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λ-Symmetry Reduction for Nonlinear ODEs without Lie Symmetries

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Abstract

In this paper some nonlinear ODEs without Lie symmetries will be analyzed from the perspective of the independent method of $\lambda-$symmetry reduction. The first integrals and the conserved form of the concerned ODEs are pointed out.
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1 Introduction

The notion of λ-symmetry has been introduced in [1] with the main purpose of obtaining this reduction even in the absence of standard Lie symmetries. For the first time, in [2] was established the connection between λ-symmetries and first integrals for ODEs of any arbitrary order. The main result states that any λ-symmetry yields an invariant for ODE under consideration..The extension of $\lambda-$symmetries to partial differential equations symmetries have been reported in [3, 4] (and called in that context μ-symmetries). A generalization of the concept of variational symmetry, based on λ- and μ-symmetries, has been studied in [5, 6]. This allows us to construct new methods of reduction for Euler–Lagrange equations.

The paper is organized as follows. In Sect. 2, we outline preliminary discussions regarding the notion of $\lambda-$symmetry. The determining equations for λ-functions associated to second order ODEs and the algorithmic way to construct the first integrals from known $\lambda-$symmetries of ODEs, are illustrated. In Sect. 3, the new method of $\lambda-$symmetry reduction is applied to some ODEs without Lie symmetries.The first integrals and conserved forms of these equations are pointed out. Finally, we present the conclusion in Section 4.

2 $\lambda-$Symmetries and Associated First Integrals

Consider an nth order ODE:

$$\Delta(x, u^{(n)}) = 0, \tag{1}$$

with $(x, u) \in M$, for some open subset $M \subset X \times U \simeq R^2$. We denote by $M^{(k)}$ the corresponding $k-$jet space $M^{(k)} \subset X \times U^{(k)}$, for $k \in N$. Their elements are $(x, u^{(k)}) = (x,$

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$u,\ u_x, ...u_{kx})$, where u_{ix}, $i=\overline{1,k}$ denotes the derivative of order i of u with respect to x. Eq. (1) can be locally written in the explicit form:

$$u_{nx}=\omega\left(x,u^{(n-1)}\right). \tag{2}$$

The vector field

$$A(x,u)=\partial_x+u_x\partial_u+...+\omega\left(x,u^{(n-1)}\right)\partial_{u_{(n-1)x}} \tag{3}$$

represents the vector field associated with Eq. (2).

Definition: Let $X=\xi(x,u)\partial_x+\eta(x,u)\partial_u$ be a vector field on M, and let $\lambda\in C^\infty(M^{(k)})$ be an arbitrary function. The $\lambda-$prolongation of order n of X, denoted by $X^{[\lambda,(n)]}$ is the vector field defined by:

$$X^{[\lambda,(n)]}=\xi(x,u)\partial_x+\sum_{i=0}^{n}\eta^{[\lambda,(i)]}(x,u^{(i+k-1)})\partial_{u_{ix}}, \tag{4}$$

where $\eta^{[\lambda,(0)]}=\eta(x,u)$ and

$$\begin{aligned}\eta^{[\lambda,(i)]}(x,u^{(i+k-1)}) &= D_x(\eta^{[\lambda,(i-1)]}(x,u^{(i+k-2)}))-(D_x\xi(x,u))u_{ix}+\\ &\lambda(\eta^{[\lambda,(i-1)]}(x,u^{(i+k-2)}))-\xi(x,u)u_{ix}.\end{aligned} \tag{5}$$

for $\overline{i=1,n}$.

Assuming that we know a first integral I, the determining equation for $\lambda\in C^\infty(M^{(k)})$, $k=\overline{0,n-1}$ is any solution of the PDE:

$$X^{[\lambda,(n-1)]}I=0, \tag{6}$$

where

$$X^{[\lambda,(n-1)]}=\sum_{i=0}^{n-1}(A+\lambda)^i(1)\partial_{u_{ix}}$$

denotes the $(n-1)$th-order λ-prolongation of the symmetry generator $X=\partial_u$, so that the characteristic $Q:=\eta-\xi u_x=1$.

