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Mean sojourn time fraction in frozen, homogenous, isotropic and self similar electrostatic turbulence

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Abstract

In the framework of the general theory of random Hamiltonian dynamical systems the relation between the mean sojourn time fraction in an arbitrary domain and the projector to the subspace of the invariant function is established. In the particular case of the random formal Hamiltonian system related to the electrostatic drift motion in homogenous magnetic field, the limiting case, when the electrostatic potential is not differentiable is studied. By this result the general form of the projector to invariant states is established in the case of homogenous, isotropic and self similar electrostatic turbulence. We prove that with probability one all of the trajectories are either unbounded (that coresponds to sub, normal or super diffusion) either are degenerated to a single point, that means that in the physical case when the self similarity is approximate only the trajectories are closed curves with small area. Implications on the electron anomalous transport in tokamak are discussed.

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1 Introduction

1.1 The physical aspects

The problem of prediction of the statistical properties of the trajectories of particles moving in a random electromagnetic field with given statistical properties is one of the key problems in the non equilibrium statistical mechanics, with implication on the future technologies related to the magnetic confinement of the plasma, astrophysics[[5]].

Our problem is related to a generic problem that appears in various studies in fluid and plasma turbulence [2]: The motion of a particle in a two-dimensional incompressible static stochastic velocity field. Also in this case the equations can be written like random Hamiltonian system with one degree of freedom. Similar mathematical formalism describe simplified models stochastic magnetic field line dynamics. The applications concern a large class of physical processes and, in particular, the particle and energy transport in hot magnetized plasmas.

The problem of describing the statistical properties of the solutions of Eq.(1) was considered in the works [5], [3], mainly because it is a starting point in the study of the non autonomous generalization of Eq.(1), with long range correlations. The mathematically equivalent problem of the stochastic magnetic field line dynamics was studied in [4], [6], [7].

In realistic situations the random electromagnetic field cannot be modelled with a temporal white noise, consequently the particle motion cannot be modelled with classical stochastic differential equations in the Itô formalism. Moreover, due to the long range time correlations [[5]] the Corsin approximation cannot be used. For the study of the transversal particle transport in constant magnetic field, with random electric field with polynomial decay of the time correlation function, in the zero Larmor radius approximation, a new method was elaborated [[3], [4]], for the study of the diffusive behavior. In this article this problem, will be studied, in the opposite extreme limit of white noise model: the model with frozen turbulence approximation. This extreme limit is important because it is also a starting point of the DCT method.

The problem studied here, in the framework of the infinite ergodic theory, [[13]], is to obtain information on the mean sojourn time of a family of particles in a given domain, when the distribution of the initial position is given. This problem has a particular case: to estimate the mean sojourn time fraction, that means the time fraction spent in the domain where from the initial condition is randomly selected according to given initial distribution. This problem appears naturally for the application in magnetic confinement fusion, where it is desirable that a large fraction of the trajectories to have bounded trajectories.

The computational complexity of this class of problems higher then the lattice QCD with fermions. It is natural to consider as the analogue of the "Ising model" of non equilibrium statistical mechanics.

In the framework of the infinite ergodic theory, when the volume of the phase space is infinite , the mean sojourn time give a partial statistical information about geometry of the trajectories. We will prove that in very general situations, the problem of mean sojourn time fraction can be expressed as suitable matrix element of projection operator to the invariant states, that appears in the von Neumann mean ergodic theorem. In the particular case of the problem of particle motion in a constant magnetic field, in the frozen turbulence and drift approximation, the mean sojourn time fraction is related to the statistical properties of the equipotential surfaces, that again can be expressed and reformulated in the term of projector on subspace of measurable functions with respect to the $\sigma-$ algebra generated by random electrostatic potential. By this reformulation the problem of the mean sojourn time fraction can be treated in the extreme, but solvable limit when the electric field is *self similar, homogenous and isotropic (SHI)*, which is only Hölder continuos but not differentiable. A class of examples of SHI random fields, the fractional Brownian field, was already studied in [24]

We prove that under the SHI assumptions the mean sojourn time fraction of a trajectory is either 0 *(in this case the particle has unbounded trajectory) , either the particle is completely trapped in a single point.*

Recall that in physical situations, when the random field is not exactly self similar at small distances less than a critical distance δl*, this means that the area enclosed by a closed trajectory has the order of magnitude* $O(\delta l^2)$ *. This property is an interesting physical manifestation of the fractal nature of SHI random fields.*

Consequently if we consider the problem of the particle transport in homogenous, isotropic random electric field that is obtained by a regularization of the SHI model, with the very short wavelength components filtered out, the mean sojourn time fraction is related to the geometry of the level surfaces of an SHI random field with continuous realizations. We prove that almost surely the connected components of the level surfaces enclose infinite or zero area.

1.2 The physical problem

In order to illustrate the our initial problem, we consider an one degree of freedom, autonomous, random Hamiltonian systems. The typical interesting case is the charged particle motion, in the zero Larmor radius approximation, transversal to constant magnetic field $\mathbf{B}$, under the effect of the random, static, electric potential $\Phi_\omega(\mathbf{x})$. The potential that contains a set of random parameters, generically denoted by ω that are elements of a probability space Ω. More exactly we denote by $(\Omega, \mathcal{A}, P)$ the probability space and its $\sigma-$algebra related to the realizations of the random electric potential in $\mathbb{R}^2$, $\mathbf{x} \equiv (x_1, x_2) \in \mathbb{R}^2$, are Cartesian coordinates perpendicular to the magnetic field . Then for $\omega \in \Omega$ the scalar function $\Phi_\omega(\mathbf{x})$ is the random potential. In this limit the dynamics is specified by an one degree of freedom random Hamiltonian system, with Hamiltonian function $H_\omega(\mathbf{x}) = \frac{c}{|\mathbf{B}|}\Phi_\omega(\mathbf{x})$ see e.g. [5]

$$\frac{dx_{\omega,i}(t)}{dt} = e_{i,j}\frac{c}{|\mathbf{B}|}\frac{\partial \Phi_\omega(\mathbf{x}_\omega(t))}{\partial x_{\omega,j}} \tag{1}$$

Here $e_{i,j}$ is the two dimensional Levi-Civita symbol. In this case the trajectories are exactly the level sets $\Phi_\omega(\mathbf{x}) = E$. In typical cases the mean sojourn time of a family of trajectory in a finite domain will be zero when the level sets are open.

