Physics AUC, vol. 21-Sp.Issue, 242-248 (2011) **PHYSICS AUC**

# On $p$-Adic Pseudodifferential Operator(s)

D. D. Dimitrijević, G. S. Djordjević, Lj. Nešić
Department of Physics, Faculty of Science and Mathematics,
University of Niš,
P.O.Box 224, 18000 Niš, Serbia,
ddrag@pmf.ni.ac.rs, gorandj@junis.ni.ac.rs, nesiclj@junis.ni.ac.rs

**Abstract**

In this paper we review some properties of a "new" pseudodifferential operator with "a rational part" (as its symbol) over the fields of $p$-adic numbers $\mathbb{Q}_p$. We present a few new integrals and discuss similarities and differences between the "new" operator and the Vladimirov operator.

## 1 Introduction

The $p$-adic numbers were discovered by K. Hensel around the end of the nineteenth century. In the course of about one hundred years, the theory of $p$-adic numbers has penetrated into several areas of mathematics, including number theory, algebraic geometry, algebraic topology and analysis. Nevertheless, there are a lot of papers where different applications of $p$-adic analysis to physical problems, especially in the area of high energy physics [1, 2, 3] and cosmology [4, 5, 6, 7], stochastics, cognitive sciences and psychology [8, 9, 10], are studied.

Why do we need $p$-adic numbers at all? Conventional description of the physical space-time uses the field $\mathbb{R}$ of real numbers. In most cases, mathematical models based on $\mathbb{R}$ provide quite satisfactory descriptions of the physical reality. However, the result of a physical measurement is always a rational number, so the use of the completion $\mathbb{R}$ of the field of rational numbers $\mathbb{Q}$ is not more than a mathematical idealization. On the other hand, by Ostrowski theorem [11], the only reasonable alternative to $\mathbb{R}$ among completions of $\mathbb{Q}$ are the fields $\mathbb{Q}_p$ of $p$-adic numbers. For this reason, it is natural to use $p$-adic analysis in physical situations, where conventional space-time geometry is known to fail, for examples in the attempts to understand the matter at sub-Planck distances or time intervals. In order to do this, at first, it is necessary to develop $p$-adic counterparts of the standard quantum mechanics and quantum field theory. However, despite considerable success obtained in recent years, many interesting problems of $p$-adic quantum mechanics are still unsolved.

The study of pseudodifferential operators over the field of $p$-adic numbers $\mathbb{Q}_p$ ($p$ is any prime number) emerged in the late 1980's, in the study of Schrödinger-type equation in $p$-adic quantum mechanics [1], [12]. The idea was to write $p$-adic Schrödinger-type equation and, by solving it, to obtain complex-valued wave function with $p$-adic argument that will contain information about the system under consideration.

$p$-Adic pseudodifferential operators we are going to consider here are expressed with the help of $p$-adic norm $|\cdot|_p$ and rational part $\{\cdot\}_p$ of $p$-adic number. The difference between these two mapping is that $|\cdot|_p : \mathbb{Q}_p \mapsto [0, +\infty) \subset \mathbb{Q}$, while $\{\cdot\}_p : \mathbb{Q}_p \mapsto [0, 1) \subset \mathbb{Q}$. This is what we should have in mind when we analyze properties of the operators.

The paper is organized as follows. After the Introduction, in Section 2 we recapitulate basic facts about $p$-adic analysis. Section 3 is devoted to $p$-adic quantum mechanics. $p$-Adic pseudodifferential operators are introduced in Section 4. In Section 5 some properties of the pseudodifferential operator with rational part of $p$-adic number as a symbol are studied. We complete this paper with remarks and conclusions concerning further investigation.

## 2 $p$-Adic analysis

As we already mentioned, all numerical experimental results belong to the field of rational numbers $\mathbb{Q}$. The completion of this field with respect to the standard norm $|\cdot|_\infty$ (absolute value) leads to the field of real numbers $\mathbb{R} = \mathbb{Q}_\infty$. Besides absolute value and $p$-adic norms $|\cdot|_p$ there are no other nonequivalent and nontrivial norms on $\mathbb{Q}$ (the Ostrowski theorem). The completion of $\mathbb{Q}$ with respect to (a concrete prime number $p$) the $p$-adic norm leads to the (corresponding) $p$-adic number field $\mathbb{Q}_p$.

