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Spatially Homogeneous Yang-Mills Theory: Instant and Light-Front Forms of Dynamics

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Abstract

Two different forms of relativistic dynamics, the instant and the light-front form, for the pure SU(2) Yang-Mills field theory in 4-dimensional Minkowski space are examined under the supposition that the gauge fields depend on the time evolution parameter only. The obtained under that restriction of gauge potential space homogeneity mechanical matrix model, sometimes called Yang-Mills classical mechanics, is systematically studied in its instant and light-front form of dynamics using the Dirac's generalized Hamiltonian approach. In the both cases the constraint content of the obtained mechanical systems is found. In contrast to its well-known instant-time counterpart the light-front version of SU(2) Yang-Mills classical mechanics has in addition to the constraints generating the SU(2) gauge transformations the new first and second class constraints also. On account of all of these constraints a complete reduction in number of the degrees of freedom is performed. In the instant form of dynamics it is shown that after elimination of the gauge degrees of freedom from the classical SU(2) Yang-Mills mechanics the resulting unconstrained system represents the ID_3 Euler-Calogero-Moser model with a certain external fourth-order potential, whereas in the light-front form it is argued that the classical evolution of the unconstrained degrees of freedom is equivalent to a free one-dimensional particle dynamics.

Keywords: Gauge theories, Yang-Mills mechanics, Hamiltonian reduction, Integrable systems.

1 Introduction

In the context of gauge field theory there is a physically very important regime when finite-dimensional systems arise. The long-wavelength approximation in the dynamics of gauge fields effectively leads to reduction of the field theory and at first has been intensively studied for the non-supersymmetric Yang-Mills theory, both from physical as well as from a purely mathematical point of view (see e.g [1]-[3] and references therein).

In the middle of 1980's analogous supersymmetric mechanical models with more than four supersymmetries were constructed from the corresponding super Yang-Mills theory [4, 5, 6]. The recent renewed interest in the supersymmetric version of Yang-Mills mechanics is motivated by the observation that the Hamiltonian of $D = 1$ SU(n) super Yang-Mills theory in the large n limit describes the dynamics of $D = 11$ supermembrane [7] and claims to the role of M-theory Hamiltonian [8].

Even the simplest of these dimensionally reduced models are still rather complicated and possesses non-trivial dynamics. It was found that the classical non-supersymmetric SU(2) Yang-Mills mechanics exhibits chaotic behavior when the dynamics takes place on a special invariant submanifold and was proved that on this submanifold there is no analytical integral of motion except the energy integral, and thus the Yang-Mills classical mechanics represents a non-integrable system (see e.g [1]-[3] and references therein).

In the present paper we study models obtained from the classical pure SU(2) Yang-Mills field theory under the supposition of gauge fields spatial homogeneity, or more concretely, we consider the models of instant and light-cone [1] SU(2) Yang-Mills classical mechanics and address the problem of its complete Hamiltonian reduction and integrability. The instant form of the classical Yang-Mills mechanics, as well as the light-cone version, follows from the instant, or respectively light-cone, form of Yang-Mills field theory when the gauge fields depend on the time evolution parameter alone. Both models, obtained under such suppositions, contain a finite number of degrees of freedom and inherit in a specific form the gauge invariance of the original Yang-Mills theory, hence they belong to the class of degenerate dynamical systems. In a previous article we already outlined that there exists a difference between the light-cone version of Yang-Mills mechanics to its instant form counterpart even in the character of the local gauge invariance [9].

After Dirac's famous paper [10] on different forms of relativistic dynamics it has been recognized that the different choice of the time evolution parameter can drastically change the content and interpretation of the theory. The present study shows that the long-wavelength approximation in instant and light-front formulation leads to the models that differ drastically even in sense of their classical integrability. Here we present results of the Hamiltonian reduction of both versions, the instant and the light-cone, of SU(2) Yang-Mills classical mechanics and demonstrate that after elimination of all ignorable coordinates the corresponding unconstrained Hamiltonian systems can be related with well known integrable models. More precisely, we shall demonstrate that the unconstrained instant form of SU(2) Yang-Mills mechanics represents the Euler-Calogero-Moser system of ID_3 type, i.e., the inverse-square interacting 3-particle system with internal degrees of freedom related to the root system of the simple D_3 Lie algebra from the classical Cartan's series, which in turn is embedded in a fourth order external potential. For the light-cone version we show that it is equivalent to the dynamics of a free particle in one dimension. We also study the complex solutions to the second class constraints and demonstrate that in this case the reduced system coincides with the well-known model of so-called conformal mechanics, introduced by V. de Alfaro, S. Fubini and G. Furlan [11].

2 Instant form of the Yang-Mills mechanics

In this Section we implement the technics of Hamiltonian reduction. Two kinds of reduction of the degrees of freedom are implemented, namely due to the gauge invariance and due to the discrete symmetry of the model. As a result we derive at the both cases an unconstrained system whose configuration space represents a certain stratified manifold.

2.1 Reduction to equivalent unconstrained matrix model using the continuous symmetry

We start with the action of Yang-Mills field theory in four-dimensional Minkowski space M_4, endowed with a metric η and represented in the coordinate free form

$$S_{YM} = \frac{1}{2}\int_{M_4} \operatorname{tr} F \wedge *F\,, \tag{1}$$

where g is the coupling constant and the su(2) algebra valued curvature two-form $F := dA + g\,A \wedge A$ is constructed from the connection one-form A. The connection and curvature, as Lie algebra valued quantities, are expressed in terms of the antihermitian su(2) algebra basis $\tau^a = \sigma^a/2i$ with the Pauli matrices $\sigma^a\,, a = 1, 2, 3,$ [2] $A = A^a\,\tau^a\,, F = F^a\,\tau^a$. The metric η enters the action through the dual field strength tensor defined in accordance with the Hodge star operation $*F_{\mu\nu} = \frac{1}{2}\,\sqrt{\eta}\,\varepsilon_{\mu\nu\alpha\beta}\,F^{\alpha\beta}$.