In this article the main result is that under SHI assumptions the mean sojourn time is either zero, that means that the with probability one the level sets are open curves, going to infinity, either.

The first problem is related to the fact that under SHI assumptions $\Phi_\omega(\mathbf{x})$ is non differentiable a.s. This aspect will be circumvented by reformulation of the initial problem in the term of statistical geometry of the level surfaces of $\Phi_\omega(\mathbf{x})$.

2 Intuitive approach: mean sojourn time, the von Neumann mean ergodic theorem, conditional expectation values

2.1 Deterministic case

2.1.1 Mean sojourn time fraction and the projector to the subspace of invariant states

In order to define the mean sojourn time the general framework of the Hamiltonian dynamics with random perturbations of the Hamiltonian function, we consider first an autonomous deterministic Hamiltonian dynamically system, for the sake of simplicity with phase space $\mathbf{M} = \mathbb{R}^{2N}$. Generalizations are obvious. We denote by $\mathbf{x} = \{x_1, ..., x_{2N}\} \equiv \{p_1, q_1, \cdot\cdot\cdot, p_N, q_N\}$ the canonical phase space coordinates, by $H(\mathbf{x})$ the Hamiltonian, by $\lambda(\mathbf{x})$ the invariant Liouville measure and by $\mathbf{x} \to g_H^t(\mathbf{x})$ the diffeomorphism group associated to $H(\mathbf{x})$ [8] Let $\rho(\mathbf{x})$ the probability density of the distribution of the initial positions of the trajectories. We will denote by $\chi_A(\mathbf{x})$ the characteristic function of the domain $A \subset \mathbf{M}$. We will denote by λ the Lebesgue measure and $L_p = L_p\left(\mathbb{R}^{2N}, d\lambda\right)$, $p \geq 1$. The proofs and the notations will be simplified by the use of Hilbert space formalism of ergodic theory [9]. In the Hilbert space L_2 we have the canonical scalar product, invariant under $g_H^t(\mathbf{x})$

$$\langle \varphi, \psi \rangle = \int_{\mathbf{M}} \varphi^*(\mathbf{x})\, \psi(\mathbf{x}) d\lambda(\mathbf{x}) \tag{2}$$

Remark 1 *In the following we suppose that $\rho(\mathbf{x})$ and $\chi_A(\mathbf{x})$ are square integrable, i.e. the probability density $\rho(\mathbf{x}) \in L_1 \cap L_2$ and $\lambda(A) < \infty$.*

For any $\phi(\mathbf{x}) \in L_2$ we define the unitary operator U_t (Koopman [9]) as follows

$$(U_t\varphi)(\mathbf{x}) = \varphi\left[g_H^t(\mathbf{x})\right] \tag{3}$$

We observe that, in particular, the function $(U_t\chi_A)(x) = \chi_A\left(g_H^t(\mathbf{x})\right) \in L_2$ describes the visit at the time t of the *finite* domain A. Consequently, the mean value $\frac{1}{T}\int_0^T (U_t\chi_A)(\mathbf{x})dt = \frac{1}{T}\int_0^T \chi_A\left[g_H^t(\mathbf{x})\right]dt$ is the mean sojourn time in A of the trajectory that started from the initial position $\mathbf{x}$. By averaging over initial positions $\mathbf{x}$ with the probability density $\rho(\mathbf{x}) \in L_1 \cap L_2$, the mean sojourn time will be

$$S_T(H, A, \rho) := \frac{1}{T}\int_0^T \langle U_t\chi_A, \rho\rangle\, dt \tag{4}$$

that represents the mean sojourn time in the domain A, during the time interval $(0, T)$, of the particles that started according to the distribution $\rho(\mathbf{x})$. For large time T the quantity $S_T(H, A, \rho)$ is one of the candidates, which can describe the trapping effect, or the degree of confinement in the domain $A \subset \mathbf{M}$.

By Liouville theorem U_t is unitary operator. We denote by $\mathcal{H}_{H,inv}$ the subspace of square integrable invariant functions with respect to the dynamics generated by the Hamiltonian function H . From von Neumann mean ergodic theorem [10], [12], [9] results that the strong limit

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T U_t\varphi dt := \widehat{P}_H\varphi \tag{5}$$

exists (as a limit in L_2) for any $\varphi \in L_2$, and $\widehat{P}_H$ is the projection operator on the subspace $\mathcal{H}_{H,inv}$. We have the equivalencies

$$\varphi \in \mathcal{H}_{H,inv} \iff \widehat{P}_H\ \varphi = \varphi \iff U_t\varphi = \varphi \quad \forall\ t \tag{6}$$

$$\iff \varphi\left(g_H^t(\mathbf{x})\right) = \varphi\left(\mathbf{x}\right), \quad \forall\ t \tag{7}$$

It follows that

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T \langle U_t\varphi_1, \varphi_2\rangle\, dt = \left\langle \varphi_1, \widehat{P}_H\ \varphi_2 \right\rangle \tag{8}$$

also exists for all $\varphi_1, \varphi_2 \in L_2$. Consequently the $T \to \infty$ limit in Eq.(4) exists when A is finite and $\rho(\mathbf{x})$ is square integrable. We will denote this limit by $S(H, A, \rho)$. According to Eq.(8) the latter limit can be rewritten in the terms of $\widehat{P}_H$

$$S(H, A, \rho) = \left\langle \chi_A, \widehat{P}_H\ \rho \right\rangle \tag{9}$$

From the definition 5 follows that the $\widehat{P}_H$ preserves the positivity: if $\varphi \in L_2$ and $\varphi(\mathbf{x}) \geq 0$ almost everywhere () then also $\left[\widehat{P}_H\varphi\right](\mathbf{x}) \geq 0$ almost everywhere .

