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Short Laser Pulse Propagation in Nonlinear Media. From Nonlinear Schrödinger Equation to Short Pulse Equation

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Abstract

The propagation of light pulses in weakly nonlinear dielectric media is discussed in two opposite limits.Firstly, when the width of the pulse is large enough, the relevant equation is the well known cubic nonlinear Schrödinger (NLS) equation. It is a generic equation describing the propagation of quasi-monochromatic waves in weakly nonlinear media, irrespective of the physical problem under study. The second case corresponds to a short pulse containing only a few oscillations of the carrier wave. Its evolution is described by the short pulse equation (SPE) which is derived in more restrictive conditions. Both NLSE and SPE are completely integrable although through different inverse scattering transform methods. An interesting equivalence between SPE and sine-Gordon equation (SGE) is noted which was used to find solutions of SPE starting from well known solutions of SGE.

1 Introduction

A fundamental problem in nonlinear optics is the study of the propagation of a light (laser) pulse through a (nonlinear) dielectric medium. From Maxwell's equations, without free electric charges nor current densities and with constants magnetic permeability $\mu = \mu_0 = (\varepsilon_0 c^2)^{-1}$, the basic equation describing the electric field evolution writes

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{E}\right) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \vec{P}}{\partial t^2}, \tag{1}$$

where $\vec{P}$ is the polarization vector of the medium which can be separated into a linear and a nonlinear part, $\vec{P} = \vec{P_L} + \vec{P_{NL}}$. We consider an isotropic medium and $\vec{E}$ and $\vec{P}$ linearly polarized $\vec{E} = \vec{e}E$, $\vec{P} = \vec{e}P$, with $\vec{e}$ the polarization vector. In general the expression of P is of the form

$$\begin{aligned} \frac{1}{\varepsilon_0}P = &\int_{-\infty}^{+\infty} \chi^{(1)}(t-\tau)E(\tau)\,\mathrm{d}\tau + \\ + &\iiint \chi^{(3)}(t-\tau_1, t-\tau_2, t-\tau_3)E(\tau_1)E(\tau_2)E(\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3, \end{aligned} \tag{2}$$

where according to the causality principle, the susceptibilities $\chi^{(i)}$ are vanishing for $t < \tau_i$ $(i = \overline{1,3})$. In this expression the term with quadratic dependence on the electrical field was omitted, as it vanishes due to the inversion symmetry, and higher order terms were neglected. A medium, thus approximated, is known as the (nonlinear) Kerr medium and it will be used through-out the present paper. Moreover we shall assume that the electrical field depends only on one space coordinate, let it be z, $\vec{E}(z,t)$, and then (1) becomes

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{+\infty} \chi^{(1)}(t-\tau)E(z,\tau)\,\mathrm{d}\tau = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}P_{NL}(E). \tag{3}$$

A general property of this equation, in the case of a weak nonlinearity, $|\chi^{(3)}||E|^3 \ll |\chi^{(1)}||E|$, is easily obtainable if one looks for solutions of the following form

$$E(z,t) = E_0(z,t) + E_1(z,t), \tag{4}$$

where $|E_1| \ll E_0$ and $E_0(z,t)$ is a plane wave

$$E_0(z,t) = A\exp\left[i(kz - \omega t)\right] + \text{cc}. \tag{5}$$

Then, considering the linear equation in the left-hand side of (3) we get

$$k(\omega) = \frac{1}{c}\sqrt{1+\hat{\chi}^{(1)}(\omega)}\omega, \tag{5'}$$

where $\hat{\chi}^{(1)}(\omega)$ is the Fourier transform of $\chi^{(1)}(t)$

$$\hat{\chi}^{(1)}(\omega) = \int_{-\infty}^{+\infty} \chi^{(1)}(t)e^{i\omega t}\,\mathrm{d}t.$$

The linear equation satisfied by $E_1(z,t)$ is

$$\frac{\partial^2 E_1}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_1}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{+\infty} \chi^{(1)}(t-\tau)E_1(\tau)\,\mathrm{d}\tau = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}P_{NL}(E_0). \tag{6}$$

Using the expression (5) for E_0, the right-hand side of (6) becomes

$$\begin{aligned}&-\frac{9\omega^2}{c^2}\hat{\chi}^{(3)}(\omega,\omega,\omega)A^3 e^{i(3kz-\omega t)} - \frac{9\omega^2}{c^2}\hat{\chi}^{(3)}(-\omega,-\omega,-\omega)(A^*)^3 e^{-i(3kz-\omega t)}\\ &-\frac{3\omega^2}{c^2}\hat{\chi}^{(3)}(-\omega,\omega,\omega)|A|^2 A e^{i(kz-\omega t)} - \frac{3\omega^2}{c^2}\hat{\chi}^{(3)}(\omega,-\omega,-\omega)|A|^2 A^* e^{-i(kz-\omega t)},\end{aligned}$$

where $\hat{\chi}^{(3)}(\omega_1,\omega_2,\omega_2)$ is the Fourier transform of the cubic susceptibility

$$\hat{\chi}^{(3)}(\omega_1,\omega_2,\omega_2) = \int_{-\infty}^{+\infty} \chi^{(3)}(t_1,t_2,t_3)e^{i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3.$$

