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The Lie Symmetries and Inverse Problem for the Nonlinear Transfer Equation

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Abstract

The paper will investigate a special type of nonlinear equation, the transfer equation with power law nonlinearities. We shall start with the Lie symmetry problem and then we shall try to generalize the equation towards a whole class of nonlinear equations with the same type of symmetries. We shall apply the so-called inverse problem and we shall see that from the symmetry point of view the transfer equation belongs to the class of nonlinear heat equations. The importance of our investigation is connected with the possibility of reducing the initial equation to a simpler one by using the similarity reduction procedure and, by that, of obtaining particular solutions which can not be deduced directly.

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1 Introduction

The aim of the paper consists in the investigation of the classical symmetries of the $2D$ nonlinear mass/heat transport equation:

$$u_t = \partial_x(\alpha x^s u_x) + \partial_y(\beta y^p u_y) + f(u) \tag{1}$$

This equation has always played an important role in the formation of a correct understanding of qualitative features of various transport processes in chemical engineering, thermophysics, and power engineering. In non-homogeneous media, the diffusion coefficients may depend on coordinates and even on temperature. There are numerous approximation formulas (among them linear, power-low and exponential) describing the dependence of the transfer coefficients on temperature or concentration. This equation was intensively studied and particular class of solutions have been noted. In the case $s \neq p \neq 2$, in "Handbook of Nonlinear Partial Differential Equations" (A. D. Polyanin and V. F. Zaitsev, Chapman & Hall/CRC Press, Boca Raton, (2004), ISBN I-58488-355-3) was shown that the equation (1) can be brought by similarity reduction to the following $1D$ equation:

$$u_t = u_{2\zeta} + \frac{M}{\zeta}u_\zeta + f(u),\ M = \frac{4 - sp}{(2-s)(2-p)} \tag{2}$$

We shall add to these results an equivalent $1D$ equation corresponding to the case $s = p = 2$. Moreover, we shall obtain for various particular cases of $s, p, f(u)$ different symmetry groups, invariants and particular solutions. This will be done in the next section, and the inverse symmetry problem for the model will be investigated in the third section. Some concluding remarks will end the paper.

2 The Lie symmetry problem

Let us consider the $2D$ transfer equation with power-law nonlinearities (1) with $\alpha = \beta = 1$, that is an equation of the form:

$$u_t = \partial_x(x^s u_x) + \partial_y(y^p u_y) + f(u) \tag{3}$$

with s, p arbitrary constants and $f(u)$ an arbitrary function.

The general expression of the classical Lie operator which leaves (3) invariant is:

$$U(x,y,t,u)=\varphi(x,y,t,u)\frac{\partial}{\partial t}+\xi(x,y,t,u)\frac{\partial}{\partial x}+\eta(x,y,t,u)\frac{\partial}{\partial y}+\sigma(x,y,t,u)\frac{\partial}{\partial u} \tag{4}$$

Following the Lie symmetry theory, the invariance condition of the equation (3) is given by the relation:

$$0=U^{(2)}[u_t-\partial_x(x^s u_x)+\partial_y(y^p u_y)+f(u)] \tag{5}$$

where $U^{(2)}$ is the second order extension of the operator (4).

The latter relation has the equivalent expression:

$$\begin{aligned}&sx^{(s-1)}\xi u_{2x}+s(s-1)x^{(s-2)}\xi u_x+py^{(p-1)}\eta u_{2y}+p(p-1)y^{(p-2)}\eta u_y+\\&+\dot{f}(u)\sigma-\sigma^t+sx^{(s-1)}\sigma^x+py^{(p-1)}\sigma^y+x^s\sigma^{2x}+y^p\sigma^{2y}=0\end{aligned} \tag{6}$$