After the supposition $\mathcal{L}_{\partial_i}\mathbf{A} = 0$ of the spatial homogeneity of the connection $\mathbf{A}$ the action (1) reduces to the action for a finite dimensional model, the so-called Yang-Mills mechanics described by the degenerate matrix Lagrangian

$$L^{IF}_{YMM} = \frac{1}{2}\operatorname{tr}\left((D_tA)(D_tA)^T\right) - V(A)\,. \tag{2}$$

[1] Hereinafter we shall use the terms light-cone form (dynamics) and light-front form (dynamics) like synonyms.

[2] Further, we shall treat in equal footing the up and down isotopic indexes denoted with a, b, c, d.

The entries of the 3×3 matrix A are nine spatial components $A_{ai} := A^a_i$ of the connection $\mathbf{A} = Y_a\tau_a dt + A_{ai}\tau_a dx^i$, where D_t denotes the covariant derivative $(D_tA)_{ai} = \dot{A}_{ai} + g\,\varepsilon_{abc}Y_bA_{ci}$. Due to the spatial homogeneity condition, all dynamical variables Y_a and A_{ai} are functions of time only. The part of the Lagrangian corresponding to the self-interaction of the gauge fields is gathered in the potential $V(A)$

$$V(A) = \frac{g^2}{4}\left(\mathrm{tr}^2(AA^T) - \mathrm{tr}(AA^T)^2\right)\,. \tag{3}$$

To express the Yang-Mills mechanics in a Hamiltonian form, let us define the phase space endowed with the canonical symplectic structure and spanned by the canonical variables (Y_a, P_{Y_a}) and (A_{ai}, E_{ai}) where

$$P_{Y_a} = \frac{\partial L}{\partial \dot{Y}_a} = 0\,, \quad E_{ai} = \frac{\partial L}{\partial \dot{A}_{ai}} = \dot{A}_{ai} + g\,\varepsilon_{abc}Y_bA_{ci}\,. \tag{4}$$

According to these definitions of the canonical momenta (4), the phase space is restricted by the three primary constraints $P_{Y_a} = 0$ and thus evolution of the system is governed by the total Hamiltonian [12, 13, 14] $H^{IF}_T = H^{IF}_C + u_{Y_a}(t)\,P_{Y_a}$, where the canonical Hamiltonian is given by

$$H^{IF}_C = \frac{1}{2}\mathrm{tr}(EE^T) + \frac{g^2}{4}\left(\mathrm{tr}^2(AA^T) - \mathrm{tr}(AA^T)^2\right) + g\,Y_a\,\mathrm{tr}(J_aAE^T) \tag{5}$$

and the matrix $(J_a)_{bc}$ is defined by $(J_a)_{bc} = -\varepsilon_{abc}$. The conservation of primary constraints $P_{Y_a} = 0$ in time entails the further condition on the canonical variables $\Phi_a = g\,\mathrm{tr}(J_aAE^T) = 0$, that reproduces the homogeneous part of the conventional non-Abelian Gauss law constraints. They are the first class constraints obeying the Poisson brackets algebra

$$\{\Phi_a, \Phi_b\} = g\,\varepsilon_{abc}\Phi_c\,. \tag{6}$$

In order to project onto the reduced phase space, we use the well-known polar decomposition for an arbitrary 3×3 matrix $A_{ai}(\phi, Q) = O_{ak}(\phi)Q_{ki}$, where Q_{ij} is a positive definite 3×3 symmetric matrix and $O(\phi_1,\phi_2,\phi_3) = e^{\phi_1J_3}e^{\phi_2J_1}e^{\phi_3J_3}$ is an orthogonal matrix. Assuming the nondegenerate character of the matrix A_{ai}, we can treat the polar decomposition as uniquely invertible transformation from the configuration variables A_{ai} to a new set of six Lagrangian coordinates Q_{ij} and three coordinates ϕ_i. As it follows from further consideration, the variables parameterizing the elements of the orthogonal group (namely the Euler angles (ϕ_1,ϕ_2,ϕ_3)) are the pure gauge degrees of freedom.

The field strength E_{ai} in terms of the new canonical pairs (Q_{ik}, P_{ik}) and (ϕ_i, P_i) is [15, 16]

$$E_{ai} = O_{ak}(\phi)\biggl(\,P_{ki} + \varepsilon_{kil}(\gamma^{-1})_{lj}\left(\xi^L_j - S_j\,\right)\biggr)\,, \tag{7}$$

where ξ^L_a are three left-invariant vector fields on $\mathrm{SO}(3,\mathcal{R})$

$$\begin{aligned}
\xi^L_1 &= \frac{\sin\phi_3}{\sin\phi_2}\,P_1 + \cos\phi_3\,P_2 - \cot\phi_2\sin\phi_3\;P_3\,,\\
\xi^L_2 &= \frac{\cos\phi_3}{\sin\phi_2}\,P_1 - \sin\phi_3\,P_2 - \cot\phi_2\cos\phi_3\;P_3\,,\\
\xi^L_3 &= P_3\,.
\end{aligned} \tag{8}$$

The vector $S_j = \varepsilon_{jmn}(QP)_{mn}$ in (7) is the spin vector of the gauge field and $\gamma_{ik} = Q_{ik} - \delta_{ik}\,\mathrm{tr}Q$.

Reformulation of the theory in terms of these variables allows one to easily achieve the Abelianization of the secondary Gauss law constraints. Using the representation (7) one can convince oneself that the variables Q_{ij} and P_{ij} make no contribution to the secondary constraints $\Phi_a = M_{ab}P_b = 0$. Hence, assuming nondegenerate character of the matrix

$$M = \begin{pmatrix} \frac{\sin\phi_1}{\sin\phi_2}\,, & \cos\phi_1\,, & -\sin\phi_1\cot\phi_2\\ -\frac{\cos\phi_1}{\sin\phi_2}\,, & \sin\phi_1\,, & \cos\phi_1\cot\phi_2\\ 0\,, & 0\,, & 1 \end{pmatrix}\,, \tag{9}$$

we find the set $P_a = 0$ of Abelian constraints is equivalent to the set Gauss law constraints.

