$

Like in the previous step, we neglect the solutions to the last equation since they have already been considered within S_1 and take

$$S_3 = 0. \tag{148}$$

Regarding the fourth-order deformation, we begin with the equation

$$sS_4 + (S_1, S_3) + \tfrac{1}{2}(S_2, S_2) = 0, \tag{149}$$

and notice that both antibrackets are vanishing, the former due to (147) and the latter because S_2 contains no antifields, so, based on the same argument like before, we find that we can set

$$S_4 = 0. \tag{150}$$

Further, it can be proved by complete induction that we can take all the remaining higher-order deformations to vanish

$$S_i = 0, \qquad i \geq 5. \tag{151}$$

Assembling the results deduced so far via expansion (21), we conclude that *the most general, nontrivial deformation of the solution to the master equation, which is consistent to all orders of perturbation theory, complies with all the working hypotheses, and provides cross-couplings between a massless tensor field with the mixed symmetry $(k, 1)$ and a vector gauge field, is nonvanishing only at orders one and two in the deformation parameter and evolves on a spacetime of dimension $D = k + 2$*

$$\bar{S} = S + \lambda S_1 + \lambda^2 S_2, \tag{152}$$

where S is the solution to thee master equation for the starting free model, (17), particularized to $D = k+2$, whereas the other two pieces read as in (140) and (144). In order to analyze the Lagrangian formulation of the emerging coupled model, we arrange the terms from $\bar{S}$ according to their increasing values of the antighost number

$$\begin{aligned}
\bar{S} = \int \Big\{ & -\tfrac{1}{2\cdot(k+1)!}\left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha} - (k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right] - \tfrac{1}{4}F^{\mathrm{V}}_{\mu\nu}F^{\mathrm{V}\mu\nu} \\
& + \lambda\tfrac{k+1}{2k}\varepsilon_{\mu_1\mu_2\ldots\mu_{k+2}}F^{\mathrm{V}\mu_1\mu_2}F^{\mu_3\ldots\mu_{k+2}} + \lambda^2\left(\tfrac{k+1}{k}\right)^2\tfrac{k!}{2}F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k} \\
& + t^{*\mu_1\ldots\mu_k|\alpha}\left[\partial_{[\mu_1}\overset{(1)}{C}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\partial_\alpha\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}\right] \\
& + V^{*\mu_1}\left(\partial_{\mu_1}\mathcal{C} + \lambda\varepsilon_{\mu_1\mu_2\ldots\mu_{k+2}}\partial^{[\mu_2}\overset{(1)}{\eta}{}^{\mu_2\ldots\mu_{k+2}]}\right) + \overset{(k-1)}{C}{}^{*\mu_1|\alpha}\partial_{(\mu_1}\overset{(k)}{\eta}_{\alpha)} \\
& + \sum_{m=1}^{k-2}\overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha}\left[\partial_{[\mu_1}\overset{(m+1)}{C}_{\mu_2\ldots\mu_{k-m}]|\alpha} + \partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m}\alpha]}\right. \\
& \left. + (-)^{k-m+1}(k-m+1)\partial_\alpha\overset{(m+1)}{\eta}_{\mu_1\ldots\mu_{k-m}}\right] \\
& + \sum_{m=1}^{k-1}\tfrac{k-m}{k-m+2}\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}}\partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m+1}]}\Big\}d^{k+2}x.
\end{aligned} \tag{153}$$

6 Lagrangian formulation of the coupled model

Investigating the structure of (153), we deduce the Lagrangian formulation together with the specific properties of the resulting coupled gauge model.

1. Cross-couplings are allowed only on a spacetime of minimum dimension, $D = k+2$, where the massless tensor field with the mixed symmetry $(k, 1)$ displays no physical degrees of freedom and meanwhile allows no self-interactions.

2. The projection of the functional $\bar{S}$ on antighost number 0 provides the nontrivially deformed Lagrangian action that satisfies all the selection rules like

$$\begin{aligned}\bar{S}_0\left[t_{\mu_1\ldots\mu_k|\alpha},V_\mu\right] =& -\tfrac{1}{2\cdot(k+1)!}\int\left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha}-(k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right]d^{k+2}x\\ &-\tfrac{1}{4}\int F^{\mathrm{V}}_{\mu\nu}F^{\mathrm{V}\mu\nu}d^{k+2}x+\tfrac{\lambda(k+1)}{2k}\int\varepsilon_{\mu_1\ldots\mu_{k+2}}F^{\mathrm{V}\mu_1\mu_2}F^{\mu_3\ldots\mu_{k+2}}d^{k+2}x\\ &+\left(\tfrac{\lambda(k+1)}{k}\right)^2\tfrac{k!}{2}\int F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}d^{k+2}x. \qquad (154)\end{aligned}$$

Due to the fact that both F^{V} and F are linear in the first-order derivatives of the fields, the Lagrangian density from (154) comprises only "mixing-component" cross-coupling terms of order one and respectively two in the coupling constant that are quadratic in the first-order derivatives of the fields. Consequently, the associated field equations are linear in the fields and with the derivative order equal to two, like the free ones.