Particular cases of the projector $\widehat{P}_H$ Suppose that there is some invariant domain $D \subset \mathbf{R}^{2N}$ under g_H^t: $g_H^t(D) = D$. Denote by $\widehat{P}_{H,D}$ the restriction of $\widehat{P}_H$ on the invariant subspace $L_2(D)$ of the L_2 functions with support in D.

In the case $E \geq 0$, $\mathbf{a} \in \mathbf{R}^{2N}$ we denote by $L_{H,E,\mathbf{a}} \subset \mathbf{R}^{2N}$ the interior of that connected component of the level subset $\{\mathbf{x}|\mathbf{x} \in \mathbf{R}^{2N},\ |H(\mathbf{x}) - H(\mathbf{a})| \leq E\}$ that contains the point $\mathbf{a}$ and by $\chi_{H,E,\mathbf{a}}(\mathbf{x})$ its characteristic function. In the must interesting cases, when $N = 1$ and there is no open domain in $\mathbf{R}^2$ where $H(\mathbf{x})$ is constant, the closure of the linear combination of functions $\chi_{H,E,\mathbf{a}}(\mathbf{x})$ span the whole invariant space $\mathcal{H}_{H,inv}$. Because $\chi_{H,E,\mathbf{a}}(\mathbf{x}) \in L_2$ iff $\lambda(L_{H,E,\mathbf{a}}) < \infty$, the following cases of invariant functions, invariant domains will appear in this study.

Generic case, confinement. If $\lambda\left(L_{H,E,\mathbf{a}}\right) < \infty$ then $\chi_{H,E,\mathbf{a}}(\mathbf{x}) \in \mathcal{H}_{H,inv} \subset L_2$. If at least such a domain exists then clearly the projection operator $\widehat{P}_H$ is non trivial.

Non generic case, confinement. In the case $E = 0$, if for some $\mathbf{a} \in L_{H,0,\mathbf{a}} \neq \emptyset$, that means , there is some open domain, containing $\mathbf{a}$ where $H(\mathbf{x})$ is constant, then in fact any open subset $D' \subset D = L_{H,0,\mathbf{a}}$ is an invariant domain and if D' is set sufficiently small such that $\lambda(D') < \infty$, then $\chi_{D'}(\mathbf{x}) \in \mathcal{H}_{H,inv}$. In this case it is clear that $\widehat{P}_{H,D} = 1_D$, where 1_D is the identity operator in $L_2(D)$

Conversely, if for some invariant domain $D \subset \mathbf{R}^{2N}$ we have for $\widehat{P}_{H,D} = 1_D$, then then any function from $L_2(D)$ is invariant, so $U_t \downharpoonleft D = 1_D$ and the Hamiltonian function is constant in the domain D .

Infinite measure invariant domains, generic case, $E > 0$. Suppose that for some E and $\mathbf{a}$ we have $\lambda\left(L_{H,E,\mathbf{a}}\right) = \infty$. In this case, for the restriction of the projector the the invariant domain $D = L_{H,E,\mathbf{a}}$ we have $\widehat{P}_{H,D} = \widehat{0}_D$.

Infinite measure invariant domains, non generic case, $E = 0$. When $D = L_{H,0,\mathbf{a}}$ has non void interior with $\lambda(D) = \infty$, like in the previous case (2.1.1) all of the finite measure subsets $D' \subset D$ are invariant, and $\widehat{P}_{H,D} = \widehat{1}_D$. In a similar manner the converse is true: if for some infinite measure invariant domain D we have $\widehat{P}_{H,D} = \widehat{1}_D$ then $H(\mathbf{x})$ is constant in D.

Reformulations

$\widehat{P}_H$ **in the term of conditional expectation values** The previous formulation has two drawback. First, the existence of the globally defined map g_H^t requires special treatment. Moreover, in the typical cases we are interested on the limiting cases when the Hamilton function is not smooth, only Holder continuos. Now we try to extend $S(H, A, \rho)$, $\widehat{P_H}$ for some limiting cases, when $H(\mathbf{x})$ is only continuos, so the flow $g_H^t(\mathbf{x})$ cannot be defined. For an arbitrary continuous function $H(\mathbf{x})$, accordingly to the previous discussion (2.1.1), in the generic case we define $\mathcal{T}_{1,H}$ the family of open sets

$$\mathcal{T}_{1,H} = \left\{ L_{H,E,\mathbf{a}} | E > 0, \mathbf{a} \in \mathbf{R}^{2N} \right\}$$

and in order to include also the non generic case we define

$$\mathcal{T}_{2,H} = \left\{ \ interior(D) |\ D \subset L_{H,0,\mathbf{a}}, \mathbf{a} \in \mathbf{R}^{2N} \right\}$$

and $\mathcal{T}_H = \mathcal{T}_{1,H} \cup \mathcal{T}_{2,H}$. Denote by $\mathcal{A}_H$ the $\sigma-$algebra generated by $\mathcal{T}_H$. For $\psi \in L_1 \cap L_2$, the conditional expectation value with respect to the $\sigma-$algebra $\mathcal{A}_H$, $\varphi(\mathbf{x}) = \mathbb{E}\left[\psi(\mathbf{x}) \ | \mathcal{A}_H\right]$ defines a bounded operator in the Hilbert space L_2 that can be extended by continuity to a projector that projects on the sub-space of $\mathcal{A}_H$- measurable functions. In the particular case when $H(\mathbf{x})$ is smooth and $\widehat{P}_H$ can be defined by the evolution map we have

$$\varphi = \left[\widehat{P}_H \ \psi \right](\mathbf{x}) \ \Leftrightarrow \varphi(\mathbf{x}) = \mathbb{E}\left[\psi(\mathbf{x}) \ | \mathcal{A}_H\right] \tag{10}$$

.This remark is important because the σ- algebra $\mathcal{A}_H$,the projector $\widehat{P}_H$ and the problem of the mean sojourn time can be defined safely also in the limiting case when $H(\mathbf{x})$ is only continuos and non differentiable.

