Physics AUC, vol. 21-Sp.Issue, 234-241 (2011) **PHYSICS AUC**

Two Component Long Wave-Short Wave Resonant Interaction in a Madelung Fluid Description*

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Abstract

The interaction between two bright solitons (short waves) and a dark one (long wave) is discussed and in the resonance condition, using a multiple scales analysis, a two component Zakharov-Yajima-Oikawa system is obtained. A Madelung fluid description is used to discuss this system and several solutions are presented.

PACS: 42.81 Dp, 05.45.Yv

1 Introduction

In many physical applications more than one single wave is propagating in a nonlinear medium, and the interaction of several waves has to be taken into account. Such examples are: 1) the propagation of solitonlike pulses in birefringent optical fibers [1]-[4]; 2) nonlinear waves in Bose-Einstein condensates [5], [6]; 3) soliton propagation through optical fiber array [1], [7]-[9]; 4) nonlinear dynamics of gravity waves in crossing sea states [10], to mention only few physical phenomena where these "'vector solitons"' play a fundamental role.

Many years ago a discussion of bright-dark soliton interaction was given by Kivshar [11]. The interesting fact of this analysis was the reduction of the problem in certain conditions to the completely integrable system of Zakharov-Yajima-Oikawa [12], [13] (see eqs. (9) and (10) of [11]). The special conditions refer to the existence of a "long wave-short wave resonance" (LW-SW resonance). This resonance phenomena has quite a large universality. In plasma physics it describes Langmuir solitons moving near the speed of sound [12], [14], in hydrodynamics it appears in the study of internal gravity waves [15] and a general study of LW-SW resonance [16], [17] in quasi-one-dimensional molecular crystals it describes the resonance between the excitonic and phonon fields in Davydov's model [18], to mention only few such examples. Recently extensions of the LW-SW resonance to two dimensions and more components have been discussed and solved by several authors [22]-[24].

In the present paper the interaction between two bright solitons (short waves) and a dark one (long wave) will be discussed and in the resonance condition a two component Zakharov-Yajima-Oikawa system will be obtained. Besides its relevance in nonlinear optics the same system is describing a Davydov model with two excitonic components [25]. In the next section the basic equations describing the three wave interaction will be written down and using a multiple scales analysis the Zakharov-Yajima-Oikawa system is obtained. In section three a Madelung fluid description is used to discuss this system and several solutions are presented. Few conclusions will be presented in the last section.

2 Basic equations and multiple scales analysis

We consider three nonlinear dispersive waves propagating in an optical fiber. Suppose that the dispersion relations of these weakly nonlinear waves are $\omega_i = \omega_i(k_i : |A_1|^2, |A_2|^2, |A_3|^2),\ i = 1, 2, 3$ and we

*Work supported by CNCSIS research project Idei 571/2008

consider $e^{i(k_0x-\omega_0t)}$ to be a basic carrier wave. Then a Taylor expansion around (k_0,ω_0) and $|A_i|=0$ of each ω_i will give

$$\omega_i-\omega_0=\left(\frac{\partial\omega_i}{\partial k_i}\right)_0(k_i-k_0)+\frac{1}{2}\left(\frac{\partial^2\omega_i}{\partial k_i^2}\right)_0(k_i-k_0)^2+\left(\frac{\partial\omega_i}{\partial|A_1|^2}\right)_0|A_1|^2+$$
$$\left(\frac{\partial\omega_i}{\partial|A_2|^2}\right)_0|A_2|^2+\left(\frac{\partial\omega_i}{\partial|A_3|^2}\right)_0|A_3|^2+... \tag{1}$$

