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Similarity reduction to wave type equations. Application to the $1D$ Rossby waves in stratified oceans

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Abstract

We present a practical way of generating the largest class of $(1+1)$-dimensional second order partial differential equations of a given form which can be reduced to an imposed ordinary wave type equation. Lie symmetry and similarity reduction procedures will be used. Homogeneous, armonic oscilator and Rossby waves will be considered as important examples. The last case is peculiary important describing the equatorial trapped waves generated in a continuously stratified ocean. A simpler first order differential equation will be proposed for replacing the nonlinear second order Rossby equation.

Keywords: Nonlinear dynamical systems, Lie symmetries, Similarity reduction procedure, Rossby type symmetries.

1 Introduction

A rich variety of complex phenomena occurring in many physical fields, including the atmospheric dynamics, are described by *linear differential equations* which allow a simple handling of the system's evolution. Although, the linearized models often do not adequately describe the dynamics of the processes as a whole and it is very simple to shift the system to a region in which the linear behavior is no longer valid. This is why, in order to capture the real behavior, to accurately estimate and control the complex systems in all their regimes, *nonlinear models* must be defined. In this case, the linear differential equations could appear as approximations to the nonlinear systems, valid under restricted conditions.

The price to be paid when nonlinearity is taken into consideration appears in the investigation of the exact solutions of the attached equations. There are not standard methods for solving nonlinear differential equations, they are usually depend on the form of the equations and on their particular symmetries. Many interesting nonlinear models have been proposed over the last years [1] and a lot of methods have been developed in order to find solutions of equations describing these nonlinear phenomena. Some of the most important methods [2] are the inverse scattering method [3], the Darboux and Bäcklund transformations [4], the Hirota bilinear method [5], the Lie symmetry analysis [6, 7, 8], etc. By applying these methods, many types of specific solutions have been obtained. For example, solitary waves or solitons, which have no analogue for linear partial differential equations, are very important for the nonlinear dynamical systems.

In this paper we shall concentrate our attention to the *Lie group method.* It is well-known that this method is a powerful and direct approach to construct many types of exact solutions of nonlinear differential equations, such as soliton solutions, power series solutions, fundamental solutions [9, 10], and so on. The existence of the operators associated with the Lie group of infinitesimal transformations allows the reduction of equations to simpler ones. The similarity reduction method for example is an important way of transforming a $(1+1)-$dimensional partial differential equation into an ordinary differential one. We shall concretely consider *the inverse symmetry problem* [11] and we shall generate the largest class of second order $(1+1)-$dimensional partial differential equations which generalize the ordinary, wave type, differential equation describing the equatorial trapped waves generated in a continuously stratified ocean. Practically, four types of waves appears in this case and have to be found among the solutions of the equation: Kelvin waves, Rossby waves, inertia-gravity waves and mixed Rossby-gravity waves. In the Boussinesq approximation and on an equatorial β-plane, the equation which describe the m-th oscillation mode of the wave's vertical velocity $\phi_m(z)$ on the direction z has the form [12]:

$$\ddot{\phi}(z) + \frac{N_m^2(z)}{C_m^2}\phi_m(z) = 0 \tag{1}$$

One consider for the velocity the boundary conditions:

$$\phi_m(z=-H) = 0 \text{ (ocean floor)}$$
$$\phi_m(z=0) = 0 \text{ (ocean surface)}$$

In the equation (1), C_m is a constant and $N_m(z)$ represents the "buoyancy frequency". Measurements made during El Niño events by 2 Japonese stations in the equatorial Pacific [12] lead to the conclusion that the buoyancy frequency $N_m(z)$ has strong variations with the water depth close to the surface, with an averaged value around $\overline{N_m(z)} = 2\cdot 10^{-4} m\cdot s^{-1}$ for $z \in [0, 300]$ m and practically vanishes for higher depths. So, the equation (1) can be linearized in one of the following forms:

$$\ddot{\phi}(z) = 0; z \geq 300 \tag{2}$$

$$\ddot{\phi}(z) + k^2\phi(z) = 0; z \in [0, 300]; k \equiv \frac{\overline{N_m(z)}}{C_m} = const. \tag{3}$$

In this paper, we shall consider the two previous wave type equations and we shall see how they can be extended towards $(1+1)-$second order differential equations which by similarity reduction take the form the initial ordinary wave equations have.