Any $p$-adic number $x \in \mathbb{Q}_p$, can be presented in the canonical form, as an infinite expansion [13]

$$x = p^\gamma \sum_{i=0}^{\infty} x_i p^i, \tag{1}$$

where $x_i$ are digits $0 \le x_i \le p-1$, $x_0 \ne 0$, and $\gamma$ is an integer, $\gamma \in \mathbb{Z}$. The $p$-adic norm of $p$-adic number $x$ in (1) is $|x|_p = p^{-\gamma}$, and it is nonarchimedean (ultrametric) one

$$|x+y|_p \le max(|x|_p, |y|_p), \quad x, y \in \mathbb{Q}_p. \tag{2}$$

Let us state the notation for the ring of $p$-adic integers, $p$-adic circle and disc, respectively:

$$Z_p = \{x \in \mathbb{Q}_p, |x|_p \le 1\}, \tag{3}$$

$$S_\gamma(a) = \{x \in \mathbb{Q}_p, |x-a|_p = p^\gamma\}, \tag{4}$$

$$B_\gamma(a) = \{x \in \mathbb{Q}_p, |x-a|_p \le p^\gamma\}, \tag{5}$$

$$\bigcup_{-\infty}^{\gamma} S_\gamma = B_\gamma. \tag{6}$$

$\mathbb{Q}_p$ is commutative group with respect to addition. Hence there exists an invariant measure (unique up to a factor) on $\mathbb{Q}_p$, i.e the Haar measure $dx$

$$d(x+a) = dx, \quad d(ax) = |a|_p dx, \tag{7}$$

which is normalized by a condition

$$\int_{Z_p} dx = \int_{B_0} dx = 1, \tag{8}$$

and the following formulae hold

$$\int_{B_\gamma} \psi(x)dx - \int_{B_{\gamma-1}} \psi(x)dx = \int_{S_\gamma} \psi(x)dx, \tag{9}$$

$$\int_{\mathbb{Q}_p} \psi(x)dx = \sum_{\gamma=-\infty}^{+\infty} \int_{S_\gamma} \psi(x)dx. \tag{10}$$

Let $O$ be an open set in $\mathbb{Q}_p$. A function $f : O \to \mathbb{C}$ is called locally constant in $O$, if for any $x \in O$ there exists $k \in \mathbb{Z}$, such that $f(x+a) = f(x)$ for all $a \in B_k$ or, equivalently, $|a|_p = p^k$.

The set of all locally constant functions in $O$ is denoted by $\varepsilon(O)$. A function $f \in \varepsilon(O)$ is called a test function in $O$ (the Bruhat-Schwartz function) if its support is compact in $O$. The set of test functions in $O$ we denote by $\mathcal{L}(O)$.

The set of functions $f : \mathbb{Q}_p \to \mathbb{C}$ for which $f(x) = 0 \Leftrightarrow x \notin O$ and

$$\| f \| = \left( \int_O |f(x)|^q dx \right)^{1/q} \le \infty \tag{11}$$

is denoted by $\mathcal{L}_q(O)$. Special case is the (Hilbert) space $\mathcal{L}_2(\mathbb{Q}_p)$ with the inner product

$$(f, g) = \int_{\mathbb{Q}_p} \overline{f}(x) g(x) dx. \tag{12}$$

The $p$-adic Fourier transform is defined as

$$F[\psi](y) = \widetilde{\psi}(y) = \int_{\mathbb{Q}_p} \psi(x)\chi_p(yx)dx, \tag{13}$$

$$F^{-1}[\widetilde{\psi}](x) = \psi(x) = \int_{\mathbb{Q}_p} \widetilde{\psi}(y)\chi_p(-yx)dy, \tag{14}$$

where $\chi_p(x) = \exp(2\pi i\{x\}_p)$ (an additive character on $\mathbb{Q}_p$) is a complex-valued continuous function. Recall that in the real case one has $\chi_\infty(x) = \exp(-2\pi ix)$. Here, $\{x\}_p$ denotes the fractional part of $p$-adic number

$$\{x\}_p = \begin{cases} 0, & \gamma \geq 0, \\ p^\gamma(x_0 + x_1p + x_2p^2 + ...x_{|\gamma|-1}p^{|\gamma|-1}), & \gamma < 0, \end{cases} \tag{15}$$

or putting it all together

$$\{x\}_p = p^\gamma(1 - \Omega(|x|_p))(x_0 + x_1p + ...x_{|\gamma|-1}p^{|\gamma|-1}), \tag{16}$$

where $\Omega(|x|_p)$ is a characteristic function of the ring of $p$-adic integers $Z_p$

$$\Omega(|x|_p) = \begin{cases} 0, & x \notin Z_p, \\ 1, & x \in Z_p. \end{cases} \tag{17}$$