This expression suggests us to look for solutions of (6) of the form [1]

$$E(z,t) = F_1(z)e^{-i\omega t} + F_1^* e^{i\omega t} + F_e(z)e^{-3i\omega t} + F_3^* e^{3\omega t} \tag{7}$$

Introducing (7) in (6) we find

$$\frac{\mathrm{d}^2 F_3}{\mathrm{d}z^2} + \frac{1-\hat{\chi}^{(1)}(3\omega)}{c^2}9\omega^2 F_3 = -\frac{9\omega^2}{c^2}\hat{\chi}^{(3)}(\omega,\omega,\omega)A^3 e^{3ik(\omega)z} \tag{8}$$

$$\frac{\mathrm{d}^2 F_1}{\mathrm{d}z^2} + \frac{1-\hat{\chi}^{(1)}(\omega)}{c^2}\omega^2 F_1 = -\frac{3\omega^2}{c^2}\hat{\chi}^{(3)}(-\omega,\omega,\omega)|A|^2 A e^{ik(\omega)z} \tag{9}$$

But

$$k(3\omega) = \frac{3\omega}{c}\sqrt{1+\hat{\chi}^{(1)}(3\omega)} \neq 3k(\omega) = \frac{3\omega}{c}\sqrt{1+\hat{\chi}^{(1)}(\omega)} \tag{9'}$$

and consequently F_3 is bounded (no resonance with the right-hand side exists). On the other hand because of the presence of the exponential term $\exp(ik(\omega)z)$ in the right-hand side of (9), a solution of the inhomogeneous equation is

$$F_1(z) = i\frac{3\omega}{c}\frac{\hat{\chi}^{(3)}(-\omega,\omega,\omega)}{\sqrt{1+\hat{\chi}^{(1)}(\omega)}}z|A|^2 A e^{ik(\omega)z} \tag{10}$$

and the asymptotic expansion (4) will fail at distances $z \sim \left(\hat{\chi}^{(3)}|A|^2\right)^{-1}$. Such terms are called "*secular terms*", and their appearance invalidates the simple expansion (4), so better asymptotic methods must be applied (we shall use in the next section the asymptotic method of the "*multiple scales*"). Momentarily let us try to eliminate these such terms by allowing the amplitude A (constant up to this point) to slowly variate along z and consider a supplemental term in the right-hand side of (6) of the form

$$-2ik\frac{\mathrm{d}A}{\mathrm{d}z}e^{i(kz-\omega t)} - 2ik\frac{\mathrm{d}A^*}{\mathrm{d}z}e^{-i(kz-\omega t)}. \tag{11}$$

We also choose the dependence $A(z)$ in such a way as to suppress the resonance term. Writing $\hat{\chi}^{(3)}(-\omega,\omega,\omega) = \hat{\chi}^{(3)}(\omega)$ we get

$$\frac{\mathrm{d}A}{\mathrm{d}z} = i\frac{3\omega}{2c}\frac{\hat{\chi}^{(3)}(\omega)}{\sqrt{1+\hat{\chi}^{(1)}(\omega)}}|A|^2A \tag{12}$$

which integrated gives

$$A = A_0\exp\left(i\frac{3\omega}{2c}\frac{\hat{\chi}^{(3)}(\omega)}{\sqrt{1+\hat{\chi}^{(1)}(\omega)}}|A|^2z\right) \tag{13}$$

and thus a "*new*" wave number may be introduced

$$\frac{c}{\omega}k(\omega;|A|^2) = \sqrt{1+\hat{\chi}^{(1)}(\omega)} + \frac{3}{2}\frac{\hat{\chi}^{(3)}(\omega)}{\sqrt{1+\hat{\chi}^{(1)}(\omega)}}|A|^2. \tag{14}$$

Using the relation between the wave number k and the refractive index, $k = n\frac{\omega}{c}$, the equation (12)-(14) allow us to define a nonlinear refractive index

$$\begin{gathered} n(\omega;|A|^2) = n_0(\omega) + n_2(\omega)|A|^2, \\ n_0(\omega) = \sqrt{1+\hat{\chi}^{(1)}(\omega)}, \quad 2n_0n_2 = 3\hat{\chi}^{(3)}(\omega). \end{gathered} \tag{15}$$