The outline of this paper is as follow: after this introductory notes, in Section 2, we shall obtain the general determining system for a chosen class of $(1+1)-$dimensional models. The system will be generated by using the Lie symmetry approach and by asking for an imposed form of the similarity reduction equation. As an application, in Section 3 we shall generate a class of $(1+1)-$second order differential equations which by similarity reduction come to the wave forms (2) and, respectively, (3). In Section 4 the inverse way will be followed. We shall try to use the considered form of $(1+1)$ equation as a simpler mathematical model of a more complicated equation describing the Rossby waves. We shall compute the form of the second order partial differential equation which admit an imposed form of symmetry, specific for the $(1+1)$ dimensional Rossby type equation. So we shall be able to replace the study of this last strongly nonintegrable equation with a simpler class of equations observing similar symmetries. Some concluding remarks will end the paper.

2 Determining equations for the Lie symmetry group

Let us consider the class of general dynamical systems described in a $(1+1)$-dimensional space (x,t) by a second order partial differential equation of the form:

$$u_t = A(x,t)u_{2x} + B(x,t)u_x + C(x,t)u \Leftrightarrow \Omega(x,t,u,u_x,u_t,u_{2x}) = 0 \tag{4}$$

Our aim is to select from the general dynamical systems described by (4) the class of differential equations which admit a similarity reduction to wave type equations of the form (2) and (3). As feed-back, the solution of the wave equations will be used in order to obtain a solution for (4). The procedure that will be followed firstly implies to obtain the system of determining equations for the Lie symmetry group of (4). Then, an additional system of partial differential equations will be generated by imposing that (4) possess a reduced similarity equation of the wave type. Finally, this last system and the Lie determining equations will be solved and coefficient functions $A(x,t)$, $B(x,t)$, $C(x,t)$ will be obtained.

In this section we shall apply the Lie symmetry approach for the equation (4). Let us consider a one-parameter Lie group of infinitesimal transformations:

$$\bar{x} = x + \varepsilon\xi(t,x,u),\ \bar{t} = t + \varepsilon\varphi(t,x,u),\ \bar{u} = u + \varepsilon\eta(t,x,u) \tag{5}$$

with a small parameter $\varepsilon \ll 1$. The Lie symmetry operator associated with the above group of transformations can be written as follows:

$$U(x,t,u) = \varphi(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{6}$$

The second order equation $\Omega(x,t,u,u_x,u_t,u_{2x}) = 0$ of the form (4) is invariant under the action of the operator (6) if and only if the following condition [6] is verified:

$$U^{(2)}(\Omega)\mid_{\Omega=0} = 0 \tag{7}$$

where $U^{(2)}$ is the second extension of the generator (6). A concrete computation shows that the coefficient functions from (4) and (6), $A(x,t), B(x,t), C(x,t), \varphi(x,t,u), \xi(x,t,u), \eta(x,t,u)$, must satisfy the equation:

$$(\varphi A_t + \xi A_x)u_{2x} + (\varphi B_t + \xi B_x)u_x + \varphi C_t u + \xi C_x u + C\eta + B\eta^x - \eta^t + A\eta^{2x} = 0 \tag{8}$$