Thus, according to Eqs.(4, **??**), in the large time limit, the mean sojourn time in the domain A , when the distribution of the initial positions is given by probability density function $\rho(\mathbf{x})$, can be written as

$$S(H, \chi_A, \rho) := \left\langle \chi_A, \widehat{P}_H \ \rho \right\rangle \tag{11}$$

The operator $\widehat{P}_H$ has all the properties of projection operator in the Hilbert space L_2. From Eq.(10) results

$$\varphi(\mathbf{x}) \geq 0 \Rightarrow (\widehat{P}_H \varphi)(\mathbf{x}) \geq 0 \quad a.e \tag{12}$$

$$\widehat{P}_H = \widehat{P}_{kH} \tag{13}$$

Thus the function $S(H, \chi_A, \rho)$, respectively the operator $\widehat{P}_H$ describe the geometrical property of the trajectories.

$\widehat{P}_H$ **in term of dense subspace.** In the case when the flow $g_H^t(\mathbf{x})$, as well as the group of unitary operators U_t is well defined, the ortogonal complement of $\mathcal{H}_{H,inv}$ is given by

$$\mathcal{H}_{H,inv}^{\perp} = \overline{span\{\varphi|\varphi = (U_t - 1)g;\ t > 0,\ g \in L_2\}} \tag{14}$$

This results from a slight modification of the Lemma [12]. By taking the limit $t \to 0$ the following formula results

$$\mathcal{H}_{H,inv}^{\perp} = \overline{span\{\varphi|\varphi = \{H,g\}_P;\quad \{H,g\}_P \in L_2\}} \tag{15}$$

2.1.2 Group action

Let G an affine subgroup of transformations acting on $\mathbf{R}^{2N}$. In the our case G is generated by translations, ortogonal transformations and dilatations. Consider $g \in G$, denote by $J(g)$ its (constant) Jacobian and by $g \to V(g^{-1})$ its unitary representation on L_2, defined on $\phi(x) \in \mathcal{H} :=L_2\left(\mathbb{R}^{2N}, d\lambda\right)$ as

$$\phi(x) \to \sqrt{J(g^{-1})}\phi(g^{-1}\mathbf{x}) := [V(g)\phi](\mathbf{x}) \tag{16}$$

. We denote by $\widehat{P}_{H\circ g}$ the projector obtained via Eq.(10), where $(H \circ g)(\mathbf{x}) :\equiv H(g(\mathbf{x}))$, by $\mathcal{M}_H$ the class of characteristic functions from L_2, of the sets from $\mathcal{T}_H$, respectively by $\mathcal{M}_{H\circ g}$ .the class of characteristic functions associated to $H(g(\mathbf{x}))$ Observe that because $V^+(g) = V^{-1}(g)$, from 16 results

$$[V^+(g)\chi_{H,E,\mathbf{a}}](\mathbf{x}) = \sqrt{J(g)}\chi_{H,E,g(\mathbf{a})}(g(\mathbf{x})) = \chi_{H\circ g,E,\mathbf{a}}(\mathbf{x}) \in \mathcal{M}_{H\circ g} \tag{17}$$

for all $\chi_{H,E,\mathbf{a}}(\mathbf{x}) \in \mathcal{M}_H$. Then we have the following lemma

Lemma 2 *.*

$$V^+(g)\widehat{P}_H V(g) = \widehat{P}_{H\circ g} \tag{18}$$

Proof. Because$V(g)$ is unitary the operator . $V^+(g)\widehat{P}_H V(g) := \widehat{P_1}$is also a projector and $V(g)\mathcal{H} = \mathcal{H}$. The subspace $M_1 = \widehat{P}_H U(g)\mathcal{H} = \widehat{P}_H\mathcal{H}$ is generated by the closure of the linear subspace spanned by functions from $\mathcal{H} = L_2\left(\mathbb{R}^{2N}, d\lambda\right)$ of the form $\chi_{H,E,\mathbf{a}}(\mathbf{x}) \in \mathcal{M}_H$. Results from 17 that subspace $M_g = \widehat{P_1}\mathcal{H} = V^+(g)\widehat{P}_H V(g)\mathcal{H} = V^+(g)M_1$ is generated by the closure of the linear span of the functions $\chi_{H\circ g,E,\mathbf{a}}(\mathbf{x}) \in \mathcal{M}_{H\circ g}$, the subspace associated to the projector $\widehat{P}_{H\circ g}$, that proves 18. In consequence, the range of the projectors $\widehat{P}_{H\circ g}$ and $\widehat{P_g}$ are identical that completes the proof of Eq.(18). ∎

2.2 Generalization to the case of random Hamiltonian system

2.2.1 Formal aspects

In the case when the Hamiltonian contains a set of random parameters, ω, from a probability space $(\Omega, \mathcal{A}, p)$, $\omega \in \Omega$ so the evolution is fixed by a random Hamiltonian $H_\omega(\mathbf{x})$. More generally, in the case when there exists at least a continuos realization of the random field then for almost all values of ω the function $\mathbf{x} \to H_\omega(\mathbf{x})$ is continuos, then the operator the operator-valued function $\omega \to \widehat{P}_{H_\omega}$ can be defined according to Eq.(10). If it is measurable, then it is natural to generalize the definition of the mean sojourn time from Eq.(11) by ensemble average

$$S(A,\rho) := \mathbb{E}_\omega\left(S(H_\omega, A, \rho)\right) = \left\langle \chi_A, \widehat{Q}_H \quad \rho \right\rangle \tag{19}$$

where the operator $\widehat{Q}_{\{H_\omega\}}$ is defined as the weak limit

$$\left\langle \varphi_1, \widehat{Q}_H\ \varphi_2 \right\rangle = \mathbb{E}_\omega \left(\left\langle \varphi_1, \widehat{P}_{H_\omega}\ \varphi_2 \right\rangle \right) \tag{20}$$

where $\mathbb{E}_\omega$ is the expectation value in $(\Omega, \mathcal{A}, p)$. Observe that by the notation $\widehat{Q}_H$ the dependence on the random Hamilton function $(\mathbf{x}, \omega) \to H(\omega, \mathbf{x}) \equiv H_\omega(\mathbf{x})$ is in explicit form.