On the other hand, the symmetry condition of (2) is given by

$$X^{[\lambda,(n)]}(u_{nx}-\omega)=0 \text{ on } u_{nx}=\omega. \tag{7}$$

This leads to the linearized symmetry condition:

$$(A+\lambda)^n(1)=\sum_{i=0}^{n-1}(A+\lambda)^i(1)\partial_{u_{ix}}\omega \tag{8}$$

When $n=2$ this condition assumes the form

$$A(\lambda)+\lambda^2=\omega_u+\lambda\omega_{u_x}. \tag{9}$$

Following [7] for second-order ODE, the procedure to construct the integral directly from λ-symmetry involves four steps:

1). Find a first integral $w(t,u,u_x)$ of $X^{[\lambda,(1)]}$, that is a particular solution of the equation

$$w_u+\lambda w_{u_x}=0; \tag{10}$$

2) Evaluate $A(w)$ and express $A(w)$ in terms of (t,w) as $A(w)=F(t,w)$;

3). Find a first integral G of $\partial_x+F(t,w)\partial_w$;

4) Evaluate $I(t,u,u_x)=G(t,w(t,u,u_x))$.

In the following sections, we will apply the algorithm presented above to some second-order ODEs which do not admit any Lie symmetry.

3 Nonlinear ODEs without Lie Symmetries

i) The Painlevé-type second-order ODE:

$$u_{2x} = \frac{u_x^2}{u} + \left(u + \frac{x}{u}\right) u_x - 1. \tag{11}$$

In this case, the determining equation for λ is (9) where A and ω are now expressed by:

$$A = \partial_x + u_x \partial_u + u_{2x} \partial_{u_x},\ \omega(x, u, u_x) = \frac{u_x^2}{u} + \left(u + \frac{x}{u}\right) u_x - 1. \tag{12}$$

We look for λ as a linear function in u_x, $\lambda = a(x, u)u_x + b(x, u)$. Inserting this expression into determining equation, we get that the functions $a(x, u)$ and $b(x, u)$ satisfy the following differential system:

$$\begin{aligned} a_u + a^2 - \frac{a}{u} + \frac{1}{u^2} &= 0, \\ a_x + b_u + 2ab - \frac{2b}{u} + \frac{x}{u^2} - 1 &= 0, \\ b_x - a - bu - \frac{bx}{u} &= 0. \end{aligned} \tag{13}$$

Solving this system with Maple Program, we find that the vector field $X = \partial_u$ is a λ-symmetry of Painlevé Eq. (21) with λ given by:

$$\lambda(x, u, u_x) = \frac{1}{u} u_x + \frac{x + u^2}{u}. \tag{14}$$

The first integral $I(x, u, u_x)$ of our model will be found by solving the equation:

$$X^{[\lambda,(1)]} I = 0 \ \Leftrightarrow I_u + \left(\frac{1}{u} u_x + \frac{x + u^2}{u}\right) I_{u_x} = 0. \tag{15}$$

It can be checked that two functionally indendent invariants of $X^{[\lambda,(1)]}$ are:

$$z = x,\ \zeta = \frac{u_x - u^2 + x}{u}. \tag{16}$$

In terms of these invariants, the original equation is written as:

$$D_z \zeta = 0, \tag{17}$$

the general solution of which is given by:

$$\zeta = k = const.,\ k \in R. \tag{18}$$

We observe that the method we have followed provides not only the first integrals, but also the conserved form of the equation without additional computation. In this example the conserved form of the resulting equation is given by:

$$D_x \left(\frac{u_x - u^2 + x}{u}\right) = 0. \tag{19}$$

Following the $\lambda-$symmetry reduction procedure, the Painlevé Eq. (21) is reduced to a first order ODE of Riccati type:

$$u_x = ku + u^2 - x, \tag{20}$$

which can be integrated and does generate solution in terms of Airy functions.

ii) Consider the nonlinear ODE [8]:

$$u_{2x} + \frac{x^2}{4u^3} + u + \frac{1}{2u} = 0. \tag{21}$$

which does not admit Lie point symmetries.