Coefficient functions η^x, η^t, η^{2x} appear in the process of extension of U towards $U^{(2)}$ and their general expressions are given in [6]. Using these expressions in (8) and asking for the vanishing of the coefficients of each monomial in the derivatives of $u(t,x)$, we obtain the following differential system:

$$\begin{gathered}\varphi_x = 0;\ \varphi_u = 0;\ \xi_u = 0;\ \eta_{2u} = 0;\ \varphi A_t + \xi A_x + A\varphi_t - 2A\xi_x = 0;\\ -\varphi B_t - \xi B_x + B\xi_x - \xi_t - B\varphi_t - 2A\eta_{xu} + A\xi_{2x} = 0\\ -\varphi C_t u - \xi C_x u - C\eta - B\eta_x + \eta_t + C\eta_u u - \varphi_t C u - A\eta_{2x} = 0\end{gathered} \tag{9}$$

The first four equations of the system (9) lead, for coefficient functions $\varphi(x,t,u), \xi(x,t,u), \eta(x,t,u)$, to the following reduced dependences:

$$\varphi = \varphi(t),\ \xi = \xi(x,t),\ \eta = M(x,t)u \tag{10}$$

Consequently, the remaining equations of (9) become:

$$\varphi A_t + \xi A_x + A\varphi_t - 2A\xi_x = 0;$$

$$-\varphi B_t - \xi B_x + B\xi_x - \xi_t - B\varphi_t - 2AM_x + A\xi_{2x} = 0 \tag{11}$$

$$-\varphi C_t - \xi C_x - BM_x + M_t - \varphi_t C - AM_{2x} = 0$$

with 6 unknown functions: $A(x,t), B(x,t), C(x,t)$ provided by the evolutionary equation (4) and $\varphi(t)$, $\xi(x,t)$, $\eta(x,t,u)$ introduced by the symmetry group of transformations (5) and described by the relations (10).

Let us consider now the similarity reduction procedure. In this section, some particular choices for the system (11) will be considered. We shall find equations describing concrete dynamical systems which admit reduction through the similarity procedure to ordinary wave type equations of the form (2) and (3). For the moment, we restrict the forms (10) of the infinitesimals $\varphi(t), \xi(x,t), \eta(x,t,u)$ to the following separable expressions:

$$\varphi = \varphi(t),\ \xi = \xi(x,t) = \xi_1(x)\xi_2(t),\ \eta = M(x,t)u = M_1(x)M_2(t)u \tag{12}$$

The Lie operator (6) becomes:

$$U(x,t,u) = \varphi(t)\frac{\partial}{\partial t} + \xi_1(x)\xi_2(t)\frac{\partial}{\partial x} + M_1(x)M_2(t)u\frac{\partial}{\partial u} \tag{13}$$

The general expressions of the invariants could be found if we should consider the characteristic equations associated with the new generator (13). These equations are:

$$\frac{dt}{\varphi(t)} = \frac{dx}{\xi_1(x)\xi_2(t)} = \frac{du}{M_1(x)M_2(t)u} \tag{14}$$

By integrating the previous equations, two invariants are obtained with the following expressions:

$$I_1 = \exp\left(\int \frac{1}{\xi_1(x)}dx - \int \frac{\xi_2(t)}{\varphi(t)}dt\right),\ I_2 = u\exp\left(-\frac{M_2(t)}{\xi_2(t)}\int \frac{M_1(x)}{\xi_1(x)}dx\right) \tag{15}$$

In the similarity reduction procedure two similarity variables have to be considered:

$$I_1 = z,\ I_2 = \phi(z) \tag{16}$$

The invariants (16) allow us, by an appropriate change of coordinates $\{u,x,t\} \to \{z,\phi(z)\}$, to reduce the initial $(1+1)$ dimensional equation (4) to an ordinary differential equation of the form:

$$\Omega'[z,\phi(z),\dot{\phi}(z),...] = 0 \tag{17}$$

3 Wave type symmetries

3.1 Homogeneous wave type equation

Our aim is now to select from the general dynamical systems described by (4) the class of differential equations for which the equation (17) can be reduced at a wave type equation of the form (2):

$$\frac{d^2\phi(z)}{dz^2} = 0 \Leftrightarrow \phi(z) = az + b \tag{18}$$

where a and b are arbitrary constants. As we already mentioned, this equation describes the waves propoagating in the ocean waters, at depth greater than 300 meters.