These general considerations show that even for a weak nonlinearity the propagation of a finite amplitude wave train in the medium is associated with a nonlinear refractive index and a nonlinear dispersion relation. We have to stress the importance of the frequency dependence of the linear susceptibility $\hat{\chi}^{(1)}(\omega)$. Indeed, if $\hat{\chi}^{(1)}$ doesn't depend on ω, one can see from (9) that $k(3\omega) = 3k(\omega)$ and the right-hand side of (7) is also resonant so the wave train would have a more complicated evolution. Usually $\hat{\chi}(\omega)$ is a complex quantity, $\hat{\chi}^(\omega) = \hat{\chi}'(\omega) + i\hat{\chi}''(\omega)$ for $\omega = \omega_r + i\omega_i$ with $\omega_i > 0$ to ensure the exponential decay of $\chi(t)$ when $t \to \infty$ – $\chi(t) \sim e^{-\omega_i t}$, and the real and imaginary part of $\hat{\chi}(\omega)$ are related through the Kramers-Krönig relation (causality condition)

$$\hat{\chi}'(\omega) = \frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\hat{\chi}''(\omega')}{\omega'-\omega}\,\mathrm{d}\omega', \quad \hat{\chi}''(\omega) = -\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\hat{\chi}'(\omega')}{\omega'-\omega}\,\mathrm{d}\omega'.$$

One assumes that the corresponding imaginary part of $n_0(\omega)$ is small and neglects altogether any imaginary part of $n_2(\omega)$.

2 Nonlinear Schrödinger Equation

At first let us consider the one-dimensional propagation of a quasi-monochromatic wave $u(x,t)$ in a weak nonlinear medium

$$\mathcal{L}u = \mathcal{N}(u), \tag{16}$$

where $\mathcal{L}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)$ is a linear differential operator with constant coefficients and $\mathcal{N}(u)$ a nonlinear operator of u and its derivatives. We assume that the linear problem $\mathcal{L}u = 0$ has plane wave solutions

$$u = A\exp(i\theta) + \text{cc.}, \quad \theta = kx - \omega(k)t.$$

Here $\omega(k)$ is given by the dispersion relation, solution of the algebraic equation

$$\ell(ik,-i\omega) = 0, \tag{17}$$

where $\ell(ik, -i\omega)$ represents the action of $\mathcal{L}$ on $\exp(i\theta)$, namely

$$\mathcal{L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \exp(i\theta) = \exp(i\theta)\ell(ik, -i\omega). \tag{18}$$

We assume as well, that for any integer $n \geq 2$

$$\ell(ink, -in\omega) \neq 0$$

(no resonance condition). The quasi-monochromatic wave is written as

$$u(x,t) = \int_{-\infty}^{+\infty} A(k,t) \exp\left[i(kw - \omega t)\right] \, \mathrm{d}k + \mathrm{cc.}, \tag{19}$$

where $A(k,t)$ is different than zero on a small compact support around a wave vector k_0

$$A(k) \equiv 0 \,\text{iff}\, |k - k_0| > \varepsilon k_0$$

The dispersion relation $\omega(k)$ can be expanded in a Taylor series around the point k_0

$$\begin{gathered} \omega(k) = \omega(k_0 + \varepsilon k_0 \nu) = \omega_0 + \sum_{n=1} \varepsilon^n k_0^n \nu^n \omega_n, \; |\nu| \leq 1 \\ \omega_n = \frac{1}{n!} \left.\frac{\mathrm{d}^n \omega(k)}{\mathrm{d}k^n}\right|_{k=k_0} \end{gathered} \tag{20}$$

and the expression (19) writes

$$\begin{gathered} u(x,t) = \varepsilon k_0 \exp\left[i(k_0 x - \omega_0 t)\right] \int_{-1}^{+1} \mathrm{d}\nu A(k_0 + \varepsilon k_0 \nu) \cdot \\ \cdot \exp\left\{i\left[k_0(\varepsilon x - \omega_1 \varepsilon t)\nu - k_0^2 \omega_2 \varepsilon^2 t \nu^2 - k_0^3 \omega_3 \varepsilon^3 t \nu^3 + \ldots\right]\right\} + \mathrm{cc.} \end{gathered} \tag{21}$$

This discussion allows us to introduce in a natural way the so-called "*slow variables*"

$$x_1 = \varepsilon x, \; t_1 = \varepsilon t, \; t_2 = \varepsilon^2 t, \; \ldots \tag{22}$$

The solution can also be written as

$$u(x,t) = \exp\left[i(k_0 x - \omega_0 t)\right] U(x_1, t_1, t_2, \ldots),$$

where $U(x_1, t_1, t_2, \ldots)$ depends only on the slow variables and therefore can be expanded in a power series of ε

$$U(x_1, t_1, t_2, \ldots) = \sum_{n=1} \varepsilon^n \Psi_n(x_1, t_1, t_2, \ldots).$$

This procedure, applied to the nonlinear equation (16), will allow us to obtain the correct asymptotic behavior of its solutions, and it is well known in mathematics under the name of "*multiple scale method*" [2]. Generally, one considers

$$u(x,t) = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \ldots \tag{23}$$

The derivatives $\partial/\partial t$, $\partial/\partial x$ have to be generalized to include the slow variables (22), namely

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial x_1}, \; \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots$$