Further generalization to the case when there exists a realization of the random field H_ωThe operator $\widehat{Q}_{\{H_\omega\}}$ is not a projector in the generic cases. Nevertheless it obeys the inequalities derived from Eqs.(12, **??**, **??**, 13)

$$0 \leq \left\langle \varphi, \widehat{Q}_{\{H_\omega\}}\ \varphi \right\rangle \leq \langle \varphi, \varphi \rangle \tag{21}$$

$$\widehat{Q}^\dagger_{\{H_\omega\}} = \widehat{Q}_{\{H_\omega\}} \tag{22}$$

$$\varphi(\mathbf{x}) \geq 0 \Rightarrow \left(\widehat{Q}_{\{H_\omega\}} \varphi \right)(\mathbf{x}) \geq 0 \tag{23}$$

$$\widehat{Q}_{\{H_\omega\}} = \widehat{Q}_{\{kH_\omega\}}\ ;\ k > 0 \tag{24}$$

where $\{kH_\omega\}$ is the statistical ensemble obtained from the ensemble $\{H_\omega\}$ by the map $H_\omega \to kH_\omega$.

Remark 3 *Iff , with probability* 1*, there are no invariant domain with finite volume, then* $\widehat{Q}_H = \widehat{0}$ *and it follows* $S(A, \rho) = 0$ *for all* A, ρ*. Otherwise the function* $S(A, \rho)$ *that is in one to one correspondence with the operator* $\widehat{Q}_H$ *give us a quantitative measure of the degree of permanent confinement of the particles, subjected to magnetic field and random electric potential. It is clear that this is a purely geometrical problem, related to the statistical properties of the ensemble of equipotential surfaces: in what fraction they are closed, what is the mean size of the closed trajectories. Remark that in the work [19] an alternative method was used to characterize the statistics of the connected components of the level those level sets that are close to maxima of the random function* H_ω *.*

2.3 Exact formulation of the problem

2.3.1 Basic notation

Despite the above formulation works for the case when the probability space $(\Omega, \mathcal{A}, p)$ is atomic and finite, in the general case there is a problem with the measurability of the map $\omega \to \left\langle \varphi_1, \widehat{P}_{\{H_\omega\}}\ \varphi_2 \right\rangle$. So we will adopt a new reformulation, that is identical with the previous in the particular case whenever the expectation value is well defined.

Let $(\Omega, \mathcal{A}, p)$ a probability space and we consider a real valued random field Φ indexed by elements of $\mathbb{R}^N$

$$\Phi : \mathbb{R}^N \times \Omega \to \mathbb{R} \tag{25}$$

$$(\mathbf{x}, \omega) \to \Phi(\mathbf{x}, \omega) \tag{26}$$

so that for all $\mathbf{x} \in \mathbb{R}^N$ the map $\omega \to \Phi(\mathbf{x}, \omega)$ is $\mathcal{A} - \mathcal{B}(\mathbb{R})$ measurable.

Moreover, concerning the random field Φ we make the additional assumption: $\Phi(\mathbf{x}, \omega)$ has a modification that is sample continuos, separable and the map $(\mathbf{x}, \omega) \to \Phi(\mathbf{x}, \omega)$ is $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{A} - \mathcal{B}(\mathbb{R})$ measurable. The continuity in probability is a sufficient condition for the existence of modifications that are separable and measurable. Moreover any dense set is separating [21]. Sufficient conditions for the existence of continuos modifications are given in [22].

Denote by $\mathcal{C}_\Phi$ the $\sigma-$algebra generated by the all of the sets $M(a,b) = \{(\mathbf{x},\omega)|\ a < \Phi(\mathbf{x},\omega) < b\}$ for all $a < b$, i.e. the smallest $\sigma-$algebra so that Φ is measurable. Clearly $\mathcal{C}_\Phi \subset \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{A}$ because Φ is measurable. We generate now a larger $\sigma-$algebra. Consider the family of all open subsets $\mathcal{T}(\mathbb{R}^N)$ and for any subset $C \in \mathcal{T}(\mathbb{R}^N)$ and $M(a,b) \in \mathcal{C}_\Phi$ we consider the (measurable) set $M(a,b,C) = M(a,b) \cap [C \times \Omega]$. To any of the sections $S(\omega,a,b,C) \subset \mathbb{R}^N$ defined as $S(\omega,a,b,C) = \{\mathbf{x}|(\mathbf{x},\omega) \in M(a,b,C)\}$ we associate the larger set $R(\omega,a,b,C)$ that contains all of the points of $S(\omega,a,b,\mathbb{R}^N)$ that are connected to the set $S(\omega,a,b,C)$ by path that is entirely in the set $S(\omega,a,b,\mathbb{R}^N)$. Because Φ has continuos realizations results that $S(\omega,a,b,C)$ is an open subset of $\mathbb{R}^N$. Let we denote $M(a,b,C) = \bigcup_{\omega \in M(a,b)} [\{\omega\} \times R(\omega,a,b,C)]$. It is **easy to prove** the following

Proposition 4 $M(a,b,C) \in \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{A}$