After having rewritten the model in this form, we are able to reduce the theory to physical phase space by a straightforward projection onto the constraint shell. The resulting unconstrained Hamiltonian, defined as a projection of the total Hamiltonian onto the constraint shell

$$H^{IF}_{YMM}(Q_{ab}\,, P_{ab}) := H^{IF}_{C}\Big|_{P_a=0\,,\ P_{Y_a}=0} \tag{10}$$

can be written in terms of Q_{ab} and P_{ab} as

$$H^{IF}_{YMM} \;=\; \frac{1}{2}\mathrm{tr}P^2 - \frac{1}{\det^2\gamma}\mathrm{tr}\,(\gamma\mathcal{M}\gamma)^2 + \frac{g^2}{4}\left(\mathrm{tr}^2Q^2 - \mathrm{tr}Q^4\right)\,, \tag{11}$$

where $\mathcal{M}_{mn} = (QP - PQ)_{mn}$ denotes the gauge field spin tensor.

2.2 Unconstrained dynamics as a motion on stratified manifold

As it was shown the unconstrained dynamics of the instant form SU(2) Yang-Mills mechanics can be identified with the dynamics of the nondegenerate matrix model (11). The configuration space $\mathcal{C}$ of the real symmetric 3×3 matrices can be endowed with the flat Riemannian metric

$$ds^2 =< Q, Q >= \mathrm{Tr}\left(dQ^2\right)\,, \tag{12}$$

whose group of isometry is formed by orthogonal transformations $Q' = R^T\,Q\,R$. Since the unconstrained Hamiltonian system (11) is invariant under the action of this rigid group, we are interested in the structure of the orbit space given as a quotient $\mathcal{C}/\,\mathrm{SO}(3,\mathcal{R})$. The important information on the stratification of the space $\mathcal{C}/\,\mathrm{SO}(3,\mathcal{R})$ of orbits can be obtained from the so-called isotropy group of points of configuration space which is defined as a subgroup of $\mathrm{SO}(3,\mathcal{R})$ leaving point x invariant $RxR^T = x$. Orbits with the same isotropy group are collected into classes, called by *strata*. So, as for the case of symmetric matrix, the orbits are uniquely parameterized by the set of ordered eigenvalues of the matrix Q $x_1 \leq x_2 \leq x_3$. One can classify the orbits according to the isotropy groups which are determined by the degeneracies of the matrix eigenvalues [17]:

1. *Principal orbit-type stratum*, when all eigenvalues are unequal $x_1 < x_2 < x_3$ with the smallest isotropy group $Z_2 \otimes Z_2$.
2. *Singular orbit type strata* forming the boundaries of orbit space with
   (a) two coinciding eigenvalues (e.g. $x_1 = x_2$), when the isotropy group is $\mathrm{SO}(2,\mathcal{R})\otimes Z_2$.
   (b) all three eigenvalues are equal ($x_1 = x_2 = x_3$), here the isotropy group coincides with the isometry group $\mathrm{SO}(3,\mathcal{R})$.

To write down the Hamiltonian describing the motion on the principal orbit stratum, we introduce the coordinates along the slices x_i and along the orbits χ. Namely, we decompose the nondegenerate symmetric matrix Q as

$$Q = \mathcal{R}^T\,(\chi_1,\chi_2,\chi_3)\,\mathcal{D}\;\mathcal{R}(\chi_1,\chi_2,\chi_3) \tag{13}$$

with the $\mathrm{SO}(3,\mathcal{R})$ matrix $\mathcal{R}$ parameterized by the three Euler angles $\chi_i := (\chi_1,\chi_2,\chi_3)$ and the diagonal matrix $\mathcal{D} = \mathrm{diag}\|x_1,x_2,x_3\|$ and consider it as point transformation from the physical coordinates Q_{ab} and P_{ab} to $(x_i\,,p_i)$ and $(\chi_i\,,p_{\chi_i})$.

The original physical momenta P_{ik} can then be expressed in terms of the new canonical pairs (x_i, p_i) and (χ_i, p_{χ_i}) as

$$P = \mathcal{R}^T\left(\sum_{s=1}^{3}\overline{\mathcal{P}}_s\,\overline{\alpha}_s + \sum_{s=1}^{3}\mathcal{P}_s\,\alpha_s\right)\mathcal{R} \tag{14}$$

with $\overline{\mathcal{P}}_s = p_s$,

$$\mathcal{P}_i = -\frac{1}{2}\frac{\xi_i^R}{x_j - x_k}, \quad (\text{cyclic} \ \ \text{permutation} \ \ i \neq j \neq k)\,. \tag{15}$$

In (14) the orthogonal basis for the symmetric 3×3 matrices $\alpha_A = (\overline{\alpha}_i, \ \alpha_i)$ $i = 1, 2, 3$ with the scalar product

$$\mathrm{tr}(\overline{\alpha}_a\,\overline{\alpha}_b) = \delta_{ab}\,, \quad \mathrm{tr}(\alpha_a\,\alpha_b) = 2\delta_{ab}\,, \quad \mathrm{tr}(\overline{\alpha}_a\,\alpha_b) = 0$$

and the $\mathrm{SO}(3,\mathcal{R})$ right-invariant Killing vectors

$$\xi_1^R = p_{\chi_1}\,, \tag{16}$$

$$\xi_2^R = -\sin\chi_1\cot\chi_2\; p_{\chi_1} + \cos\chi_1\; p_{\chi_2} + \frac{\sin\chi_1}{\sin\chi_2}\; p_{\chi_3}\,, \tag{17}$$

$$\xi_3^R = \ \cos\chi_1\cot\chi_2\; p_{\chi_1} + \sin\chi_1\; p_{\chi_2} - \frac{\cos\chi_1}{\sin\chi_2}\; p_{\chi_3}\,. \tag{18}$$