It is necessary to write some properties of fractional part of $p$-adic number which will be used below. First of all, $\{\cdot\}_p$ is a mapping $\mathbb{Q}_p \mapsto [0, 1)$, such that

$$\{x + x'\}_p = \{x\}_p + \{x'\}_p - N(x, x'), \tag{18}$$

where $N(x, x')$ could be either 0 or 1. If one of the number belongs to $Z_p$ and the other one to $\mathbb{Q}_p/Z_p$ than $N(x, x') = 0$. A special case is

$$\{-x\}_p = 1 - \{x\}_p, \quad x \notin Z_p. \tag{19}$$

It is important to note that real and $p$-adic numbers are unified in the form of the adeles [14]. An adele is an infinite sequence

$$a = (a_\infty, a_2, ..., a_p, ...), \tag{20}$$

where $a_\infty \in \mathbb{Q}_\infty$, and $a_p \in \mathbb{Q}_p$, with restriction to $a_p \in Z_p$ ( $Z_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$) for all but a finite set $S$ of primes $p$. If we introduce $\mathcal{A}(S) = \mathbb{Q}_\infty \times \prod_{p\in S} \mathbb{Q}_p \times \prod_{p\notin S} Z_p$ then the space of all adeles is $\mathcal{A} = \bigcup_S \mathcal{A}(S)$, which is a topological ring.

Even mathematical analysis over $\mathbb{Q}_p$ and $\mathcal{A}(S)$ ($p$-adic integration, Fourier transformation, $p$-adic series, etc.) is just a relatively small part of the whole of mathematics, or of general mathematical physics, and is still too comprehensive to be discussed here. We recommend reference [1] to a reader interested in this part of modern mathematical physics, for details and references therein.

## 3 $p$-Adic quantum mechanics

$p$-Adic quantum mechanics has been developed in two different ways: in the first one, wave function is complex valued function of $p$-adic variable [1], and in the second one, $p$-adic wave function depends on $p$-adic variable [9]. If we prefer a simultaneous (adelic) treatment of standard quantum mechanics and all $p$-adics, including usual probabilistic interpretation developed in standard quantum mechanics, then the first formulation appears as unique one.

It is well known that the procedure of quantization is not unique. In foundations of standard quantum mechanics (over $\mathbb{R}$) one usually starts with a representation of the canonical commutation relation

$$[\hat{x}, \hat{k}] = i\hbar, \tag{21}$$

where $\hat{x}$ is a spatial coordinate operator and $\hat{k}$ is the corresponding momentum operator.

In formulation of $p$-adic quantum mechanics [15] operators action

$$\hat{x}\psi(x) \to x\psi(x), \quad \hat{k}\psi(x) \to -i\hbar\frac{d\psi(x)}{dx}, \tag{22}$$

has no meaning for $x \in \mathbb{Q}_p$ and $\psi(x) \in \mathbb{C}$, and there is no possibility to define $p$-adic momentum $\hat{k}$ (as well as Hamiltonian) operator, because in the real case they are infinitesimal generators of space and time translations, but, since $\mathbb{Q}_p$ is totally disconnected field, these infinitesimal transformations become meaningless [1], [15].

Despite this unpleasant facts, there is a way out. Finite transformations remain meaningful in $p$-adic case, i.e. we are forced to use Weyl operators of finite transformations, which, in an analogy with real case, should be symbolically written as

$$\hat{Q}_p(\alpha)\psi(x) = \chi_p(\alpha)\psi(x), \tag{23}$$

$$\hat{K}_p(\beta)\psi(x) = \psi(x+\beta). \tag{24}$$

The commutation relation (21) for operators $\hat{Q}_p$ and $\hat{K}_p$ now takes the form

$$\hat{Q}_p(\alpha)\hat{K}_p(\beta) = \chi_p(\alpha\beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha). \tag{25}$$

It is possible to introduce the product of unitary operators (a unitary representation of the Heisenberg-Weyl group)

$$W_p(z) = \chi_p(-\frac{1}{2}xk)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad z = \mathbb{Q}_p \times \mathbb{Q}_p. \tag{26}$$