In the continuation we denote by $\mathcal{D}_\Phi$ the $\sigma-$algebra generated by all of the sets $M(a,b,C)$, $a,b \in \mathbb{R}$, $C \in \mathbb{R}^N$

$$\mathcal{D}_\Phi = \bigvee_{a,b\in\mathbb{R},\ C\in\mathcal{T}(\mathbb{R}^N)} M(a,b,C) \tag{27}$$

In the Hilbert space $\mathcal{H}^*$=$L^2(\mathbb{R}^N \times \Omega, \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{A}, d\lambda(\mathbf{x})dp(\omega)\ )$ we have the canonical scalar product: for any $\varphi_{1,2} \in \mathcal{H}^*$ we have

$$\langle \varphi_1, \varphi_2 \rangle_{tot} := \int_\Omega dp(\omega) \int_{\mathbb{R}^N} d\lambda(\mathbf{x}) \varphi_1^*(\mathbf{x},\omega)\varphi_2(\mathbf{x},\omega)$$

We have the subspace $\mathcal{H}_\Phi = \{\Psi|\ \Psi \in \mathcal{H}^*\ \Psi\ is\ \mathcal{D}_\Phi{-}\mathcal{B}(\mathbb{R})\ measurable\}$. The corresponding projector, from $\mathcal{H}_{tot}$ to $\mathcal{H}_\Phi$ will be denoted by $\widehat{P}_\Phi$. Because the measure p is finite, From the Hilbert space $\mathcal{H}$ =$L^2(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), d\lambda(\mathbf{x}))$ with the canonical scalar product ($\psi_1, \psi_2 \in \mathcal{H}$

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^N} d\lambda(\mathbf{x}) \psi_1^*(\mathbf{x})\psi_2(\mathbf{x})$$

we have the tautologic isometric inclusion in $\mathcal{H}_{tot}$, denoted by $V : \mathcal{H} \to \mathcal{H}^*$.

Then we define the extension of the operator $\widehat{Q}_\Phi$, via its matrix elements $\left\langle \psi_1, \widehat{Q}_\Phi\ \psi_2 \right\rangle_{\mathcal{H}}$, with $\psi_{1,2} \in \mathcal{H}$. Recall that the definition defined given by Eq.(20) is correct only in the particular case when the probability field $(\Omega, \mathcal{A}, p)$ is atomic. We have

$$\left\langle \psi_1, \widehat{Q}_\Phi\ \psi_2 \right\rangle_{\mathcal{H}} := \left\langle V\psi_1, \widehat{P}_\Phi\ V\psi_2 \right\rangle_{\mathcal{H}^*} \tag{28}$$

and it is easy to see that it is consistent with Eq.(20).

2.3.2 Mean sojourn time and geometry of random energy surfaces

A more general approach is to define a dynamical system

Because the topological properties are also invariants under the symmetry group, it follows that the **statistical properties of the connected components of level sets** has the same invariance as the statistical ensemble of the functions that it defines. This is the content of the following

Theorem 5 *Let $\Phi_\omega(\mathbf{x})$ a random field in $\mathbb{R}^N$ with symmetry group G i.e. $\Phi_\omega(\mathbf{x}) \overset{d}{=} \Phi_\omega\left(g\left(\mathbf{x}\right)\right)$ for all $g \in G$. Denote by $U(g)$ the representation of the group G in $L^2\left(\mathbb{R}^N, d^N\mathbf{x}\right)$ defined as*

Now we consider the case when the statistical ensemble is invariant under previous affine group. In the case when g is Euclidean motion, this invariance means $H_\omega\left(g(\mathbf{x})\right) \overset{distribution}{=} H_\omega\left(\mathbf{x}\right)$, which means that for any measurable functional $J_{\{H_\omega(g(\mathbf{x}))\}}$ depending on the random function $H_\omega\left(\mathbf{x}\right)$, we have

$$J_{\{H_\omega(\mathbf{x})\}} = J_{\{H_\omega(g(\mathbf{x}))\}} \tag{29}$$

so from Eq.(29) in the particular case $J_{\{H_\omega(\mathbf{x})\}} = \mathbb{E}_\omega\left(\widehat{P}_{\{H_\omega(\mathbf{x})\}}\right)$ results

$$\widehat{Q}_{\{H_\omega(g(\mathbf{x}))\}} = \mathbb{E}_\omega\left(\widehat{P}_{\{H_\omega(g(\mathbf{x}))\}}\right) = \mathbb{E}_\omega\left(\widehat{P}_{\{H_\omega(\mathbf{x})\}}\right) = \widehat{Q}_{\{H_\omega(\mathbf{x})\}} \tag{30}$$

Similarly, in the case of the dilatations, we suppose that the random field is self similar with Hurst exponent H, so we consider $g(x) = \lambda x$. Self-similarity means $H_\omega\left(\lambda\mathbf{x}\right) \overset{distribution}{=} \lambda^H H_\omega\left(\mathbf{x}\right)$ so we obtain in a similar manner

$$\widehat{Q}_{\{H_\omega(\lambda\mathbf{x})\}} = \left\langle \widehat{P}_{\{H_\omega(\lambda\mathbf{x})\}} \right\rangle_\omega = \left\langle \widehat{P}_{\{\lambda^H H_\omega(\mathbf{x})\}} \right\rangle_\omega \tag{31}$$

On the other hand, we have in general, for any constant a and any continuos function $H\left(\mathbf{x}\right)$ the obvious relation $\widehat{P}_{\{aH(\mathbf{x})\}} = \widehat{P}_{\{H(\mathbf{x})\}}$, because the level sets of $H\left(\mathbf{x}\right)$ and $aH\left(\mathbf{x}\right)$ are identical, so from Eq. (31) we obtain similar to Eq.(30)

$$\widehat{Q}_{\{H_\omega(g(\mathbf{x}))\}} = \widehat{Q}_{\{H_\omega(\mathbf{x})\}} \tag{32}$$

so finally Eq. (32) is valid for all group G.