are introduced. In terms of these variables the physical Hamiltonian reads

$$H^{IF}_{YMM} = \frac{1}{2}\sum_{a=1}^{3} p_a^2 + \frac{1}{4}\sum_{a=1}^{3} k_a^2\xi_a^2 + V^{(3)}(x)\,, \tag{19}$$

where

$$k_a^2 = \frac{1}{(x_b + x_c)^2} + \frac{1}{(x_b - x_c)^2}\,, \qquad \text{cyclic} \quad a \neq b \neq c \tag{20}$$

and

$$V^{(3)} = \frac{1}{2}\sum_{a<b} x_a^2 x_b^2\,. \tag{21}$$

Note that the potential term in (21) has a symmetry beyond the cyclic one and potential $V^{(3)}(x_1, x_2, x_3)$ can be rewritten as

$$V^{(3)}(x_1, x_2, x_3) = \frac{\partial W^{(3)}}{\partial x_a}\frac{\partial W^{(3)}}{\partial x_a}\,, \qquad a = 1,\,2,\,3 \tag{22}$$

with the superpotential $W^{(3)} = x_1 x_2 x_3$.

From (19) we conclude that the reduced Hamiltonian H_{YMM} on the principal orbit stratum is exactly the Hamiltonian of the Euler-Calogero-Moser system of type ID_3, i.e., it describes the inverse-square interacting 3-particle system with internal degrees of freedom which is embedded in the fourth order external potential (22) and is related to the root system of the simple Lie algebra D_3 [18].

Using the methods elaborated in [15, 16] and repeat the above consideration for these singular strata we can derive the following unconstrained Hamiltonians:

<u>Four-dimensional stratum</u>

The Hamiltonian after the redefinition $x := x_1$ and $y := x_3$

$$H^{(4)}_{Sing} = \frac{1}{2}\,p_x^2 + p_y^2 + \frac{1}{4}\frac{l^2}{(x-y)^2} + \frac{g^2}{2}(x^4 + 2\,x^2\,y^2)\,, \tag{23}$$

where the constant l^2 denotes the value of the square of particle internal spin, coincides with the Hamiltonian of the mass deformed IA_2 Calogero-Moser model embedded in an external potential.

<u>One-dimensional stratum</u>

The Hamiltonian reduces to the form $x := x_1 = x_2 = x_3$

$$H^{(1)}_{Sing} = \frac{1}{2}\,p_x^2 + 3/2g^2\,x^4\,, \tag{24}$$

which explicitly shows that all “angular variables” are the cyclic coordinates.

2.3 Reduction using the discrete symmetry

In order to obtain the Yang-Mills mechanics from the Euler-Calogero-Moser system via reduction by discrete symmetries it is useful to represent the later in the form of a nondegenerate matrix model. For this reason, in the beginning of the subsection, we give very shortly a few facts concerning the Euler-Calogero-Moser system. The Euler-Calogero-Moser model is a spin generalization of the Calogero-Moser system where the articles are described by their coordinates x_i and momenta p_i together with internal degrees of freedom of angular momentum type $l_{ij} = -\, l_{ji}$ [19].

2.3.1 Euler-Calogero-Moser system as a free motion on space of symmetric matrices

Let us consider the Hamiltonian system with the phase space spanned by the $N \times N$ symmetric matrices X and P with the noncanonical symplectic form

$$\{X_{ab}, P_{cd}\} = \frac{1}{2}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) . \tag{25}$$

The Hamiltonian of the system defined as

$$H = \frac{1}{2}\mathrm{tr}P^2 \tag{26}$$

describes a free motion in the matrix configuration space. The following statement is fulfilled:
The Hamiltonian (26) rewritten in special coordinates coincides with the Euler-Calogero-Moser Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2} \tag{27}$$

with nonvanishing Poisson brackets for the canonical variables [3]

$$\{x_i, p_j\} = \delta_{ij}\,, \tag{29}$$

$$\{l_{ab}, l_{cd}\} = \frac{1}{2}\left(\delta_{ac}l_{bd} - \delta_{ad}l_{bc} + \delta_{bd}l_{ac} - \delta_{bc}l_{ad}\right)\,, \tag{30}$$

To find the adapted set of coordinates in which the Hamiltonian (26) coincides with the Euler-Calogero-Moser Hamiltonian (27), let us introduce new variables

$$X = O^{-1}(\theta)Q(x)O(\theta)\,, \tag{31}$$

where the orthogonal matrix $O(\theta)$ is parameterized by the $\frac{N(N-1)}{2}$ variables, e.g., the Euler angles $(\theta_1, \cdots, \theta_{\frac{N(N-1)}{2}})$ and $Q = \mathrm{diag}\|x_1, \cdots, x_N\|$ denotes a diagonal matrix. This point transformation induces the canonical one which we can obtain using the generating function

$$F_4 = \left[P, x_1, \cdots, x_N, \theta_1, \cdots, \theta_{\frac{N(N-1)}{2}}\right] = \mathrm{tr}[X(x,\theta)P]\,. \tag{32}$$

Using the representation

$$P = O^{-1}\left[\sum_{a=1}^{N} \bar{\alpha}_a \bar{P}_a + \sum_{i<j=1}^{\frac{N(N-1)}{2}} \alpha_{ij} P_{ij}\right] O\,, \tag{33}$$

[3] An analogous model has been introduced in [20] where the internal degrees of freedom satisfy the following Poisson brackets relations

$$\{l_{ab}, l_{cd}\} = \delta_{bc}l_{ad} - \delta_{ad}l_{cb}\,. \tag{28}$$

where the matrices $(\bar{\alpha}_a, \alpha_{ij})$ form an orthogonal basis in the space of the symmetric $N \times N$ matrices under the scalar product

$$\mathrm{tr}(\bar{\alpha}_a \bar{\alpha}_b) = \delta_{ab}\,, \qquad \mathrm{tr}(\alpha_{ij}\alpha_{kl}) = 2\delta_{ik}\delta_{jl}\,, \qquad \mathrm{tr}(\alpha_a \alpha_{ij}) = 0\,, \tag{34}$$

one can find that $\bar{P}_a = p_a$ and components P_{ab} are represented via the $O(N)$ right invariant vectors fields l_{ab}

$$P_{ab} = \frac{1}{2}\frac{l_{ab}}{x_a - x_b}\,. \tag{35}$$

From this, it is clear that the Hamiltonian (26) coincides with the Euler-Calogero-Moser Hamiltonian (27).