Replacing $\omega_i-\omega_0\simeq -i\frac{\partial}{\partial t}$, $\quad k_i-k_0\simeq i\frac{\partial}{\partial x}$, after a translation of coordinate ($x\to x-\left(\frac{\partial\omega_3}{\partial k_3}\right)_0 t$), the following nonlinear system of three interacting waves is obtained

$$\begin{aligned}
&i\frac{\partial A_1}{\partial t}+iV_1\frac{\partial A_1}{\partial x}+\frac{\alpha_1}{2}\frac{\partial^2A_1}{\partial x^2}+\alpha_2|A_1|^2A_1+\alpha_3|A_2|^2A_1+\alpha_4|A_3|^2A_1=0\\
&i\frac{\partial A_2}{\partial t}+iV_2\frac{\partial A_2}{\partial x}+\frac{\beta_1}{2}\frac{\partial^2A_2}{\partial x^2}+\beta_2|A_1|^2A_2+\beta_3|A_2|^2A_2+\beta_4|A_3|^2A_2=0\\
&i\frac{\partial A_3}{\partial t}+\frac{\gamma_1}{2}\frac{\partial^2A_3}{\partial x^2}+\gamma_2|A_1|^2A_3+\gamma_3|A_2|^2A_3+\gamma_4|A_3|^2A_3=0.
\end{aligned}\tag{2}$$

Here we denoted $V_i=\left(\frac{\partial\omega_i}{\partial k_i}\right)_0-\left(\frac{\partial\omega_3}{\partial k_3}\right)_0$, $\quad i=1,2$ and the constants $\alpha_1,\beta_1,\gamma_1$ are related to derivatives of ω_i with respect to k_i (ex. $\alpha_1=-\left(\frac{\partial^2\omega_1}{\partial k_1^2}\right)_0,.$) while $\alpha_2,...\gamma_4$ to the derivatives with respect to $|A_i|^2$ (ex. $\alpha_2=\left(\frac{\partial\omega_1}{\partial|A_1|^2}\right)...$).

Further on we shall consider channel 3 with normal dispersion and 1 and 2 with anomalous dispersion [1]. Then following Kivshar [11] it is convenient to introduce new field variables

$$A_1=\Psi_1e^{i\delta_1t},\qquad A_2=\Psi_2e^{i\delta_2t},\qquad A_3=(u_0+a(x,t))e^{i(\Gamma t+\phi(x,t))}$$
$$\delta_i=\left(\frac{\partial\omega_i}{\partial|A_3|^2}\right)_0u_0^2,\qquad \Gamma=\left(\frac{\partial\omega_3}{\partial|A_3|^2}\right)_0u_0^2.$$

($u_0,\ a$ real quantities) and the system (2) becomes

$$\begin{aligned}
&i\frac{\partial\Psi_1}{\partial t}+iV_1\frac{\partial\Psi_1}{\partial x}+\frac{\alpha_1}{2}\frac{\partial^2\Psi_1}{\partial x^2}+\left(\alpha_2|\Psi_1|^2+\alpha_3|\Psi_2|^2\right)\Psi_1+2\alpha_4u_0a\Psi_1+\alpha_4a^2\Psi_1=0\\
&i\frac{\partial\Psi_2}{\partial t}+iV_2\frac{\partial\Psi_2}{\partial x}+\frac{\beta_1}{2}\frac{\partial^2\Psi_2}{\partial x^2}+\left(\beta_2|\Psi_1|^2+\beta_3|\Psi_2|^2\right)\Psi_2+2\alpha_4u_0a\Psi_2+\beta_4a^2\Psi_2=0\\
&\frac{\partial^2a}{\partial t^2}+\gamma_1\gamma_4u_0^2\frac{\partial^2a}{\partial x^2}+\frac{\gamma_1^2}{4}\frac{\partial^4a}{\partial x^4}+\frac{\gamma_1}{2}\frac{\partial^2}{\partial x^2}(\gamma_2u_0|\Psi_1|^2+\gamma_3u_0|\Psi_2|^2)
\end{aligned}\tag{3}$$
$$+(\textit{higher order nonlinear terms in } (a,\phi) \textit{ and their derivatives})=0.$$

The linear part of the a equation corresponds to an acoustic field with dispersion relation ($\gamma_1<0,\ \gamma_4>0$)

$$\omega=ck\sqrt{1+\frac{\gamma_1^2}{4c^2}k^2}\simeq ck\left(1+\frac{\gamma_1^2}{8c^2}k^2\right)$$

and phase velocity $c=\omega/k$, where $c^2=|\gamma_1|\gamma_4u_0^2$.