We use again Eq. (32) and obtain finally for all elements of the group G

$$U(g)\widehat{Q}_{\{H_\omega(\mathbf{x})\}}\, U^+(g) = \widehat{Q}_{\{H_\omega(\mathbf{x})\}} \tag{33}$$

From Eq.(33) result the following

3 The main theorem

Theorem 6 *Under the previous conditions , we have for some $\alpha \in [0, 1]$*

$$\widehat{Q}_{\{H_\omega(\mathbf{x})\}} = \alpha\widehat{1} \tag{34}$$

Proof. We simplify the notation $\widehat{Q}_{\{H_\omega(\mathbf{x})\}} := \widehat{Q}$. Because Eq.(33) can be rewritten as $\left[U(g), \widehat{Q}\right] = 0$ to prove this lemma it is to prove that the representation $g \to U(g)$ is irreductible. By representing the vectors from $\psi(\mathbf{x}) \in L^2(R^{2N})$ by their Fourier transform $\widetilde{\psi}(\mathbf{k})$, the actions of the translation subgroup $T \subset G$ acts as a multiplication: if g_a is the translation with vector $\mathbf{a}$ then we have $\left(U(g_\mathbf{a})\widetilde{\psi}\right)(k) = \exp(i\mathbf{k}\cdot\mathbf{a})\widetilde{\psi}(k)$, i.e. to the translation operator is represented as a multiplication operator so in the Fourier representation, so we obtain

$$\left(\widehat{Q}\ \exp\left(i\mathbf{k}\cdot\mathbf{a}\right)\widetilde{\psi}\right)(\mathbf{k}) = \exp\left(i\mathbf{k}\cdot\mathbf{a}\right)\left(\widehat{Q}\ \widetilde{\psi}\right)(\mathbf{k}) \tag{35}$$

Results that for any function $m(\mathbf{k}) \in L^2(R^{2N}) \cap L^\infty(R^{2N})$ we have

$$\left(\widehat{Q}\; m(\mathbf{k})\, \widetilde{\psi}\right)(\mathbf{k}) = m(\mathbf{k}) \left(\widehat{Q}\; \widetilde{\psi}\right)(\mathbf{k}) \tag{36}$$

In particular the Eq.(36) is valid when $m(\mathbf{k})$ is the characteristic function of a bounded domain and according to the Lemma it follows that $\widehat{Q}$ can be represented in the $\mathbf{k}$ space by multiplication operator with an L^∞ function $q(\mathbf{k})$.

$$\left(\widehat{Q}\; \widetilde{\psi}\right)(\mathbf{k}) = q(\mathbf{k})\, \widetilde{\psi}(\mathbf{k}) \tag{37}$$

From 21 results

$$0 \leq q(\mathbf{k}) \leq 1 \tag{38}$$

We impose now the Eq.(33) for ortogonal (rotation) subgroup. It result that $q(\mathbf{k}) = f(\|\mathbf{k}\|)$. A dilatation with λ, g_λ ,has the following action

$$\left(\widehat{U}(g_\lambda)\widetilde{\psi}\right)(\mathbf{k}) = (\lambda)^N \widetilde{\psi}(\lambda\mathbf{k})$$

Consequently from the relation $\left[U(g_\lambda), \widehat{Q}\right] = 0$ results

$$f(\|\mathbf{k}\|) = f(\|\lambda\mathbf{k}\|).$$

Consequently $f(\|\mathbf{k}\|)$ is constant and .$\widehat{Q}$ is a scalar operator. Because $\left\|\widehat{Q}\right\|_{\mathcal{H}} \leq 1$ and $\widehat{Q} \geq 0$ results that $0 \leq f(\|\mathbf{k}\|) \equiv \alpha \leq 1$ ■

Remark 7 *In this proof the symmetry group cannot be reduced. It is interesting the effect of small asymmetric perturbation of the random field*

The relation 34 means that a fraction α of the particles starting from any domain remains in their initial position, having trivial dynamics, and the remaining $1-\alpha$ fraction escapes to infinity.

Remark 8 *In principle the trivial dynamics represented by the fraction α can be interpreted as being related to the quiescent period of the intermittent time dynamics, whose sampling by regular time intervals is represented by the statistical ensemble of the frozen turbulence.*

4 Conclusions

The relation between the geometry of a random Hamiltonian field and the large time dynamics of the associated random dynamical system is established. By this connection the large time dynamics of charged particle transport in the frozen turbulence, in constant magnetic field, in the drift approximation, is studied. The main result is that in the case of random Hamiltonian field with one degree of freedom,whose statistical properties are invariant under Euclidean symmetries and dilatations, the trajectories of the particles are either unbounded, either degenerated to a single point.

5 Appendix

The following result is well known, nevertheless for the sake of completeness we will give a (possible) simplified proof.

Lemma 9 *Let* $(M, \mathcal{A}, \mu)$ *a measure space and* $\widehat{Q}$ *a bounded linear operator acting in the Hilbert space* $L^2(M, d\mu)$. *For all* $m(x) \in L^\infty(M, d\mu)$ *we denote by* $\widehat{m}$ *the associated multiplication operator in* $L^2(M, d\mu)$, $\chi_A(x)$ *is the characteristic function of* $A \in \mathcal{A}$ *and* $\widehat{\chi_A}$ *the associated projection operator in* $L^2(M, d\mu)$. *If for all* $A \in \mathcal{A}$ *we have*

$$\widehat{Q}\widehat{\chi_A} = \widehat{\chi_A}\widehat{Q} \tag{39}$$

then there exists $m(x) \in L^\infty(M, d\mu)$ *such that* $\widehat{Q} = \widehat{m}$.