The integration of the Hamilton equations of motion

$$\dot{X} = P\,, \qquad \dot{P} = 0 \tag{36}$$

derived with the help of Hamiltonian (26), gives the solution of the Euler-Calogero-Moser Hamiltonian system as follows: for the x-coordinates we need to compute the eigenvalues of the matrix $X = X(0) + P(0)t$, while the orthogonal matrix O, which diagonalizes X, determines the time evolution of internal variables.

2.3.2 The reduction

Now we are ready to demonstrate how the SU(2) Yang-Mills classical mechanics arises from the higher dimensional matrix model after projection onto a certain invariant submanifold determined by the discrete symmetries. Here we shall apply the procedure of constructing generalizations of the Calogero-Sutherland-Moser class of models elaborated in [21]. This method consists in the implementation of some appropriate method of reduction of a given Calogero-Sutherland-Moser system to an invariant submanifold of the phase space using the discrete symmetries of the initial model.

Let us consider the Hamiltonian system with the phase space spanned by the $N \times N$ symmetric matrices X and P with the noncanonical symplectic form (25). We assume here that N is even. The Hamiltonian of the system defined as

$$H = \frac{1}{2}\mathrm{tr}P^2 + V^{(N)}(X) \tag{37}$$

describes a motion on the matrix configuration space and differs from the considered in preceding section by the inclusion of the external potential $V(X)$. We specify the external potential V in superpotential form

$$V^{(N)} = -\frac{1}{4}\,\mathrm{tr}\left(\frac{\partial W^{(N)}}{\partial X}\right)^2 \tag{38}$$

with superpotential $W^{(N)}$ given as

$$W^{(N)} = i\sqrt{\mathrm{det}X}\,. \tag{39}$$

After passing to the new variables (31) and repeating the same machinery as before one can convince that the Hamiltonian (26) coincides with the Euler-Calogero-Moser Hamiltonian embedded in external potential (38)

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N}\frac{l_{ij}^2}{(x_i - x_j)^2} + V^{(N)}(x_1, x_2, \dots, x_N)\,. \tag{40}$$

For the description of discrete symmetries of the Hamiltonian (40) it is convenient to use the Cartesian form of "angular variables"

$$l_{ab} = y_a\pi_b - y_b\pi_a \tag{41}$$

with canonically conjugated variables y_a, π_a.

One can easily check that the Hamiltonian (40) possesses the following discrete symmetries:

- Parity P

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} -x_i \\ -p_i \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} -y_i \\ -\pi_i \end{pmatrix}, \tag{42}$$

- Permutation symmetry M (M is the element of the permutation group S_N)

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} x_{M(i)} \\ p_{M(i)} \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} y_{M(i)} \\ \pi_{M(i)} \end{pmatrix} \tag{43}$$

.

Let us now consider the certain invariant submanifold of the phase space of the matrix model (37) and find out the corresponding reduced system. One can verify that the submanifold defined by the constraints

$$\chi_a := \frac{1}{\sqrt{2}}\,(x_a + x_{N-a+1}) = 0\,, \qquad \bar{\chi}_a := \frac{1}{\sqrt{2}}\,(y_a + y_{N-a+1}) = 0\,, \tag{44}$$

$$\Pi_a := \frac{1}{\sqrt{2}}\,(p_a + p_{N-a+1}) = 0\,, \qquad \bar{\Pi}_a := \frac{1}{\sqrt{2}}\,(\pi_a + \pi_{N-a+1}) = 0 \tag{45}$$

is the invariant submanifold of the system (37). Indeed, because the Hamiltonian (37) possesses the discrete symmetry mentioned above this manifold is invariant under the action

$$D = P \times M\,, \tag{46}$$

where M is specified as $M(a) = N - a + 1$. Among the functions (χ_a, Π_a) and $(\bar{\chi}_a, \bar{\Pi}_a)$, $a = 1, \cdots, \frac{N}{2}$ the nonvanishing Poisson brackets are

$$\{\chi_a, \Pi_b\} = \delta_{ab}\,, \qquad \{\bar{\chi}_a, \bar{\Pi}_b\} = \delta_{ab}\,. \tag{47}$$

This means that the functions (44) form the canonical set of second class constraints. According to the Dirac method in order to project onto the arbitrary invariant manifold described by the second class constraints it is enough to pass to new symplectic form replacing the Poisson brackets by the new so-called Dirac brackets [12, 13]. One can easy verify that for canonical constraints (44) the corresponding fundamental Dirac brackets are

$$\{x_a, p_b\}_D = \frac{1}{2}\delta_{ab}\,, \qquad \{y_a, \pi_b\}_D = \frac{1}{2}\delta_{ab}\,. \tag{48}$$

After the introduction of these new brackets, one can treat all constraints in the strong sense, letting the constraint functions vanish. As result the system with Hamiltonian (40) reduces to the following one

$$H_{red} = \frac{1}{2}\sum_{a=1}^{\frac{N}{2}} p_a^2 + \frac{1}{2}\sum_{a\neq b}^{\frac{N}{2}} l_{ab}^2 k_{ab}^2 + \frac{g^2}{2}\sum_{a\neq b}^{\frac{N}{2}} x_a^2 x_b^2\,, \tag{49}$$

where

$$k_{ab}^2 = \frac{1}{(x_a + x_b)^2} + \frac{1}{(x_a - x_b)^2}\,. \tag{50}$$

Expression (49) for $N = 6$ coincides with the Hamiltonian of the SU(2) Yang-Mills mechanics noting that after projection onto the constraint shell (44)-(45), the potential (38) reduces to the potential of Yang-Mills mechanics.