Now, dynamics of a $p$-adic quantum model can be described by a unitary evolution operator $U(t)$ without using the Hamiltonian operator (and infinitesimal displacement). Therefore, $U(t)$ is formulated in terms of its kernel $\mathcal{K}_t(x,y)$

$$U_p(t)\psi(x) = \int_{\mathbb{Q}_p} \mathcal{K}_t(x,y)\psi(y)(x,y)dy. \tag{27}$$

The kernel $\mathcal{K}_t$ of the $p$-adic evolution operator is defined by the functional integral

$$\mathcal{K}_t = \int \chi_p\left(\int_0^t -\frac{1}{h}L(\dot{x},x)dt\right)\prod_t dx(t), \tag{28}$$

with properties analogous to the properties existing in standard quantum mechanics. In that way, $p$-adic quantum mechanics is given by a triple [1]

$$(\mathcal{L}_2(\mathbb{Q}_p), W_p(z), U_p(t)), \tag{29}$$

where $\mathcal{L}_2(\mathbb{Q}_p)$ is the Hilbert space on $\mathbb{Q}_p$, $W_p(z)$ is the unitary representation of the Heisenberg-Weyl group on $\mathcal{L}_2$ and $U_p(t)$ is the unitary representation of the evolution operator on $\mathcal{L}_2(\mathbb{Q}_p)$.

But, if one wants to write $p$-adic version of a Schrödinger-type equation, than the formalism of $p$-adic pseudodifferential operator should be engaged. Next Section is, for this reason, dedicated to $p$-adic pseudodifferential operator.

At the end of this Section, it is worth to mention that in many cases $p$-adic quantum cosmological models can be considered as a quantum mechanical models in $p$-adic (and adelic) approach [4], [16, 17].

# 4 $p$-Adic pseudodifferential operator(s)

In the real case, pseudodifferential operators are a generalization of differential operators. The idea is to think of a differential operator acting upon a function as the inverse Fourier transform of a polynomial in the Fourier variable times the Fourier transform of the function. This integral representation leads to a generalization of differential operators, which correspond to functions other than polynomials in the Fourier variable (as far as the integral converges).

A pseudodifferential operator in the $p$-adic case is defined in the analogous way as in the real case, that is, it is a maping $A : \psi \mapsto A(\psi)$, for $\psi : \mathbb{Q}_p \mapsto \mathbb{C}$

$$A\psi(x) = \int_{\mathbb{Q}_p} a(x,y)\tilde{\psi}(y)\chi_p(-yx)dy, \tag{30}$$

where $a(x,y)$ is a symbol of the pseudodifferential operator and $\tilde{\psi}(y)$ is the Fourier transform of a complex function $\psi(x)$.

Vladimirov introduced a pseudodifferential operator [1], [12],

$$D^{\alpha}_{VL}\psi(x) = \int_{\mathbb{Q}_p} |y|^{\alpha}_p \tilde{\psi}(y)\chi_p(-yx)dy, \tag{31}$$

with the symbol $|\cdot|^{\alpha}_p$ which is locally constant function in $\varepsilon(\mathbb{Q}_p/\{0\})$, and

$$F[D^{\alpha}_{VL}\psi(x)](y) = |y|^{\alpha}_p \tilde{\psi}(y). \tag{32}$$

An example of the orthogonal basis of eigenfunctions for Vladimirov operator one can find in [1] and [18] (and references therein).

This pseudodifferential operator leads to nonstationary Schrödinger-type equation [15] with potential $V(x)$ (we will omit here subscript "VL")

$$D_t\psi(x,t) = \frac{1}{|4|_p} D^2_x \psi(x,t) + V(x)\psi(x,t). \tag{33}$$

For $V(x) = 0$, one has free-particle equation with the solution which can be interpreted as a decomposition into plane waves

$$\psi(x,t) = \int_{\mathbb{Q}_p} \rho(k)\chi_p(\frac{k^2}{4}t - kx)dk. \tag{34}$$

While in the real case there is a solution of the Cauchy problem, such a solution does not exist in $p$-adic case [19], since

$$D_t G(x,t) - \frac{1}{|4|_p} D^2_x G(x,t) \neq \delta(x,t). \tag{35}$$

As a consequence of inequality relation (35), it is fully justified to explore other possibilities for pseudodifferential operators in $p$-adic quantum mechanics [20, 21].