Proof. The first consequence of Eq. 39 is the invariance of the subspace $L^2(A, d\mu) \subset L^2(M, d\mu)$, $A \in \mathcal{A}$, with respect to the action of the operator $\widehat{Q}$. Denote $\widehat{Q} = \widehat{Q}_1 + i\widehat{Q}_2$ where $\widehat{Q}_1 = \left(\widehat{Q} + \widehat{Q}^\dagger\right)/2$, $\widehat{Q}_2 = \left(\widehat{Q} - \widehat{Q}^\dagger\right)/2i$ are self adjoint. Because the multiplication operator $\widehat{\chi_A}$ is self adjoint we have also $\widehat{Q}_{1,2}\widehat{\chi_A} = \widehat{\chi_A}\ \widehat{Q}_{1,2}$. Results that it is sufficient to prove the lemma for the case when $\widehat{Q}$ is self-adjoint.{ *Moreover, if* $\widehat{Q}$ *is self adjoint we have the decomposition* $\widehat{Q} = \widehat{Q}_+ - \widehat{Q}_-$, *where* $\widehat{Q}_\pm \geq 0$ *and by continuous functional calculus theorem we have also* $\widehat{Q}_\pm\widehat{\chi_A} = \widehat{\chi_A}\ \widehat{Q}_\pm$, *so it is sufficient to prove for the case when* $\widehat{Q} \geq 0$, *like in the our lemma.*}. Select a cyclic vector $\psi \in L^2(M, d\mu)$ with respect to the algebra of characteristic functions, more exactly $\{\widehat{\chi_A}\psi | A \in \mathcal{A}\}$ generate a dense subspace in $L^2(M, d\mu)$. With $\psi \in L^2(M, d\mu), \psi(x) > 0$ and $A \in \mathcal{A}$ we define the measure $\mu_{1,\psi}(A) = \int_A |\psi(x)|^2 d\mu(x)$ and the signed measure $\mu_{2,\psi}(A) = \int_A \psi^*(x) \left(\widehat{Q}\psi\right)(x)\, d\mu(x)$. With the notation $\langle \varphi, \psi \rangle = \int_{\mathbf{M}} \varphi^*(\mathbf{x})\, \psi(\mathbf{x}) d\mu(\mathbf{x})$ we have

$$\mu_{1,\psi}(A) = \langle \widehat{\chi_A}\psi, \widehat{\chi_A}\psi \rangle = \|\widehat{\chi_A}\psi\|^2 \tag{40}$$

$$\mu_{2,\psi}(A) = \left\langle \widehat{\chi_A}\psi, \widehat{\chi_A}\widehat{Q}\psi \right\rangle_{\mathcal{H}} \tag{41}$$

From Eqs.39 and 41 result

$$\mu_{2,\psi}(A) = \left\langle \widehat{\chi_A}\psi, \widehat{Q}\widehat{\chi_A}\psi \right\rangle_{\mathcal{H}} \tag{42}$$

so $\mu_{2,\psi}(A)$ is a measure if $\widehat{Q} \geq 0$ .By Eq.42 , Schwarz inequality, and definition of the operator norm results

$$|\mu_{2,\psi}(A)| \leq \|\widehat{\chi_A}\psi\| \left\|\widehat{Q}\widehat{\chi_A}\psi\right\| \leq \|\widehat{\chi_A}\psi\|^2 \ \left\|\widehat{Q}\right\|_{\mathcal{H}} \tag{43}$$

So, by Eqs. 40, 43 we have

$$|\mu_{2,\psi}(A)| \leq \mu_{1,\psi}(A) \left\|\widehat{Q}\right\|_{\mathcal{H}} \tag{44}$$

Results that the $\mu_{2,\psi}$ is absolutely continuous with respect to the measure $\mu_{1,\psi}$. It follows from Radon-Nicodým theorem that exists a function $m_\psi(x)$ such that

$$\mu_{2,\psi}(A) = \int_A m_\psi(x) d\mu_{1,\psi}(x) = \int_A m_\psi(x) |\psi(x)|^2 d\mu(x) \tag{45}$$

or, by Eqs.42,

$$\left\langle \widehat{\chi_A}\psi, \widehat{Q}\widehat{\chi_A}\psi \right\rangle_{\mathcal{H}} = \langle \widehat{\chi_A}\psi, \widehat{m_\psi}\widehat{\chi_A}\psi\rangle_{\mathcal{H}} \tag{46}$$

Let $B, C \in \mathcal{A}$ and $A = B \cap C$ .From Eqs39, 46 we obtain

$$\left\langle \widehat{\chi_B}\psi, \widehat{Q}\widehat{\chi_C}\psi \right\rangle_{\mathcal{H}} = \left\langle \widehat{\chi_A}\psi, \widehat{Q}\widehat{\chi_A}\psi \right\rangle_{\mathcal{H}} = \langle \widehat{\chi_A}\psi, \widehat{m_\psi}\widehat{\chi_A}\psi\rangle_{\mathcal{H}} = \langle \widehat{\chi_B}\psi, \widehat{m_\psi}\widehat{\chi_C}\psi\rangle_{\mathcal{H}} \tag{47}$$

Because $B, C \in \mathcal{A}$ are arbitrary, from the cyclicity of ψ results that Eq.47 can be extended: we have for all $\varphi_1, \varphi_2 \in L^2(M, d\mu)$:

$$\left\langle \varphi_1, \widehat{Q}\varphi_2 \right\rangle_{\mathcal{H}} = \langle \varphi_1, \widehat{m_\psi}\varphi_2\rangle_{\mathcal{H}} \tag{48}$$

which means $\widehat{Q} = \widehat{m_\psi}$. The uniqueness of the function $m_\psi(x)$ with respect to the choice of ψ results from the invariance of the subspace $L^2(A, d\mu)$, $A \in \mathcal{A}$ under the action of the operator $\widehat{Q}$. In this case the essential range of the function $m_\psi(x)$, restricted to the (arbitrary small) domain A is the spectrum of the operator $\widehat{Q}_A$, the restriction of the operator $\widehat{Q}$ to its invariant subspace $L^2(A, d\mu)$, that does not depend on ψ. ■

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