2.3.3 Lax pair for Yang-Mills mechanics in zero coupling limit

The conventional perturbative scheme of non-Abelian gauge theories starts with the zero approximation of the free theory. However, the limit of the zero coupling constant is not quite trivial. If the coupling constant in the initial Yang-Mills action vanish, the non-Abelian gauge symmetry reduces to the $U(1) \times U(1) \times U(1)$ symmetry. For the zero coupling constant limit the Lagrangian of Yang-Mills mechanics (2) reduces to the nondegenerate Lagrangian describing nine free nonrelativistic particle system. On the other hand after the elimination of the gauge degrees of freedom the unconstrained Hamiltonian (19) in this limit coincides with the Euler-Calogero-Moser Hamiltonian. The argumentation to the equivalence to free particle motion on the level of unconstrained formulation is based on the presentation of the equations of Yang-Mills mechanics in the Lax form. The relation between (40) and (49) allows one to construct the Lax pair for the free part of the Hamiltonian (49) ($g = 0$) using the known Lax pair for the $\mathrm{IA_N}$ Euler-Calogero-Moser system.

According to the work of S. Wojciechowski [19] , the Lax pair for the system with Hamiltonian

$$H_{ECM} = \frac{1}{2}\sum_{a=1}^{N} p_a^2 + \frac{1}{2}\sum_{a\neq b}^{N} \frac{l_{ab}^2}{(x_a - x_b)^2} \tag{51}$$

is

$$L_{ab} = p_a \delta_{ab} - (1-\delta_{ab})\frac{l_{ab}}{x_a - x_b}\,, \tag{52}$$

$$A_{ab} = (1-\delta_{ab})\frac{l_{ab}}{(x_a - x_b)^2}\,. \tag{53}$$

and the equations of motion in Lax form are

$$\dot{L} = [A, L]\,, \tag{54}$$

$$\dot{l} = [A, l]\,, \tag{55}$$

where the matrix $(l)_{ab} = l_{ab}$.

The introduction of Dirac brackets allows one to use the Lax pair of higher dimensional Euler-Calogero-Moser model (namely A_6) for the construction of Lax pairs (L_{YMM}, A_{YMM}) of free Yang-Mills mechanics by performing the projection onto the constraint shell (44)-(45)

$$L^{ECM}_{6\times 6}|_{CS} = L_{YMM}\,, \qquad A^{ECM}_{6\times 6}|_{CS} = A_{YMM}\,. \tag{56}$$

Thus, the explicit form of the Lax pair matrices for the free SU(2) Yang-Mills mechanics is given by the following 6×6 matrices

$$L_{YMM} = \left(\begin{array}{ccc|ccc}
p_1 & -\frac{l_{12}}{x_1-x_2} & -\frac{l_{13}}{x_1-x_3} & \frac{l_{13}}{x_1+x_3} & \frac{l_{12}}{x_1+x_2} & 0 \\
-\frac{l_{12}}{x_1-x_2} & p_2 & -\frac{l_{23}}{x_2-x_3} & \frac{l_{23}}{x_2+x_3} & 0 & -\frac{l_{12}}{x_1+x_2} \\
-\frac{l_{13}}{x_1-x_3} & -\frac{l_{23}}{x_2-x_3} & p_3 & 0 & -\frac{l_{23}}{x_2+x_3} & -\frac{l_{13}}{x_1+x_3} \\
\hline
\frac{l_{13}}{x_1+x_3} & \frac{l_{23}}{x_1+x_2} & 0 & -p_3 & -\frac{l_{23}}{x_2-x_3} & -\frac{l_{13}}{x_1-x_3} \\
\frac{l_{12}}{x_1+x_2} & 0 & -\frac{l_{23}}{x_2+x_3} & -\frac{l_{23}}{x_2-x_3} & -p_2 & -\frac{l_{12}}{x_1-x_2} \\
0 & -\frac{l_{12}}{x_1+x_2} & -\frac{l_{13}}{x_1+x_3} & -\frac{l_{13}}{x_1-x_3} & -\frac{l_{12}}{x_1-x_2} & -p_1
\end{array}\right) \tag{57}$$

and

$$A_{YMM} = \left(\begin{array}{ccc|ccc}
0 & \frac{l_{12}}{(x_1-x_2)^2} & \frac{l_{13}}{(x_1-x_3)^2} & -\frac{l_{13}}{(x_1+x_3)^2} & -\frac{l_{12}}{(x_1+x_2)^2} & 0 \\
-\frac{l_{12}}{(x_1-x_2)^2} & 0 & \frac{l_{23}}{(x_2-x_3)^2} & -\frac{l_{23}}{(x_2+x_3)^2} & 0 & \frac{l_{12}}{(x_1+x_2)^2} \\
-\frac{l_{13}}{(x_1-x_3)^2} & -\frac{l_{23}}{(x_2-x_3)^2} & 0 & 0 & \frac{l_{23}}{(x_2+x_3)^2} & \frac{l_{13}}{(x_1+x_3)^2} \\
\hline
\frac{l_{13}}{(x_1+x_3)^2} & \frac{l_{23}}{(x_1+x_2)^2} & 0 & 0 & -\frac{l_{23}}{(x_2-x_3)^2} & -\frac{l_{13}}{(x_1-x_3)^2} \\
\frac{l_{12}}{(x_1+x_2)^2} & 0 & -\frac{l_{23}}{(x_2+x_3)^2} & \frac{l_{23}}{(x_2-x_3)^2} & 0 & -\frac{l_{12}}{(x_1-x_2)^2} \\
0 & -\frac{l_{12}}{(x_1+x_2)^2} & -\frac{l_{13}}{(x_1+x_3)^2} & \frac{l_{13}}{(x_1-x_3)^2} & \frac{l_{12}}{(x_1-x_2)^2} & 0
\end{array}\right) \tag{58}$$