## 5 $p$-Adic pseudodifferential operator with fractional part

Through consideration of commutation relation (26) one comes to idea that it is possibility to introduce a new pseudodifferential operator. B. Dragovich was the first one who proposed pseudodifferential operator with a rational part of $p$-adic number as a symbol instead of $p$-adic norm. This operator acts on the character as [20]

$$\mathcal{D}_x \chi_p(\frac{\alpha x}{h}) = \{\beta\frac{d}{dx}\}_p \chi_p(\frac{\alpha x}{h}) = 2\pi i\{\frac{\alpha\beta}{h}\}_p \chi_p(\frac{\alpha x}{h}). \tag{36}$$

In accordance with (36), and to provide the form of Vladimorov operator, the action of a new operator is proposed to be

$$\{\beta\frac{d}{dx}\}^n_p \psi(x) = \{\beta\frac{d}{dx}\}^n_p F^{-1}[\tilde{\psi}(k)](x), \tag{37}$$

that is

$$\{\beta\frac{d}{dx}\}_p^n\psi(x) = (2\pi i)^n\int_{\mathbb{Q}_p}\{-\frac{\beta k}{h}\}_p^n\tilde{\psi}(k)\chi_p(-\frac{kx}{h})dk. \tag{38}$$

Any application of this operator in the investigation of some particular $p$-adic models requires calculation of many new integrals. Recall that $\{\cdot\}_p$ maps $\mathbb{Q}_p$ in some set of rational numbers on the interval $[0,1)$. We list here a few of these results:

$$\int_{S_\gamma}\{x\}_p\chi_p(-\beta x)dx =$$
$$\begin{cases} 0, & for \quad \gamma\le 0 \quad or \quad M>0 \\ p^\gamma\frac{p-1}{2p}, & for \quad \gamma>0, M\le 0, \gamma\le |M| \\ \frac{1}{\chi_p(-\beta p^{-\gamma})-1}-\frac{p^M-1}{2}, & for \quad \gamma>0, M\le 0, \gamma=|M|+1 \\ \frac{1}{\chi_p(-\beta p^{-\gamma})-1}-\frac{1}{\chi_p(-\beta p^{-\gamma+1})-1}, & for \quad \gamma>0, M\le 0, \gamma\ge |M|+2, \end{cases} \tag{39}$$

where $|\beta|_p = p^M$;

$$\int_{B_\gamma}\{x\}_p\chi_p(-\beta x)dx = \begin{cases} 0, & for \quad \gamma\le 0 \quad \vee \quad M>0 \\ \frac{p^\gamma-1}{p}, & for \quad \gamma>0, \quad |\beta|_p p^\gamma\le 1 \\ \Omega(|\beta|_p)\frac{1}{\chi_p(-\beta p^{-\gamma})-1}, & for \quad \gamma>0, \quad |\beta|_p p^\gamma>1. \end{cases} \tag{40}$$

It can be shown that the pseudodifferential operator (we omit here all constants) acting as

$$\mathcal{D}_x\psi(x) = \int_{\mathbb{Q}_p}\{-y\}_p\tilde{\psi}(y)\chi_p(-yx)dy, \tag{41}$$

is selfadjoint operator, namely

$$(\mathcal{D}\psi,\phi) = (\psi,\mathcal{D}\phi)\,. \tag{42}$$

To prove this, we start with the definition of an inner product (12) in $\mathcal{L}_2(\mathbb{Q}_p)$ and we will use Fubini's theorem (which allow us change the order of the integration [1]):

$$\begin{gathered} \triangleright \qquad (\mathcal{D}\psi,\phi) = \int_{\mathbb{Q}_p}(\overline{\mathcal{D}\psi})(x)\phi(x)dx = \\ \int_{\mathbb{Q}_p}\left(\overline{\int_{\mathbb{Q}_p}\{-y\}_p\tilde{\psi}(y)\chi_p(-yx)dy}\right)\phi(x)dx = \\ \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\{-y\}_p\overline{\tilde{\psi}}(y)\chi_p(yx)\phi(x)dydx = \int_{\mathbb{Q}_p}\{-y\}_p\overline{\tilde{\psi}}(y)\tilde{\phi}(y)dy = \\ \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\{-y\}_p\overline{\psi}(x)\chi_p(-yx)\tilde{\phi}(y)dxdy = \int_{\mathbb{Q}_p}\overline{\psi}(x)(\mathcal{D}\phi)(x)dx = \\ (\psi,\mathcal{D}\phi)\,. \qquad \triangleleft \end{gathered} \tag{43}$$

Although at the first glance it is very similar to Vladimirov operator, this two pseudodifferential operators have very diferent properties. According to that, one can show that $p$-adic Schrödinger-type equation for the free quantum mechanical particle, with the "new" pseudodifferential operator gives very interesting relation between "$p$-adic" energy $E$ and "$p$-adic" momentum $k$ [20]

$$\{E\}_p \sim \{k\}_p^2, \tag{44}$$

namely, energy for free particle possesses rather discrete that continual dependence on momentum.