As in the previous example, we look for $\lambda-$symmetries of (21) with function $\lambda(x, u, u_x) = m(x, u)u_x + p(x, u)$. In this case, the determining equation for λ is equivalent with the differential system:

$$\begin{aligned} m_u + m^2 &= 0, \\ m_x + p_u + 2mp &= 0, \\ p_x + p^2 - m\left(\frac{x^2}{4u^3} + u + \frac{1}{2u}\right) - \frac{3x^2}{4u^4} - \frac{1}{2u^2} + 1 &= 0. \end{aligned} \tag{22}$$

Solving this system, we find that the vector field $X = \partial_u$ is a λ-symmetry of Eq. (22) with λ given by:

$$\lambda(x, u, u_x) = \frac{1}{u}u_x + \frac{x}{u^2}. \tag{23}$$

The first integral $I(x, u, u_x)$ of our model will be found by solving the equation:

$$X^{[\lambda,(1)]}I = 0 \;\Leftrightarrow I_u + \left(\frac{1}{u}u_x + \frac{x}{u^2}\right) I_{u_x} = 0. \tag{24}$$

Thereby, we find for $X^{[\lambda,(1)]}$ two functionally indendent invariants:

$$z = x,\;\; w = \frac{u_x}{u} + \frac{x}{2u^2}. \tag{25}$$

In terms of these invariants, the original equation becomes:

$$D_x w + w^2 + 1 = 0, \tag{26}$$

The general solution of the reduced equation (17) is given in implicit form by:

$$\arctan(w) + x = C, C \in R. \tag{27}$$

Consequently, the concerned equation (21) can be express in the conserved form:

$$D_x\left(\arctan\left(\frac{u_x}{u} + \frac{x}{2u^2}\right) + x\right) = 0. \tag{28}$$

4 Conclusions

In this paper, reductions of several second-order ODEs through λ-symmetry approach are presented. Because this technique provides an algorithm to construct the first integral directly from λ-symmetry, it represents an alternative approach to the problem of determining first integrals and associated integrating factors. The method may simplify the computations derived by other methods.

Several examples show how the method works in practice. They includ two models (11) and (21) that have no Lie point symmetries. For the mentioned models we determine the λ-symmetries linear in u_x, more exactly (14) and (23) and the corresponding first integrals (16) and (25) respectively. The conserved forms of the models under consideration are also highlighted.

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References

[1] Muriel, C. and J. L. Romero. "C^∞-symmetries and reduction of equations without Lie point symmetries." J. Lie theory 13 (1) (2003): 167.

[2] Muriel, C., and J. L. Romero. "Integrating Factors and λ–Symmetries." Journal of Nonlinear Mathematical Physics 15 (sup3) (2008): 300.

[3] Gaeta, G. and P. Morando. "On the geometry of lambda-symmetries and PDE reduction." Journal of Physics A: Mathematical and General 37 (27) (2004): 6955.

[4] Cicogna, G., G. Gaeta, P. Morando. "On the relation between standard and μ-symmetries for PDEs." Journal of Physics A: Mathematical and General 37 (40) (2004): 9467.

[5] Muriel, C., J. L. Romero, and P. J. Olver. "Variational C^∞-symmetries and Euler–Lagrange equations." Journal of Differential Equations 222 (1) (2006): 164.

[6] Cicogna, G. and G. Gaeta. "Noether theorem for μ-symmetries." Journal of Physics A: Mathematical and Theoretical 40 (39) (2007): 11899.

[7] Muriel C. and J.L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, J. Phys. A: Math. Theor. 42 (2009) 365207.

[8] Muriel, C. and J. L. Romero. "New methods of reduction for ordinary differential equations." IMA Journal of Applied mathematics 66 (2) (2001): 111.