Then the linear operator $\mathcal{L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$ becomes

$$\mathcal{L}\left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial x_1}, \; \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots\right)$$

and can be expanded in a Taylor series about $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$

$$\begin{aligned}\mathcal{L} \to & \mathcal{L}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) + \varepsilon\left(\mathcal{L}_1\frac{\partial}{\partial x_1} + \mathcal{L}_2\frac{\partial}{\partial t_1}\right) \\ & + \frac{1}{2}\varepsilon^2\left(\mathcal{L}_{11}\frac{\partial^2}{\partial x_1^2} + 2\mathcal{L}_{12}\frac{\partial^2}{\partial x_1 \partial t_1} + \mathcal{L}_{22}\frac{\partial^2}{\partial t_2^2}\right) + \ldots\end{aligned} \tag{24}$$

Here the indexes 1 and 2 refer to partial differentiation of $\mathcal{L}$ with respect to $\partial/\partial x$ and $\partial/\partial t$ respectively ($\mathcal{L}$ is viewed as a polynomial in $\partial/\partial x$ and $\partial/\partial t$ and differentiated accordingly with respect to these variables). Assuming that the nonlinear operator $\mathcal{N}(u)$ starts with cubic terms (Kerr nonlinearity), the substitution of (23) and (24) into (16), in different orders of ε, will lead to

$$\begin{aligned}&\mathcal{O}(\varepsilon): \quad \mathcal{L}\Phi^{(1)} = 0 \\ &\mathcal{O}(\varepsilon^2): \quad \mathcal{L}\Phi^{(2)} = -\left(\mathcal{L}_1\frac{\partial}{\partial x_1} + \mathcal{L}_2\frac{\partial}{\partial t_1}\right)\Phi^{(1)} \\ &\mathcal{O}(\varepsilon^3): \quad \mathcal{L}\Phi^{(3)} = -\left(\mathcal{L}_1\frac{\partial}{\partial x_1} + \mathcal{L}_2\frac{\partial}{\partial t_1}\right)\Phi^{(2)} - \\ &-\frac{1}{2}\left(\mathcal{L}_{11} + 2\mathcal{L}_{12}\frac{\partial^2}{\partial x_1 \partial t_1} + \mathcal{L}_{22}\frac{\partial^2}{\partial t_1^2} + 2\mathcal{L}_2\frac{\partial}{\partial t_2}\right)\Phi^{(1)} + \begin{array}{c}\text{cubic}\\ \text{nonlinear}\\ \text{terms}\end{array}.\end{aligned} \tag{25}$$

In the first equation (25) we recover the linear equation with plane wave solutions

$$\Phi^{(1)} = A(x_1, t_1, t_2, \ldots)\exp(i\theta) + \text{cc.},$$

with the phase $\theta(x, t) = kx - \omega(k)t$, $\omega(k)$ - the solution of the algebraic equation (17) and the amplitude A depending only on the slow variables.

Substituting $\Phi^{(1)}$ into the second equation (25) we obtain

$$\mathcal{L}\Phi^{(2)} = -i\left(\ell_\omega\frac{\partial A}{\partial t_1} - \ell_k\frac{\partial A}{\partial x_1}\right)\exp(i\theta) + \text{cc.} \tag{26}$$

We use the fact that $\Phi^{(1)}$ depends on the fast variables (x, t) only through the argument of the exponential and we replace $\partial/\partial x$ by ik, respectively $\partial/\partial t$ by $-i\omega$; then

$$\begin{aligned}\mathcal{L}_1 e^{i\theta} &\to -i\frac{\partial}{\partial k}\ell(ik, -i\omega) = -i\ell_k \\ \mathcal{L}_2 e^{i\theta} &\to i\frac{\partial}{\partial \omega}\ell(ik, -i\omega) = i\ell_\omega.\end{aligned} \tag{27}$$

The $\exp(i\theta)$ term in the right-hand side of (26) is a secular term and to keep our perturbation calculus valid we impose that $A(x_1, t_1, t_2, \ldots)$ evolves according to the equation

$$\ell_\omega\frac{\partial A}{\partial t_1} - \ell_k\frac{\partial A}{\partial x_1} = 0. \tag{28}$$

The total differentiation of the dispersion relation (17) with respect to k gives

$$\begin{aligned}&\ell_k + \left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)\ell_\omega = 0 \\ &\ell_{kk} + 2\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)\ell_{k\omega} + \left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)^2\ell_{\omega\omega} + \left(\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\right)\ell_\omega = 0.\end{aligned} \tag{29}$$

Using the first equation (29) in (28) we see that $A(x_1, t_1, t_2, \ldots)$ depends on (x_1, t_1) only through the combination

$$\xi = x_1 - \left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)t_1 \tag{30}$$

and hence, on the first (order) slow space-time scales, the wave travels at group velocity $\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)$. Since the right-hand side of (26) vanishes, $\Phi^{(2)}$ has also the form of a plane wave with an amplitude depending only on the slow variables

$$\Phi^{(2)} = B(x_1, t_1, t_2, \ldots)e^{i\theta} + \mathrm{cc}.$$

Introducing the expressions of $\Phi^{(1)}$ and $\Phi^{(2)}$ in the third equation (25), in order to remove the secular behavior one has to assume that $B(x_1, t_1, t_2, \ldots)$ depends on (x_1, t_1) through the variable ξ and $A(\xi, t_2, \ldots)$ satisfies the following relation

$$i\ell_\omega \frac{\partial A}{\partial t_2} - \frac{1}{2}\left[\ell_{kk} + 2\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)\ell_{k\omega} + \left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)^2 \ell_{\omega\omega}\right]\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2 A = 0$$

which, using the second equation (29), writes

$$i\frac{\partial A}{\partial t_2} + \frac{1}{2}\left(\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\right)\frac{\partial^2 A}{\partial \xi^2} + \gamma|A|^2 A = 0. \tag{31}$$