The equations of motion for the SU(2) Yang-Mills mechanics in the zero constant coupling limit read in a Lax form as

$$\dot{L}_{YMM} = [A_{YMM}, L_{YMM}]\,, \tag{59}$$
$$\dot{l}_{YMM} = [A_{YMM}, l_{YMM}]\,, \tag{60}$$

where the matrix l_{YMM} is

$$l_{YMM} = \left(\begin{array}{ccc|ccc} 0 & l_{12} & l_{13} & -l_{13} & -l_{12} & 0 \\ -l_{12} & 0 & l_{23} & -l_{23} & 0 & l_{12} \\ -l_{13} & -l_{23} & 0 & 0 & l_{23} & l_{13} \\ \hline l_{13} & l_{23} & 0 & 0 & -l_{23} & -l_{13} \\ l_{12} & 0 & -l_{23} & l_{23} & 0 & -l_{12} \\ 0 & -l_{12} & -l_{13} & l_{13} & l_{12} & 0 \end{array}\right). \tag{61}$$

3 Light-front form of the Yang-Mills mechanics

In this Section we give the formulation of the SU(2) light-cone Yang-Mills classical mechanics, calculate all constraints and separate them into the first and second class ones. After that we are ready to perform a Hamiltonian reduction and to find the unconstrained version of the model.

3.1 Model formulation

To formulate the light-cone version of the theory let us introduce the basis vectors in the tangent space $T_P(M_4)$

$$e_\pm := \frac{1}{\sqrt{2}}\,(e_0 \pm e_3)\,, \qquad e_\perp := (e_k\,, k = 1, 2)\,. \tag{62}$$

The first two vectors have directions along the light-cone and the corresponding coordinates are referred usually as the light-cone coordinates $x^\mu = \left(x^+, x^-, x^\perp\right)$

$$x^\pm := \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right)\,, \qquad x^\perp := x^k\,, \quad k = 1, 2\,. \tag{63}$$

The non-zero components of the metric η in the light-cone basis (e_+, e_-, e_k) are $\eta_{+-} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1$. The connection one-form in the light-cone basis is given as $A = A_+\,dx^+ + A_-\,dx^- + A_k\,dx^k$.

By definition the Lagrangian of light-cone Yang-Mills mechanics follows from the corresponding Lagrangian of Yang-Mills theory if one supposes that the components of the connection one-form A depend on the light-cone "time variable" x^+ alone $A_\pm = A_\pm(x^+)\,, A_k = A_k\,(x^+)$. The substitution of this *ansatz* into the classical action (1) defines the Lagrangian of the SU(2) light-cone Yang-Mills classical mechanics

$$L^{LC}_{YMM} = \frac{1}{2}\left(F^a_{+-}\,F^a_{+-} + 2\,F^a_{+k}\,F^a_{-k} - F^a_{12}\,F^a_{12}\right)\,, \tag{64}$$

where the light-cone components of the field-strength tensor are given by [4]

$$F^a_{+-} = \frac{\partial A^a_-}{\partial x^+} + \epsilon^{abc}\,A^b_+\,A^c_-\,, \tag{65}$$
$$F^a_{+k} = \frac{\partial A^a_k}{\partial x^+} + \epsilon^{abc}\,A^b_+\,A^c_k\,, \tag{66}$$
$$F^a_{-k} = \epsilon^{abc}\,A^b_-\,A^c_k\,, \tag{67}$$
$$F^a_{ij} = \epsilon^{abc}\,A^b_i\,A^c_j\,, \quad i, j, k = 1, 2\,. \tag{68}$$

Hence, the SU(2) light-cone Yang-Mills classical mechanics is a finite dimensional system with configuration coordinates $A_\pm\,, A_k$ whose evolution with respect to the time $\tau := x^+$ is determined by the Lagrangian (64).

[4]Hereinafter the coupling constant g will be set equal to 1.

3.2 Generalized Hamiltonian dynamics

Performing the Legendre transformation [5] we obtain the canonical Hamiltonian

$$H_C^{LC} = \frac{1}{2}\,\pi_a^-\pi_a^- \,-\, \epsilon^{abc}\, A_+^b \left(A_-^c\,\pi_a^- + A_k^c\,\pi_a^k \right) + V(A_k) \tag{69}$$

with a potential term

$$V(A_k) = \frac{1}{2}\,\left[\left(A_1^b A_1^b \right)\,(A_2^c A_2^c) - \left(A_1^b A_2^b \right)\,(A_1^c A_2^c) \right]\,. \tag{70}$$

The Hessian in this case is $\det \|\partial^2 L/\partial \dot{A}\,\partial \dot{A}\| = 0$ hence the Lagrangian system (64) is degenerate. Following the Dirac's approach for treating systems with constraints we arrive at the set of constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}, \chi_{k\perp}^b$ satisfying the Poisson bracket algebra

$$\{\varphi_a^{(2)}\,, \varphi_b^{(2)}\} = \epsilon_{abc}\,\varphi_c^{(2)} \;\;, \{\chi_{i\perp}^a\,, \chi_{j\perp}^b\} = -2\,\epsilon^{abc}\,A_-^c\,\eta_{ij} \;\;, \{\varphi_a^{(2)}\,, \chi_{k\perp}^b\} = \epsilon^{abc}\,\chi_{k\perp}^c\,. \tag{71}$$

The other Poisson brackets are equal to zero.

From these relations we conclude that the model has 8 first-class constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}$ and 4 second-class constraints $\chi_{k\perp}^a$. Counting the degrees of freedom taking into account all these constraints, we obtain that instead of 24 constrained phase space degrees of freedom there are $24-2(5+3)-4=4$ unconstrained degrees of freedom, in contrast to the instant form of the SU(2) Yang-Mills classical mechanics where the number of the unconstrained canonical variables is 12.