The coefficient functions $\sigma^t,\sigma^x,\sigma^y,\sigma^{2x},\sigma^{2y}$ appear in the second extension $U^{(2)}$ and their general expressions are given by the general formulas:

$$\begin{aligned}\sigma^t &= \mathcal{D}_t[\sigma-\varphi u_t-\xi u_x-\eta u_y]+\varphi u_{2t}+\xi u_{xt}+\eta u_{yt}\\ \sigma^x &= \mathcal{D}_x[\sigma-\varphi u_t-\xi u_x-\eta u_y]+\varphi u_{tx}+\xi u_{2x}+\eta u_{xy}\\ \sigma^y &= \mathcal{D}_y[\sigma-\varphi u_t-\xi u_x-\eta u_y]+\varphi u_{ty}+\xi u_{xy}+\eta u_{2y}\\ \sigma^{2x} &= \mathcal{D}_{2x}[\sigma-\varphi u_t-\xi u_x-\eta u_y]+\varphi u_{txx}+\xi u_{xxx}+\eta u_{xxy}\\ \sigma^{2y} &= \mathcal{D}_{2y}[\sigma-\varphi u_t-\xi u_x-\eta u_y]+\varphi u_{tyy}+\xi u_{xyy}+\eta u_{yyy}\end{aligned} \tag{7}$$

By replacing the expressions (7) in condition (6) and asking for the vanishing of the coefficients of each monomial in the derivatives of $u(t,x)$, we obtain the following partial differential system:

$$\begin{gathered}\varphi_x=0\\ \varphi_y=0\\ \varphi_u=0\\ \xi_u=0\\ \eta_u=0\\ \sigma_{2u}=0\\ y^p\xi_y+x^s\eta_x=0\\ -x\varphi_t+2x\xi_x-s\xi=0\\ -y\varphi_t+2y\eta_y-p\eta=0\\ py^{(p-1)}[\eta_y-\varphi_t]+sx^{(s-1)}\eta_x-p^{(p-1)}y^{(p-2)}\eta-\eta_t+x^s\eta_{2x}+y^p[\eta_{2y}-2\sigma_{uy}]=0\\ sx^{(s-1)}[\xi_x-\varphi_t]+py^{(p-1)}\xi_y-s(s-1)x^{(s-2)}\xi-\xi_t+y^p\xi_{2y}+x^s[\xi_{2x}-2\sigma_{ux}]=0\\ f(u)\varphi_t-\sigma_t-f(u)\sigma_u+sx^{(s-1)}\sigma_x+py^{(p-1)}\sigma_y+\dot{f}(u)\sigma+x^s\sigma_{2x}+y^p\sigma_{2y}=0\end{gathered} \tag{8}$$

with the unknown functions $\varphi(x,y,t,u)$, $\xi(x,y,t,u)$, $\eta(x,y,t,u)$, $\sigma(x,y,t,u)$, $f(u)$.

For solving the previous system, let us choose for the source function $f(u)$ the following expressions:

$Case(1): f(u)=u^r$. In this case, the solution is:

$$\varphi=at+b,\ \xi=\frac{a}{2-s}x,\ \eta=\frac{a}{2-p}y,\ \sigma=\frac{a}{1-r}u \tag{9}$$

with a, b, $s\neq 2$, $p\neq 2$, $r\neq 1$ arbitrary constants.

The Lie symmetry operator becomes:

$$U^{(1)}(x,y,t,u)=(at+b)\frac{\partial}{\partial t}+\left(\frac{a}{2-s}\right)\frac{\partial}{\partial x}+\left(\frac{a}{2-p}\right)\frac{\partial}{\partial y}+\left(\frac{a}{1-r}u\right)\frac{\partial}{\partial u} \tag{10}$$

$Case(2): f(u) = e^{qu}$. This case generates the solution:

$$\varphi = at + b,\ \xi = \frac{a}{2-s}x,\ \eta = \frac{a}{2-p}y,\ \sigma = -\frac{a}{q} \tag{11}$$

with a, b, $s \neq 2$, $p \neq 2$, $q \neq 0$ arbitrary constants.