Here we considered only the simplest form of a cubic nonlinearity $\sim |u|^2 u$ (which does not generate higher harmonics) and as before we used the relation (27) to replace the derivative $\mathcal{L}_{11}$, $\mathcal{L}_{12}$, $\mathcal{L}_{22}$ with the corresponding derivatives of $\ell(ik, -i\omega)$. The equation (31) is the well-known cubic nonlinear Schrödinger equation, a generic equation describing the propagation of quasi-monochromatic waves in weakly nonlinear media. The literature existent on this topic is very vast and we'll mention just a few references where more informations and applications can be found [1,3–11]. It represents a completely integrable system with multisoliton solution that can be determined through the "*Inverse Scattering Transform*" (IST) [4–7,9]. The solution depends on the relative sign of $\alpha = (\partial^2\omega/\partial k^2)$ and γ. If their product is positive(focusing case) bright soliton solutions (vanishing at $|\xi| \to \infty$) exist. We give below the expression of the one-soliton solution

$$A(\xi, t_2) = 2\eta \frac{\exp\left\{-i\left[2v\xi - 4(v^2 - \eta^2)t_2 + \varphi_0\right]\right\}}{\cosh\left[2\eta(\xi + 4vt_2 - \xi_0)\right]}, \tag{32}$$

where η is the amplitude and v is the soliton velocity. If $\alpha\gamma < 0$ (defocussing Kerr medium) dark-soliton solution manifest (having a finite limit when $|\xi| \to \infty$).

3 Short Pulse Equation (SPE)

The key assumption made in the derivation of the nonlinear Schrödinger equation (in nonlinear optics as well as in any other contexts) is that the pulse width is large in comparison to the oscillations of the carrier frequency. In most applications this assumption is well satisfied. But with the technological advance of creating shorter and shorter pulses (e. g. the chirped pulse amplification technique [12]) with durations in the femtosecond range, this assumption is no longer valid. Indeed if a pulse contains only a few oscillations of the carrier wave, the hypothesis of a slowly varying amplitude is meaningless, and new approaches are needed to describe the propagation of these ultra-short pulses in dielectric media. One possibility is to complete the NLS equation with additional higher order nonlinear terms (quintic or derivative nonlinear terms [13–15]). Recently a new approach, based on the fact that the pulse is broad in the Fourier space, was developed by several authors [16–19], its merit being that it leads to a new completely integrable equation as the generic equation describing (in certain simplifying conditions) the propagation of short pulses in (cubic) nonlinear dielectric media.

The generic equation describing the propagation of linearly polarized light of frequency far from any resonant frequencies of the medium, is given in (13). We'll assume that the Fourier transform of the linear susceptibility can be approximated by a polynomial in λ (λ being the wavelength, related to the angular frequency ω through the usual relation $\omega = 2\pi/\lambda$), namely [18]

$$\hat{\chi}^{(1)}(\lambda) \simeq \hat{\chi}_0^{(1)} - \hat{\chi}_2^{(1)}\lambda^2, \tag{33}$$

where $\hat{\chi}_0^{(1)}$, $\hat{\chi}_2^{(1)}$ are constants. Then the Fourier transform of the linear part of (3) writes ($z \to x$, $E \to 0$)

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\chi_0^{(1)}}{c^2}\omega^2 \hat{u} - (2\pi)^2 \chi_2^{(1)} \hat{u}.$$

As for the nonlinear term, we expect that only the instantaneous contribution will affect the propagation of short pulses and thus we consider $P_{NL} = \chi^{(3)} u^3$ with $\chi^{(3)}$ constant. Thus the equation under study has the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_\ell^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c_2^2} u + \chi^{(3)} \frac{\partial^2 u^3}{\partial t^2}, \tag{34}$$

where $c_\ell = c/\sqrt{1+\hat{\chi}_0^{(1)}}$, $c_2 = 1/\left(2\pi\sqrt{\hat{\chi}_2^{(1)}}\right)$ and we'll take $c_\ell = 1$ for simplicity.

The expression (33) represents the opposite approximation to the one used in the previous section in deriving the NLS equation (an expansion in power series of frequency was applied). As discussed in [18] these approximations are well satisfied for infrared light in silica fibers.