3.3 Unconstrained version of the light-cone classical mechanics

Now we shall perform a Hamiltonian reduction of the degrees of freedom rewriting the theory in terms of special coordinates adapted to the action of this gauge symmetry. To do this let us organize the configuration variables A_k^a and A_-^a in one 3×3 matrix A_{ab} whose entries of the first two columns are A_k^a and third column is composed by the elements A_-^a

$$A_{ab} := \|A_1^a\,, A_2^a\,, A_-^a\|\,, \tag{72}$$

and the momentum variables similarly

$$\Pi_{ab} := \|\pi^{a1}\,, \pi^{a2}\,, \pi^{a-}\|. \tag{73}$$

Like in the previous Section in order to find an explicit parametrization of the orbits with respect to the gauge symmetry action, it is convenient to use a polar decomposition [18] for the matrix

$$A = OS\,, \tag{74}$$

and then the main-axes decomposition for the symmetric 3×3 matrix S

$$S = \mathcal{R}^T\,(\chi_1, \chi_2, \chi_3)\,\mathcal{Q}\,\mathcal{R}(\chi_1, \chi_2, \chi_3) \tag{75}$$

with $\mathcal{Q} = \mathrm{diag}\|q_1\,, q_2\,, q_3\|$ and orthogonal matrix $\mathcal{R}(\chi_1, \chi_2, \chi_3) = e^{\chi_1 J_3} e^{\chi_2 J_1} e^{\chi_3 J_3}$, parameterized by three Euler angles (χ_1, χ_2, χ_3).

It is in order to make a few remarks on the change of variables in (74). It is well-known that the polar decomposition is valid for an arbitrary matrix. However, the orthogonal matrix in (74) is uniquely determined only for an invertible matrix A

$$O = AS^{-1}\,, \qquad S = \sqrt{A^T A}\,. \tag{76}$$

The non-degenerate 3×3 matrices can be identified with an open set of the $\mathbb{R}^9$ using the entries of the matrix A_{ab} as corresponding Cartesian coordinates and in this case the polar decomposition (74) is a

[5] To simplify the formulas we shall use overdot to denote derivative of a function with respect to light-cone time τ.

uniquely invertible transformation from these Cartesian coordinates to a new set of coordinates, the entries of positive matrix S and the angles parameterized the orthogonal matrix O. For degenerate matrices a more sophisticated analysis is necessary. Here we note only that the set of $n \times n$ matrices with rank k is a manifold with dimension $k(2n - k)$, but in contrast the no-degenerate case the atlas of the manifold now necessarily contains several charts. Hence, for degenerate matrices A the representation (74) has to be replaced by a more elaborated construction.

Projection of the canonical Hamiltonian (69) to the constraint surface gives

$$H^{LC}_{YMM} = H^{LC}_{C}(\chi_1 = \frac{\pi}{2}\,, p_{\chi_1} = 0\,, \chi_2 = \frac{\pi}{2}\,, p_{\chi_2} = 0) = \frac{1}{2}\left(p_1^2\ + q_2^2\, q_3^2\right)\,. \tag{77}$$

It may be checked that the constraints $\chi^a_{k\perp}$ lead to the conditions on the "diagonal" canonical pairs $(q_k\,, p_k)$. Namely, the canonical momenta p_2 and p_3 are vanishing $p_2 = 0\,, p_3 = 0\,,$ while the corresponding coordinates q_2 and q_3 are subject to the constraint

$$q_2^2 + q_3^2 = 0 \tag{78}$$

as well the constraint

$$2\, q_1\, q_2\, q_3 - \xi^L_3 = 0\,. \tag{79}$$

Obviously, the *real* solution of the equation (78) is the only trivial one $q_2 = q_3 = 0$. For this solution according to the constraint (79) ξ^L_3 turns to be zero and thus the Hamiltonian (77) reduces further to a Hamiltonian of a free one-dimensional particle motion.

Let us consider the analytic continuation of the constraint (78) into a complex domain and explore its *complex* solution $q_2 = \pm\, i\, q_3$. Expressing q_3 from the equation (79)

$$q_3 = \frac{1 \mp i}{2}\sqrt{\frac{\xi^L_3}{q_1}}\,, \tag{80}$$

we find that (q_1, p_1) and (χ_3, p_{χ_3}) remain *real* unconstrained variables whose Dirac brackets are the canonical ones

$$\{q_1, p_1\}_D = 1\,, \qquad \{\chi_3, p_{\chi_2}\}_D = 1\,. \tag{81}$$

Therefore the dynamics of the unconstrained pairs (q_1, p_1) and (χ_3, p_{χ_3}) is given by the standard Hamilton equations with the Hamiltonian (77). Remarking that the ξ^L_3 is conserved we conclude that (77) coincides with the Hamiltonian of conformal mechanics

$$H^{LC}_{YMM} = \frac{1}{2}\left(p_1^2\ + \frac{\kappa^2}{q_1^2}\right) \tag{82}$$

with "coupling constant" $\kappa^2 = \left(\xi^L_3/2\right)^2$ determined by the value of the gauge spin, while the gauge field coupling constant g controls the scale for the evolution parameter.

From the equation (80) it follows that the quantity κ is the parameter which measures the deviation from the real classical trajectories. They all are laying in the subspace of matrices with $\det\|A\| = 0$ and are described as the integral curves of the Hamiltonian (77) with vanishing coupling constant $\kappa = 0$, and therefore indeed correspond to a free particle motion.

4 Concluding remarks

To conclude, we have considered the instant and the light-cone form of SU(2) Yang-Mills field theory supposing that the gauge potentials in the classical action are functions only of the time evolution parameter. As we have demonstrated this *ansatz* effectively reduces the field theory to a degenerate Lagrangian mechanical systems whose unconstrained versions significantly differ from each other. Comparing with the instant form dynamics, the light-cone version of the classical Yang-Mills mechanics has a more complicated description considered as a constrained system. Applying the Dirac's

Hamiltonian method, we found that in the case of light-cone form of the dynamics the constraint content of the theory is richer: there is, apart from the expected constraints which are generators of the SU(2) gauge transformations, a new set of first and second class constraints. The presence of the new constraints leads to an essential decreasing of the number of the "true" degrees of freedom and finally to its classical integrability.

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