The previous solution, written in terms of the initial variable (x,t), leads to the following form of the solution $u(x,t)$ of (4):

$$u(x,t)=\left[a\exp\left(\int\frac{1}{\xi_1(x)}dx-\int\frac{\xi_2(t)}{\varphi(t)}dt\right)+b\right]\exp\left(\frac{M_2(t)}{\xi_2(t)}\int\frac{M_1(x)}{\xi_1(x)}dx\right) \tag{19}$$

For convenience reasons, we shall impose the following relations to be valid:

$$\frac{\xi_2(t)}{\varphi(t)}=q,\ \frac{M_2(t)}{\xi_2(t)}=v,\ \int\frac{1}{\xi_1(x)}dx\equiv P(x),\int\frac{M_1(x)}{\xi_1(x)}dx\equiv R(x) \tag{20}$$

with q,v arbitrary constants.

In terms of notations (20), the infinitesimals (12) and the solution (19) become:

$$\varphi=\varphi(t),\ \xi=q\frac{\varphi(t)}{\dot{P}(x)},\ \eta=qv\frac{\varphi(t)\dot{R}(x)}{\dot{P}(x)}u \tag{21}$$

$$u(x,t)=[a\exp(P(x)-qt)+b]\exp(vR(x)) \tag{22}$$

The solution (22) must verify the equation (4) which describes the analyzed model. This condition generates a differential system of the form:

$$\begin{aligned}0&=q+2vA(x,t)\dot{R}(x)\dot{P}(x)+v^2A(x,t)[\dot{R}(x)]^2+A(x,t)\ddot{P}(x)+A(x,t)[\dot{P}(x)]^2+\\&+vA(x,t)\ddot{R}(x)+B(x,t)\dot{P}(x)+vB(x,t)\dot{R}(x)+C(x.t)\\0&=v^2A(x,t)[\dot{R}(x)]^2+vA(x,t)\ddot{R}(x)+vB(x,t)\dot{R}(x)+C(x,t)\end{aligned} \tag{23}$$

For an unitary analysis, it is necessary to describe the differential system (11), obtained in the previous subsection, in terms of the functions $P(x)$ and $R(x)$ introduced by (20). Using the expressions (21) we obtain the following differential system:

$$\begin{gathered}\varphi A_tP_x^2+q\varphi A_xP_x+\varphi_tAP_x^2+2q\varphi AP_{2x}=0\\\varphi B_tP_x^4+q\varphi B_xP_x^3+q\varphi BP_{2x}P_x^2+q\varphi_tP_x^3+\varphi_tBP_x^4+\\+2vq\varphi AR_{2x}P_x^3-2vq\varphi AR_xP_{2x}P_x^2+q\varphi AP_{3x}P_x^2-2q\varphi AP_xP_{2x}^2=0\\\varphi C_tP_x^4+q\varphi C_xP_x^3+qv\varphi BR_{2x}P_x^3-qv\varphi BR_xP_{2x}P_x^2+\varphi_tCP_x^4+vq\varphi AR_{3x}P_x^3-\\+vq\varphi AR_xP_{3x}P_x^2-2vq\varphi AR_{2x}P_{2x}P_x^2++2qv\varphi AR_xP_xP_{2x}^2-qv\varphi_tR_xP_x^3=0\end{gathered} \tag{24}$$

Conclusion: Our problem is to find the class of $(1+1)$ evolutionary equations of type (4) which could be reduced by the similarity approach to an ordinary wave type equation. Solving this problem is equivalent with searching the solutions of the system described by equations (23) and (24).

Remark 1: The system (23)-(24) can be solved following two paths: (i) choosing a concrete dynamical system, that is to say concrete expressions for the functions $A(x,t)$, $B(x,t)$, $C(x,t)$ and trying to find out if this equation admits or not solution of the type (22). Now the unknown functions of the system are $\varphi(t)$, $P(x)$, $R(x)$ defined by (20); (ii) considering $A(x,t)$, $B(x,t)$, $C(x,t)$ as unknown functions and choosing $\varphi(t)$, $P(x)$, $R(x)$. This is the way we shall follow in the next section.