In the multiple scales analysis of the equation (34) besides the usual "*slow variables*"

$$x_n = \varepsilon^n x, \tag{35}$$

an ultra-fast variable

$$T = \frac{t-x}{\varepsilon} \tag{36}$$

is introduced (the order of magnitude of the small parameter ε is $1/\lambda$). Then $u(x,t)$ may be expanded in a power series

$$u(x,t) = \varepsilon A_0(T, x_1, x_2, \ldots) + \varepsilon^2 A_1(T, x_1, x_2, \ldots) + \ldots \tag{37}$$

For $x = 0$

$$u(x=0,t) = \varepsilon A_0(t/\varepsilon) + \varepsilon^2 A_1(t/\varepsilon) + \ldots$$

and for functions $A_0(t/\varepsilon)$, $A_1(t/\varepsilon)$ decaying at infinity, the function $u(x=0,t)$ vanishes fast enough to describe a real short pulse when $\varepsilon \ll 1$. As

$$\frac{\partial}{\partial t} \to \frac{1}{\varepsilon}\frac{\partial}{\partial T}, \quad \frac{\partial}{\partial x} \to -\frac{1}{\varepsilon}\frac{\partial}{\partial T} + \varepsilon \frac{\partial}{\partial x_1} + \ldots$$

it is easily seen that the terms of order ε^{-1} cancel each other, no terms of order ε^0 exist and in order ε we get

$$2\frac{\partial^2}{\partial T \partial x_1} A_0 = \frac{1}{c_2^2} A_0 + \chi^{(3)} \frac{\partial^2}{\partial T^2} A_0^3. \tag{38}$$

This is the Short Pulse Equation that was sought. With an appropriate scaling and the redefinition of the variables, it may be written in the standard form

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \tag{39}$$

where the subscripts x, t represent partial derivation with respect to that variable.

Soon after this equation was deduced and proven to describe better (numerically) the propagation of short pulses in weakly nonlinear dielectric media [18,19], it was shown that it is completely integrable [20] and it was solved using the inverse scattering transform method [20,21] (Wadati, Konno, Ichikawa variant [23]).

An interesting equivalence between SPE and sine-Gordon equation (SGE) was found and various solutions of SGE were used to derive solutions for SPE [23–26].

The Hamiltonian structure and the short pulse equation hierarchy were discussed in [27,28]. Recently a vector short pulse equation was studied in [29] and an integrable discretization was investigated in [30]. It is worth mentioning that the equation (39) appeared some time before in the differential geometry as one of Rabelo's equations describing pseudospherical surfaces [31–33].

In the followings we shall discuss briefly the equivalence between SPE and SGE, and using simple solutions of SGE we shall derive the corresponding solutions of SPE. According to [25,28] we introduce a new dependent variable r

$$r^2 = 1 + u_x^2 \tag{40}$$

and write the equation (39) as

$$u_{xt} = u(1 + u_x^2) + \frac{1}{2}u^2 u_{xx}. \tag{41}$$

Multiplying (41) by u_x, and using

$$r r_t = u_x u_{xt}, \quad r r_x = u_x u_{xx}$$

the SPE (41) may be easily expressed as a conservative law

$$r_t = \left(\frac{1}{2}u^2 r\right)_x \tag{42}$$

Let us define the hodograph transformation

$$\mathrm{d}y = r\mathrm{d}x + \frac{1}{2}u^2 r\mathrm{d}t, \quad \mathrm{d}\tau = \mathrm{d}t. \tag{43}$$

Then

$$\frac{\partial}{\partial x} \to r\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial \tau} + \frac{1}{2}u^2 r\frac{\partial}{\partial y}$$

and in terms of the new variables the equations (40) and (41) write

$$\begin{aligned} r^2 &= 1 + r^2 u_y^2 \\ r_\tau &= r^2 u u_y. \end{aligned} \tag{44}$$

We introduce a new dependent variable $z = z(y,\tau)$ through the relation

$$u_y = \sin z. \tag{45}$$

Using this relation in the first equation (44) we get

$$\cos z = \frac{1}{r}.$$

Differentiating this equation with respect to τ we obtain successively

$$\sin z z_\tau = \frac{1}{r^2}r_\tau = u u_y = u\sin z,$$

so that

$$u = z_t, \tag{46}$$

which introduced in (45) transforms it into the SGE

$$z_{y\tau} = \sin z. \tag{47}$$

But the hodograph transformation (43) is invertible, namely

$$\mathrm{d}x = \frac{1}{r}\mathrm{d}y - \frac{1}{2}u^2\mathrm{d}\tau, \quad \mathrm{d}t = \mathrm{d}\tau \tag{48}$$

and $x(y,\tau)$ is determined by solving the following system of linear partial differential equations

$$\begin{aligned} x_y &= \frac{1}{r} = \cos z, \quad x_\tau = -\frac{1}{2}u^2 \\ x(y,\tau) &= \int \mathrm{d}y \cos z + \mathcal{C}, \end{aligned} \tag{49}$$

with $\mathcal{C}$ an integration constant. Then if $z(y,\tau)$ is a solution of the SGE (47), the corresponding solution of the SPE is given by (in parametric form)

$$u(y,\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} z(y,\tau), \tag{50}$$

with $\tau = t$ and $x(y,\tau)$ given by (49). The success of the analysis relies on whether one can perform the integration over the variable y in (49).