3. By projecting the solution $\bar{S}$ on antighost number 1, taking into account notation (33), and recalling that the ghost combination $\overset{(1)}{\eta}$ comes from the gauge parameters $\overset{(1)}{\epsilon}$, it follows that the deformed generating set of infinitesimal gauge transformations corresponding to the Lagrangian action (154) coincides with that from the free limit at the level of the $(k,1)$ sector, but is different with respect to the spin-1 sector

$$\bar{\delta}_{\overset{(1)}{\theta},\overset{(1)}{\epsilon},\xi}V_{\mu_1}=\partial_{\mu_1}\xi+\lambda\varepsilon_{\mu_1\ldots\mu_{k+2}}\partial^{[\mu_2}\overset{(1)}{\epsilon}{}^{\mu_3\ldots\mu_{k+2}]}. \qquad (155)$$

As such, we can state that only the gauge transformations of the vector field get modified during the deformation procedure, strictly in order one of perturbation theory, by a term linear in the antisymmetric first-order derivatives of the antisymmetric gauge parameters from the $(k,1)$ sector.

4. We can organize the last three terms from the right-hand side of relation (154) and rewrite the cross-coupled Lagrangian action like

$$\begin{aligned}\bar{S}_0\left[t_{\mu_1\ldots\mu_k|\alpha},V_\mu\right] =& -\tfrac{1}{2\cdot(k+1)!}\int\left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha}-(k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right]d^{k+2}x\\ &-\tfrac{1}{4}\int\bar{F}_{\mu_1\mu_2}\bar{F}^{\mu_1\mu_2}d^{k+2}x, \qquad (156)\end{aligned}$$

in terms of a deformed antisymmetric tensor of order two whose free limit is nothing but the Abelian field strength of the vector field

$$\bar{F}_{\mu_1\mu_2}=F^{\mathrm{V}}_{\mu_1\mu_2}-\tfrac{\lambda(k+1)}{k}\varepsilon_{\mu_1\ldots\mu_{k+2}}F^{\mu_3\ldots\mu_{k+2}}. \qquad (157)$$

The main property of $\bar{F}$ is expressed by its gauge invariance with respect to the deformed gauge transformations (in spite of its dependence on the first-order derivatives of t), just like its free limit

$$\bar{\delta}_{\overset{(1)}{\theta},\overset{(1)}{\epsilon},\xi}\bar{F}_{\mu_1\mu_2}=0. \qquad (158)$$

More precisely, the deformed field strength of the gauge vector field represents the quantity with the minimum number of derivatives with the property of being gauge invariant with respect to the coupled gauge transformations.

5. The projection of the functional $\bar{S}$ on agh ≥ 2 does not depend on λ (or, in other words, coincides with the projection of the solution to the classical master equation from the free limit, S) and therefore neither the Abelian gauge algebra associated with the generating set of gauge transformations corresponding to the free action nor the original reducibility functions/relations are modified at the level of the coupled model.

6. The supplementary requirement of PT invariance with respect to the deformed model eliminates the cross-couplings.

7. The problem of obtaining consistent interactions essentially depends in this case on the spacetime dimension in the sense that if we require $D > k + 2$ (for instance in order to have nonvanishing degrees of freedom in the mixed-symmetry sector), then it is not possible to couple nontrivially the two sorts of fields under study.

7 Conclusion

The final conclusion of this paper is that, under the standard hypotheses on interacting gauge field theories, there can be introduced nontrivial interactions between a massless tensor field with the mixed symmetry $(k, 1)$ and an Abelian vector field in $D = k + 2$ spacetime dimensions (mixing-component-type terms). It is nevertheless possible that the relaxation of the working hypotheses, such as giving up the invariance under spacetime translations (which is a component of the Poincaré invariance) or slightly adjusting the derivative order assumption in the sense of still asking the preservation of the differential order of the free field equations with respect to the coupled model, but renouncing the condition on the maximum derivative order of the interacting Lagrangian, might generate a broader spectrum of nontrivial cross-couplings.

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