The Lie generator (4) takes the expression:

$$U^{(2)}(x,y,t,u) = (at+b)\frac{\partial}{\partial t} + \left(\frac{a}{2-s}\right)\frac{\partial}{\partial x} + \left(\frac{a}{2-p}\right)\frac{\partial}{\partial y} - \left(\frac{a}{q}\right)\frac{\partial}{\partial u} \tag{12}$$

$Case(3): f(u) = u$, $s = p = 2$. The analyzed model is now described by the evolutionary equation:

$$u_t = x^2 u_{2x} + y^2 u_{2y} + 2xu_x + 2yu_y + u \tag{13}$$

By a computational way, the solution has a more complicated expression:

$$\begin{aligned}
\varphi &= \frac{c_1}{2}t^2 + c_2 t + c_3 \\
\xi &= \frac{x}{2}\left[(2c_6 + 4c_5 + c_2)\ln y + (c_1 t + c_2)\ln x + 2c_6 t + 2c_7\right] \\
\eta &= -2y\left[\left(\frac{c_2}{4} + \frac{c_6}{2} + c_5\right)\ln x + \left(-\frac{c_1}{4}t - \frac{c_2}{4}\right)\ln y + \left(\frac{c_1}{2} - \frac{c_2}{2} + c_4 + \frac{c_6}{2}\right)t - \frac{c_{12}}{2}\right] \\
\sigma &= \frac{1}{\sqrt{xy}}\left[c_9 c_{11} e^{(1+c_1+c_2)t}\left(c_7 x^{\left(\frac{1}{2}\sqrt{1+4c_1}\right)} + c_8 x^{\left(-\frac{1}{2}\sqrt{1+4c_1}\right)}\right) y^{\left(\frac{1}{2}\sqrt{1+4c_2}\right)} + \right. \\
&\quad + c_{10} c_{11} e^{(1+c_1+c_2)t}\left(c_7 x^{\left(\frac{1}{2}\sqrt{1+4c_1}\right)} + c_8 x^{\left(-\frac{1}{2}\sqrt{1+4c_1}\right)}\right) y^{\left(-\frac{1}{2}\sqrt{1+4c_2}\right)} + \\
&\quad + u\sqrt{xy}\left(-\frac{1}{8}c_1 \ln^2 y + \left(\left(\frac{1}{2} - \frac{t}{4}\right)c_1 - c_2 + c_4 - c_5\right)\ln y - \frac{1}{8}c_1 \ln^2 x + \right. \\
&\quad \left.\left. + \left(c_5 - \frac{c_1 t}{4}\right)\ln x + c_6 + \frac{c_1 t^2}{4} + c_4 t\right)\right]
\end{aligned} \tag{14}$$

The solution (14) generates 8 nonvanishing independent Lie operators as follows:

$$\begin{aligned}
U_1 &= \frac{t^2}{2}\partial_t + \frac{x}{2}t\ln x\partial_x + yt\left(\frac{1}{2}\ln y - 1\right)\partial_y + \frac{u}{8}\left(\ln^2 y + \ln^2 x + 2(t-2)\ln y + 2t\ln x - 2t^2\right)\partial_u \\
U_2 &= t\partial_t + \frac{x}{2}(\ln x + \ln y)\partial_x - \frac{y}{2}(\ln x - \ln y - 2t)\partial_y - (\ln y)u\partial_u,\ U_3 = \partial_t \\
U_4 &= -2ty\partial_y + u(\ln y + t)\partial_u,\ U_5 = 2x\ln y\partial_x - 2y\ln x\partial_y + u(\ln x - \ln y)u\partial_u \\
U_6 &= x(\ln y + t)\partial_x - y(\ln x + t)\partial_y + u\partial_u \\
U_7 &= x\partial_x,\ U_8 = y\partial_y
\end{aligned} \tag{15}$$

All the invariants associated to each of these Lie operators could be obtained.