# 6 Conclusion

Just a part of the results presented here, express very complex mathematical nature of pseudodifferential operator with rational part. It serves a good starting point for investigation of its spectral properties, related to the standard quantum mechanical models, as well as to other parts of physics treated by means of $p$-adic analysis. As we noted, a lot of new $p$-adic integrals (containing rational part of $p$-adic number) should be calculated, besides many of them we already done, in order to reveal new properties of this operator and new physical models.

## Acknowledgements

D. D. D. and G. S. Dj. would like to thank Prof. R. Constantinescu and the organizers for their kind hospitality. In addition, all authors express their gratitude to Prof. B. Dragovich for useful discussions. This work is partially supported by the Ministry of Science of the Republic of Serbia under Grants 174020 and 176021. Work on this paper is also supported in a part by the Southeastern European Network in Mathematical and Theoretical Physics (SEENET-MTP) and the ICTP Project PRJ-09.

## References

[1] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.

[2] I. V. Volovich, CERN-TH 4781 (1987).

[3] I. V. Volovich, *Class. Quantum. Gravity* **4**, 83 (1987).

[4] D. D. Dimitrijevic, G. S. Djordjevic, Lj. Nesic, *Fortschr. Phys.* **56**, No. 4–5, 412 (2008).

[5] D. D. Dimitrijevic, G. S. Djordjevic, Lj. Nesic, *Physics AUC*, **18**, 166 (2008).

[6] G. S. Djordjevic, D. D. Dimitrijevic, *Physics AUC* **20**, Part I, 9 (2010).

[7] N. Barnaby, T. Biswas and J. M. Cline, *JHEP* **0704**, 056 (2007).

[8] I. Ya. Aref'eva, B. G. Dragovic, I. V. Volovich, *Phys. Lett.* **B** 209(4), 445 (1998).

[9] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer Acad. Publ., Dordrecht (1997).

[10] A. Yu. Khrennikov, *Information Dynamics in Cognitive, Psychological, Social and Anomalous Phenomena*, Kluwer Academic, Dordrecht (2004).

[11] A. Ostrowski, *Acta Math.* **41**, 271 (1918)

[12] V. S. Vladimirov, *Usp. Math. Nauk.* **43**, 17 (1988); *Russ. Math. Surveys*, v. 43, no. **5**, 19 (1988); *Leningrad Math. J.*, v. 2, no. **6**, 1261 (1991); *Acad. Sci. Izv., Math.*, v. 41, no. **1**, 55 (1993).

[13] W. H. Schikhof, *Ultrametric Calculus*, Cambridge Univ. Press (1984)

[14] I. M. Gel'fand, I. M. Graev, I. I. Piatetskii-Shapiro, *Representation Theory and Automorphic Functions*, Saunders, London (1966).

[15] V. S. Vladimirov and I. V. Volovich, *Commun. Math. Phys.* **123**, 659 (1989).

[16] G. S. Djordjevic, B. Dragovich, Lj. D. Nesic and I. V. Volovich, *Int. J. Mod. Phys.* **A17**, 1413 (2002).

[17] G. Djordjevic, B. Dragovich and Lj. Nesic, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **6**, 179 (2003)

[18] S. V. Kozyrev, *Izv. Ross. Akad. Nauk Ser. Mat.* **66**, no. 2, 149 (2002).

[19] A. N. Kochubei, *J. Math. Phys.* **34**, 3420 (1993).

[20] D. Dimitrijevic, G. S. Djordjevic and B. Dragovich, *Bul. Jour. Phys.* **3** Vol. 27, 50 (2000).

[21] D. D. Dimitrijević, G. S. Djordjević and Lj. Nešić, Proc. of the BPU5: Fifth General Conference of the Balkan Physical Union, August 25-29, V. Banja, Serbia and Montenegro, 1231 (2003).