Two solutions of SPE will be determined starting from simple solutions of SGE. As the first example we consider the one-kink solution of SGE (47)

$$z(y,\tau) = z(y+\tau) = 4 \arctan\left[\exp(y+\tau)\right].$$

Using (50) we find

$$u(y,\tau) = \frac{2}{\cosh(y+\tau)}. \tag{51}$$

Also the integration in (49) is easily performed giving

$$x(y,\tau) = x(y+\tau) = (y+\tau) - 2\tanh(y+\tau). \tag{52}$$

We used $\cos 4\alpha = 1 - 8\sin^2\alpha\cos^2\alpha$, $\alpha = \arctan\xi$, $\xi = \exp(y+\tau)$ and $\sin\alpha = \xi/\sqrt{1+\xi^2}$, $\cos\alpha = 1/\sqrt{1+\xi^2}$ resulting $\cos 4\alpha = 1 - \frac{2}{\cosh^2(y+\tau)}$. The solution $u(x,t)$ is the one loop soliton moving from right to left with the velocity $c = 1$. From (51) and (52) we get (for $t = 0$)

$$u_x = \frac{u_y}{\mathrm{d}x/\mathrm{d}y} = \frac{2\sinh y}{\cosh^2 y - 2},$$

so that the solution has two discontinuity points for $y = \pm y_0$, $\cosh y_0 = \sqrt{2}$, $x(\pm y_0) \simeq \mp 0.5328$ (see figure 1). It is a multivalued solution and therefore it is not convenient for applications in optical

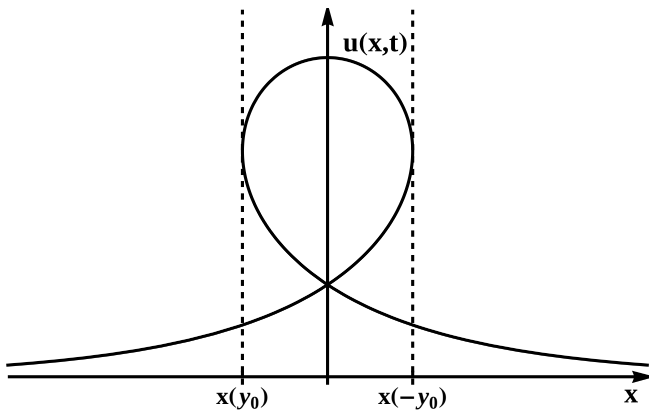

Figure 1: One loop solution $u(x)$ of SPE for $t = 0$.

fibers.

A convenient solution may be obtained if one starts from a breather solution of SGE

$$z(y,t) = -4\arctan\left(\frac{m\sin\psi}{n\cosh\phi}\right), \tag{53}$$

where

$$n = \sqrt{1-m^2}, \quad \psi = n(y-t), \quad \phi = m(y+t) \tag{54}$$

and m is a real parameter, $0 < m < 1$. The solution of SPE is now given by [23, 25]

$$\begin{aligned} u(y,t) &= 4mn\frac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} \\ x(y,t) &= (y+t) + 2mn\frac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi}, \end{aligned} \tag{55}$$

which is nonsingular and single-valued if $m < m_{cr} = \sin(\pi/8)$ [23], representing a true wave packet. A typical pulse is represented in figure 2 for $m = 0.27 < m_{cr}$ at two different times.

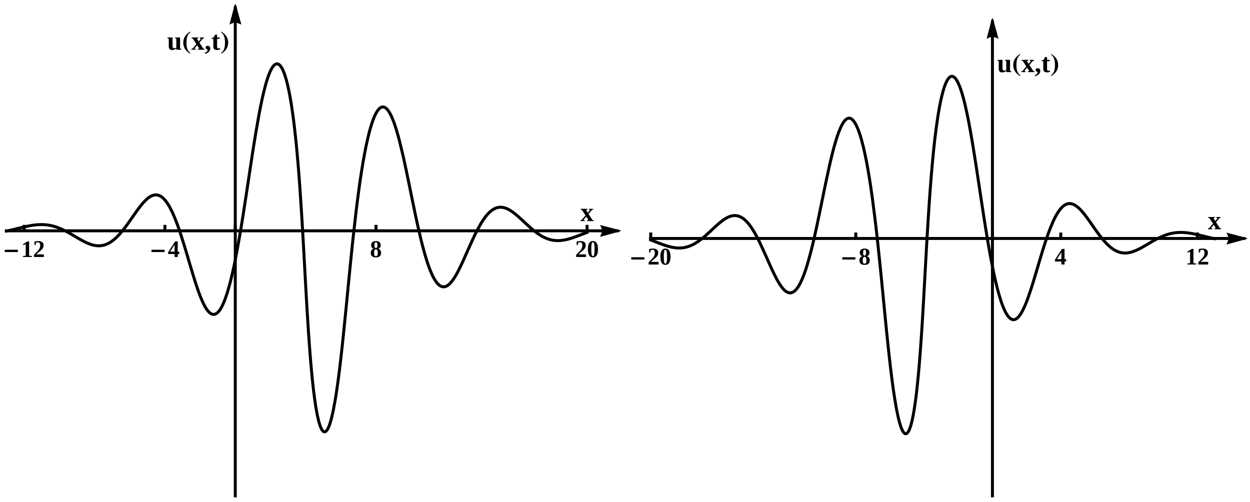

Figure 2: The pulse solution $u(x,t)$ of SPE for $m = 0.27$: (*left*) $t = -4.5$, (*right*) $t = 4.5$.