For example, for operator U_4, the invariants could be found by integrating the following characteristic equations:

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{-2ty} = \frac{du}{u(\ln y + t)} \tag{16}$$

They are the expressions:

$$I_1^{(4)} = t,\ I_2^{(4)} = x,\ I_3^{(4)} = uy^{\frac{1}{2}\left(1 + \frac{1}{2t}\ln y\right)} \tag{17}$$

Following a similar way, the invariants generated by each of the the remaining Lie operators could be obtained. Let us pointed out the invariants associated to another interesting operators U_5 and U_6.Their expressions are:

$$I_1^{(5)} = t,\ I_2^{(5)} = \left(x^{\ln x} y^{\ln y}\right)^{1/2},\ I_3^{(5)} = ux^{\frac{1}{2}\left(1 - \frac{\ln x}{\ln y}\right)}$$

$$I_1^{(6)} = t,\ I_2^{(6)} = x^{\left(t+\frac{\ln x}{2}\right)} y^{\left(t+\frac{\ln y}{2}\right)},\ I_3^{(6)} = yu^{(\ln x+t)}$$

Remark 1: It is important to note that, by a computational way, the equation (13) admits a separable solution which has the form:

$$u = q_1 e^{mt} \left(q_2 x^{\frac{-1+\sqrt{-1+4v}}{2}} + q_3 x^{\frac{-1-\sqrt{1+4v}}{2}} \right) \left(q_4 y^{\frac{-1+\sqrt{-3+4(m-v)}}{2}} + q_5 y^{\frac{-1-\sqrt{-3+4(m-v)}}{2}} \right) \tag{18}$$

with $q_i, i = \overline{1,5}$ and m, v arbitrary constants.

Remark 2: By similarity reduction method another separable solutions for evolutionary equation (13) could be obtained.

For example, let us review the invariants generated by U_4. Taking into account the last invariant , we assume a similarity solution of the form:

$$u = h(t,x) y^{-\frac{1}{2}(1+\frac{1}{2t}\ln y)} \tag{19}$$

By replacing it into equation (13) is obtained that $h(t,x)$ is a solution for the following $1D$ partial differential equation:

$$4th_t - 4tx^2 h_{2x} - 8txh_x + 2(1 + \frac{9}{2}t)h = 0 \tag{20}$$

By a computational way, the previous reduced equation has the solution:

$$h(t,x) = \rho_1 t^{-\frac{1}{2}} e^{\frac{kt}{2}} \left(\rho_2 x^{(-1+\sqrt{10+2k})/2} + \rho_3 x^{(-1-\sqrt{10+2k})/2} \right) \tag{21}$$

with $\rho_i, i = \overline{1,3}$ and k arbitrary constants.

3 The inverse symmetry problem

We start with an general differential equation from which the analyzed equation (3) could be obtained. It has the form:

$$u_t = C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} + E(x,y,t,u)u_x + F(x,y,t,u)u_y + G(u) \tag{22}$$

with $C(x,y,t,u)$, $D(x,y,t,u)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $G(u)$ arbitrary functions of their arguments.

Following the symmetry theory, the invariance condition under the action of the Lie operator (4), will have the form:

$$0 = U^{(2)}[u_t - C(x,y,t,u)u_{2x} - D(x,y,t,u)u_{2y} - E(x,y,t,u)u_x - F(x,y,t,u)u_y - G(u)] \tag{23}$$

The latter condition has the equivalent expression:

$$\begin{aligned} 0 \;=\; & -C_t\varphi u_{2x} - D_t\varphi u_{2y} - F_t\varphi u_y - E_t\varphi u_x - G_t\varphi - C_x\xi u_{2x} - D_x\xi u_{2y} - F_x\xi u_y \\ & -E_x\xi u_x - C_y\eta u_{2x} - D_y\eta u_{2y} - F_y\eta u_y - E_y\eta u_x - C_u\sigma u_{2x} \\ & -D_u\sigma u_{2y} - F_u\sigma u_y - E_u\sigma u_x - G_u\sigma + \sigma^t - C\sigma^{2x} - D\sigma^{2y} - E\sigma^x - F\sigma^y \\ & -G_u\sigma + \sigma^t - C\sigma^{2x} - D\sigma^{2y} - E\sigma^x - F\sigma^y \end{aligned} \tag{24}$$