Remark 2: In the case (ii) the general solutions obtained by computational method can be expressed as:

$$A(x,t) = F\left(\frac{qt - P(x)}{q}\right)\exp\left[G(x)\right]$$
$$B(x,t) = \frac{\left[-2F\left(\frac{qt-P(x)}{q}\right)\left(\frac{1}{2}[\dot{P}(x)]^2 + v\dot{R}(x)\dot{P}(x) + \frac{1}{2}\ddot{P}(x)\right)\exp\left[-G(x)\right] - q\right]}{\dot{P}(x)}$$
$$C(x,t) = \frac{v\left[F\left(\frac{qt-P(x)}{q}\right)\left[\dot{R}(x)\ddot{P}(x) + \dot{P}(x)(-\ddot{R}(x) + \dot{R}(x)(v\dot{R}(x) + \dot{P}(x)))\right]\exp^{[-G(x)]} + q\dot{R}(x)\right]}{\dot{P}(x)} \tag{25}$$

where

$$G(x) = -\int^{x} \frac{2(D^{(2)})(P)(a)\varphi\left(\frac{P(a)+qt-P(x)}{q}\right)q + [D(P)(a)]^2 D(\varphi)\left(\frac{P(a)+qt-P(x)}{q}\right)}{D(P)(a)\varphi\left(\frac{P(a)+qt-P(x)}{q}\right)q}da \tag{26}$$

These solutions are valid for arbitrary constants q, v and for an arbitrary function $F\left(\frac{qt-P(x)}{q}\right)$.

Conclusion: The equation (4) can be seen as a generalization of the wave type equation describing the high depth waves in the ocean water if the coefficients have the form (25). As an example, a possible equation of this form is the Foker-Planck equation of the form:

$$u_t = u_{2x} + xu_x + u \tag{27}$$

3.2 Harmonic oscillators

Let us consider now that the similarity reduction equation is an ordinary oscillator type equation of the form (3):

$$\frac{d^2\phi(z)}{dz^2} + k^2\phi(z) = 0 \tag{28}$$

It has the solution:

$$\phi(z) = a\sin(kz) + b\cos(kz) \tag{29}$$

Here k, a and b are arbitrary constants.

To write down the previous solution in terms of the initial variable $u(x,t)$ means that the solution of (4) should have the form:

$$u(x,t) = a\exp\left(\frac{M_2(t)}{\xi_2(t)}\int\frac{M_1(x)}{\xi_1(x)}dx\right)\sin\left(\sqrt{k}\exp\left(\int\frac{1}{\xi_1(x)}dx - \int\frac{\xi_2(t)}{\varphi(t)}dt\right)\right) + \tag{30}$$
$$+ b\exp\left(\frac{M_2(t)}{\xi_2(t)}\int\frac{M_1(x)}{\xi_1(x)}dx\right)\cos\left(\sqrt{k}\exp\left(\int\frac{1}{\xi_1(x)}dx - \int\frac{\xi_2(t)}{\varphi(t)}dt\right)\right)$$

For convenience reasons, we shall impose again the following relations to be valid:

$$\frac{\xi_2(t)}{\varphi(t)} = q,\ \frac{M_2(t)}{\xi_2(t)} = v,\ \int\frac{1}{\xi_1(x)}dx \equiv P(x), \int\frac{M_1(x)}{\xi_1(x)}dx \equiv R(x) \tag{31}$$

with q, v arbitrary constants.