4 Conclusions

In the present paper we discussed two opposite limits of a light pulse propagation in a weak nonlinear dielectric medium. In the first case when the pulse width is large enough, containing many oscillations of the carrier wave, the relevant equation describing its propagation is the well known NLSE. The conditions in which it is derived are quite general and this explains its presence in many areas of physics. In the opposite case, the pulse is so short that it contains just a few oscillations and its propagation is described by the SPE. Although the conditions are less general, the SPE represents an attractive result as it is also completely integrable. An interesting link between SPE and SGE was found, namely that solutions of SPE can be constructed starting from well known solutions of SGE. Using this method a well-behaving pulse solution was obtained from the breather solutions of SGE. It should be very interesting to extend these considerations for less restrictive conditions and thus determine the relevant equation describing the short pulse propagation in more general situations.

References

[1] A. C. Newell, J. V. Moloney, *Nonlinear Optics* (Addison-Wesley Publ. Comp., Redwood, 1992).

[2] A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).

[3] D. J. Benney, A. C. Newell, J. Math. and Phys. **46**, 363 (1967).

[4] A. C. Newell, *Solitons in Mathematics and Physics* (SIAM, Philadelphia, 1985).

[5] M. J. Ablowitz, H. Segur, *Solitons and the Inverse Spectral Transformations* (SIAM, Philadelphia, 1981).

[6] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982).

[7] V. E. Zakharov, A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).

[8] A. Hasegawa, *Optical Solitons in Fibers*, *Springer Tracts Modern Phys.*, vol. 116 (Springer, Berlin, 1989).

[9] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, Stud. Appl. Math. **53**, 249 (1974).

[10] Y. S. Kivshar, G. P. Agrawal, *Optical Solitons. From Fibers to Photonic Crystals* (Academic Press, Amsterdam, 2003).

[11] A. Hasegawa, Y. Kodama, *Solitons in Optical Communications* (Clarendon Press, Oxford, 1995).

[12] D. Strikland, G. Mourou, Opt. Comm. **56**, 219 (1985).

[13] K. J. Blow, D. Wood, IEEE J. Quant. Elect. **25**, 2665 (1989).

[14] P. V. Mamyshev, S. V. Chernikov, Opt. Lett. **15**, 1076 (1990).

[15] S. V. Chernikov, P. V. Mamyshev, J. Opt. Soc. Am. B **8**, 1633 (1991).

[16] D. Alterman, J. Rauch, Phys. Lett. A **264**, 390 (2000).

[17] D. Alterman, J. Rauch, SIAM J. Math. Anal. **34**, 1477 (2003).

[18] T. Schäfer, C. E. Wayne, Physica D **196**, 90 (2004).

[19] Y. Chung, C. K. R. T. Jones, T. Schäfer, C. E. Wayne, Nonlinearity **18**, 1351 (2005).

[20] A. Sakovich, S. Sakovich, J. Phys. Soc. Japan **74**, 239 (2005).

[21] K. K. Victor, B. B. Thomas, T. C. Kofane, J. Phys. A: Math. Theor. **40**, 5585 (2007).

[22] M. Wadati, K. Konno, Y. H. Ichikawa, J. Phys. Soc. Japan **47**, 1698 (1979).

[23] A. Sakovich, S. Sakovich, J. Phys. A: Math. Gen. **39**, L361 (2006).

[24] E. J. Parker, Chaos Sol. Fractals **38**, 154 (2008).

[25] Y. Matsuno, J. Math. Phys. **49**, 073508 (2008).

[26] V. K. Kuetche, T. B. Bouetou, T. C. Kofane, Phys. Lett. A **372**, 665 (2008).

[27] J. C. Brunelli, Phys. Lett. A **353**, 475 (2006).

[28] J. C. Brunelli, J. Math. Phys. **46**, 123507 (2005).

[29] S. Sakovich, J. Phys. Soc. Japan **77**, 123001 (2008).

[30] B. F. Feng, K. Maruno, Y. Ohta, J. Phys. A: Math. Theor. **43**, 265202 (2010).

[31] M. L. Rabelo, Stud. Appl. Math. **81**, 221 (1989).

[32] R. Beals, M. Rabelo, K. Tenenblat, Stud. Appl. Math. **81**, 125 (1989).

[33] A. Sakovich, S. Sakovich, SIGMA **3**, 086 (2007).