Following the algorithm exposed in the first section, a more general differential system is generated:

$$\varphi_x = 0, \varphi_y = 0, \varphi_u = 0, \eta_u = 0, \xi_u = 0, \sigma_{2u} = 0, D\xi_y + C\eta_x = 0$$

$$-C\varphi_t - \varphi C_t + 2C\xi_x - \xi C_x - \eta C_y - C_u\sigma = 0$$
$$0 = -D\varphi_t - \varphi D_t + 2D\eta_y - \eta D_y - \xi D_x - D_u\sigma = 0 \tag{25}$$

$$-F\varphi_t-\eta_t+E\eta_x+F\eta_y-F_t\varphi-F_x\xi-F_y\eta-F_u\sigma+C\eta_{2x}+D\eta_{2y}-2D\sigma_{yu}=0$$
$$-E\varphi_t-\xi_t+E\xi_x+F\xi_y-E_t\varphi-E_x\xi-E_y\eta-E_u\sigma+C\xi_{2x}+D\xi_{2y}-2C\sigma_{xu}=0$$
$$-G\varphi_t+\sigma_t+G\sigma_u-E\sigma_x-F\sigma_y-G_u\sigma-C\sigma_{2x}-D\sigma_{2y}=0$$

Another approach for solving this system is given by the *inverse symmetry problem.* It allows us to find all the equations which are equivalent from the point of view of the symmetry group they admit.

As an application, let us impose the Lie symmetries generated by the operator U_4 obtained in the first section. In other words, in the system (25) we know the functions:

$$\varphi=\xi=0,\ \eta=-2ty,\ \sigma=u\left(\ln y+t\right) \tag{26}$$

and we shall determine the coefficient functions $C(x,y,t,u)$, $D(x,y,t,u)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $G(u)$.

In these conditions, by solving the system (25) is obtained the general solution:

$$\begin{aligned}
C(x,y,t,u) &= A\left(x,t,uy^{1/2}e^{\frac{1}{4t}\ln^2 y}\right)\\
D(x,y,t,u) &= (1+n_1t)y^2\\
E(x,y,t,u) &= B\left(x,t,uy^{1/2}e^{\frac{1}{4t}\ln^2 y}\right)\\
F(x,y,t,u) &= (2+2n_1t+n_1\ln y)y\\
G(u) &= (n_2-n_1\ln u)u
\end{aligned} \tag{27}$$

where n_1,n_2 are arbitrary constants and A,B are arbitrary functions of their arguments.

Remark 3: By choosing:

$$C(x,y,t,u)=x^2,\ D(x,y,t,u)=y^2,\ E(x,y,t,u)=2x,\ F(x,y,t,u)=2y,\ G(u)=u \tag{28}$$

the nonlinear $2D$ nonlinear transfer equation (13) has been discovered.

4 Conclusions

The $2D$ mass/heat transport equation is an important equation in mathematical physics, both because of their practical applications and of a model of nonlinear differential equation which can be investigated using the Lie symmetry properties.

We made a double investigation:(i) the direct approach, that is the determination of the symmetry group and of the associated invariant quantities.

(ii) the inverse approach, which consists in determining of the general class of evolutionary equations with the same symmetry group.

In the first case we concluded that for the equation corresponding to $s=p=2$ there are 8 independent symmetry operators which generate an open algebra. For $s\neq p\neq 2$ we found two interesting cases with 5-parameters Lie group of symmetry.

The inverse problem leads us to a general class of equations with a similar symmetry group with those generated by one of the symmetry operator of the nonlinear transport equation with power-law nonlinearities.

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