In terms of notations (31), the infinitesimals (12) and the solution (30) become:

$$\varphi = \varphi(t),\ \xi = q\frac{\varphi(t)}{\dot{P}(x)},\ \eta = qv\frac{\varphi(t)\dot{R}(x)}{\dot{P}(x)}u \tag{32}$$

$$u(x,t) = [a\sin(k\exp(P(x)-qt)) + b\cos(k\exp(P(x)-qt))]\exp(vR(x)) \tag{33}$$

The solution (33) must verify the equation (4) which describes the analyzed model. This condition generates the vanishing of the coefficient function $A(x,t)$ and two other differential equations of the form:

$$\begin{aligned} A(x,t) &= 0 \\ q + B(x,t)\dot{P}(x) &= 0 \\ vB(x,t)\dot{R}(x) + C(x,t) &= 0 \end{aligned} \tag{34}$$

For an unitary analysis, it is again necessary to describe the general differential system (11) obtained in the previous section, in terms of the functions $P(x)$ and $R(x)$ introduced by (31). Taking into account the equations (32), we obtain the following differential system:

$$\begin{aligned} 0 &= \varphi B_t P_x^4 + q\varphi B_x P_x^3 + q\varphi B P_{2x} P_x^2 + q\varphi_t P_x^3 + \varphi_t B P_x^4 \\ 0 &= \varphi C_t P_x^4 + q\varphi C_x P_x^3 + qv\varphi B R_{2x} P_x^3 - qv\varphi B R_x P_{2x} P_x^2 + \\ &+ \varphi_t C P_x^4 - qv\varphi_t R_x P_x^3 \end{aligned} \tag{35}$$

The system (34)-(35) can be solved following two paths: (i) by choosing a concrete dynamical system, that is to say concrete expressions for the functions $B(x,t)$, $C(x,t)$ and trying to find out if this equation admits or not solution of the type (33). Now the unknown functions of the system are $\varphi(t)$, $P(x)$, $R(x)$ defined by (31); (ii) by considering $B(x,t)$, $C(x,t)$ as unknown functions and by choosing $\varphi(t)$, $P(x)$, $R(x)$.

This second case is the way we are interested in to follow and, in this case, the general solutions obtained by computational way can be expressed as:

$$B(x,t) = \frac{-q}{\dot{P}(x)},\ C(x,t) = \frac{vq\dot{R}(x)}{\dot{P}(x)} \tag{36}$$

or in terms of the coefficient functions $\varphi(t)$, $\xi(x,t)$, $M(x,t)$ which appear in the general Lie symmetry operator (13), in the equivalent forms:

$$B(x,t) = \frac{-\xi(x,t)}{\varphi(t)} = -q\xi_1(x),\ C(x,t) = \frac{M(x,t)}{\varphi(t)} = qvM_1(x) \tag{37}$$

4 Rossby type symmetries

The equation for coupled gravity, inertial and Rossby waves in a rotating, stratified atmosphere using the β-plane approximation (which simplifies the spherical geometry whilst retaining the essential dynamics) and the Boussinesq approximation which filters out higher frequency acoustic waves can be written in $(2+1)-$dimensions in the form [13]:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_y=-\beta\frac{\partial u_y}{\partial x} \tag{38}$$

As we mentioned, this equation describes the coupling between the inertial, the gravity and the Rossby waves, but also the shallow water in an ocean of depth H. It was proven that reducing the model to $(1+1)-$dimensions, (x,t), it admits some very simple Lie symmetries of the form:

$$\varphi(t)=ct+c_1,\ \xi(x,t)=cx+c_2,\ \eta(u)=-3cu \tag{39}$$

with c, c_1, c_2 arbitrary constants. Despite this fact, it is still difficult to find explicit solutions for the equation (38). This is why, we shall consider another approach: we shall impose the Lie symmetries (39) to our general equation (4) and we shall try to find the class of equations which observe them. This means that we have in fact to impose the Rossby symmetries (39) to the system (11). It will take the form:

$$\begin{aligned}(ct+c_1)A_t+(cx+c_2)A_x+3cA=0\\ -(ct+c_1)B_t-(cx+c_2)A_x-2cB=0\\ -(ct+c_1)C_t-(cx+c_2)C_x-cC=0\end{aligned} \tag{40}$$

with the unknown functions $A(x,t)$, $B(x,t)$, $C(x,t)$.

This system admits the solutions:

$$\begin{aligned}A(x,t)&=\frac{F\left(x(ct+c_1)-c_2t\right)}{(ct+c_1)^3}=\frac{F\left(x\varphi(t)-c_2t\right)}{[\varphi(t)]^3}\\ B(x,t)&=\frac{G\left(x(ct+c_1)-c_2t\right)}{(ct+c_1)^2}=\frac{G\left(x\varphi(t)-c_2t\right)}{[\varphi(t)]^2}\\ C(x,t)&=\frac{H\left(x(ct+c_1)-c_2t\right)}{ct+c_1}=\frac{H\left(x\varphi(t)-c_2t\right)}{\varphi(t)}\end{aligned} \tag{41}$$

with F, G, H arbitrary functions of their arguments. These expressions give us equations of the form (4) which are equivalent from the point of view of their symmetries with the Rossby equation.

Let us consider in (41) the particular case: $c=1$, $c_1=c_2=0$.Therefore, the master equation (4) becomes:

$$u_t=\frac{F(tx)}{t^3}u_{2x}+\frac{G(tx)}{t^2}u_x+\frac{H(tx)}{t}u \tag{42}$$

It is already known that the solution of $(1+1)$ Rossby equation has, in the previous particular case, the following similarity solution:

$$u(x,t)=mx^3+st^3-3sx^2t-\left(3m+\frac{\beta}{2}\right))xt^2 \tag{43}$$

with β, m, s arbitrary constants.

The further question under discussion is: what conditions must be imposed to the class of evolutionary equations (42) in order to accept (43) as a solution?

A concrete computation generates the conditions:

$$F(tx)\equiv 0,\ G(tx)=tx,\ H(tx)\equiv 3\Leftrightarrow A(x,t)\equiv 0,\ B(x,t)=\frac{x}{t},\ C(x,t)=\frac{3}{t} \tag{44}$$

So, an equation simpler as the $(1+1)$ Rossby equation, with the same symmetry group is:

$$tu_t = xu_x + 3u \tag{45}$$

5 Conclusions

The problem of finding exact solutions for nonlinear differential equations plays an important role in the study of nonlinear dynamics. There are many ways of tackling with it. One of them is based on the Lie symmetry method. This method supposes to find the symmetries of the system and, on this basis, to determine the general or some particular solutions of the equations. There is a direct approach in which the symmetries of a given equation are obtained, but also an inverse problem has been formulated [11]. A step forward for this latter approach is represented by the use of similarity reduction, a procedure which allows the reduction of the number of degrees of freedom and, by that, it simplifies the problem of solving the equation. This paper used this approach and determined a class of $(1+1)$ dimensional second order differential equations which can be reduced to ordinary wave-type equations with simple solutions. Using the Lie symmetry and the similarity reduction procedures, some particular cases of the equation (4) arise as good candidates of equations which could be used as generalization of the linear wave type equations describing complex atmospheric phenomena. Moreover, following our method, we were also able to use the $(1+1)$ differential equation (4) as a possible substitute of a more complicated, third order differential equation, which is currently used in describing the propagation of special type of waves in the ocean waters, the nonintegrable Rossby equation. We found the equation (45) which has the same symmetry group and the same similarity solution as the $(1+1)$ Rossby equation. The paper is important both by these results, but also as a methodological approach in finding class of differential equations which could generalize simpler linear wave type equations. The similarity reduction allows to recover the solution of a most complicated problem, defined in a space with more than one dimensions, starting from a particular form of solution of the reduced equation. It is an important approach considering that the most part of the atmospheric phenomena can be described by linear equations in a first approximation only. The nonlinearity is prevalent in such complex systems. We tackled out a particular case, considering that the coefficient functions appearing in the symmetry operators are separable. The problem can be extended for other cases, too.

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