We shall perform a multiple scales analysis of the system (3) [11]. We introduce new scaled variables

$$t\Rightarrow\epsilon t,\quad x\Rightarrow\sqrt{\epsilon}(x-ct)$$

and new functions

$$a\Rightarrow\epsilon a,\quad \phi\Rightarrow\epsilon\phi,\quad \Psi_1\Rightarrow\epsilon^{\frac{3}{4}}\Psi_1,\quad \Psi_2\Rightarrow\epsilon^{\frac{3}{4}}\Psi_2.$$

Then in order $\frac{5}{2}$ in ϵ from a equation we obtain

$$-2c\frac{\partial a}{\partial t}+\frac{\gamma_1}{2}\frac{\partial}{\partial x}\left(\gamma_2u_0|\Psi_1|^2+\gamma_3u_0|\Psi_2|^2\right)=0. \tag{4}$$

All the nonlinear terms in a equation contribute to higher order in ϵ. In the order $\frac{5}{4}$ from Ψ_i equations we obtain $V_1 = V_2 = c$. This is the well known long wave-short wave (LW-SW) resonance condition:" *the group velocity V of the SW is equal to the phase velocity of the LW*" [16]. In the next order $\left(\frac{7}{4}\right)$ in ϵ from the Ψ equations we get

$$i\frac{\partial \Psi_1}{\partial t} + \frac{\alpha_1}{2}\frac{\partial^2 \Psi_1}{\partial x^2} + 2\alpha_4 u_0 a \Psi_1 = 0$$
$$i\frac{\partial \Psi_2}{\partial t} + \frac{\beta_1}{2}\frac{\partial^2 \Psi_2}{\partial x^2} + 2\beta_4 u_0 a \Psi_2 = 0 \tag{5}$$

The equations (4)+(5) represent an 1-dimensional 2-components Zakharov [12], Yajima-Oikawa [13] system. As mentioned in the Introduction the same system in the same LW-SW resonance condition was obtained in a Davydov model with two excitonic modes coupled with a phonon field [25]. The same line of reasoning was used in [24] for three interacting waves in 2-dimensions.

3 Madelung fluid description

The special case ($\alpha_i = \beta_i, \quad \gamma_2 = \gamma_3$) is completely integrable [24] and will be considered in the following. In this case, simplifying the notations, the system (5) is written in the following form ($\gamma > 0, \ \beta > 0$)

$$\frac{\partial a}{\partial t} + \gamma \frac{\partial}{\partial x}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = 0$$
$$i\frac{\partial \Psi_i}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_i}{\partial x^2} + \beta \Psi_i a = 0, \qquad i = 1, 2. \tag{6}$$

The Ψ_i equations will be transformed using a Madelung fluid description [26], [27].

$$\Psi_i = \sqrt{\rho_i}\, e^{i\theta_i},$$

where ρ_i, θ_i are real functions of (x, t) and moreover ρ_i are positive quantities. Introducing this expression into a-equation this becomes

$$\frac{\partial a}{\partial t} + \gamma \frac{\partial}{\partial x}(\rho_1 + \rho_2) = 0, \tag{7}$$

while from the Ψ_i equations, after the separation of real and imaginary parts, we obtain

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(v_i \rho_i) = 0 \tag{8}$$

which is a continuity equation for the fluid densities $\rho_i = |\Psi_i|^2$ with $v_i(x,t) = \frac{\partial \theta_i(x,t)}{\partial x}$ the fluid velocities components and

$$-\frac{\partial \theta_i}{\partial t} + \frac{1}{2}\frac{1}{\sqrt{\rho_i}}\frac{\partial^2 \sqrt{\rho_i}}{\partial x^2} - \frac{1}{2}\left(\frac{\partial \theta_i}{\partial x}\right)^2 + \beta a = 0. \tag{9}$$

Derivating this last expression with respect to x the following equations of motion for the fluid velocities v_i are obtained

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x}\right) v_i = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho_i}}\frac{\partial^2 \sqrt{\rho_i}}{\partial x^2}\right) + \beta \frac{\partial a}{\partial x}. \tag{10}$$

In the right hand side of (10) $a(x,t)$ plays the role of external potential, and the first term is the derivative of the so called Bohm potential, $\frac{1}{2}\frac{1}{\sqrt{\rho_i}}\frac{\partial^2 \sqrt{\rho_i}}{\partial x^2}$, and contains all the diffraction effects (quantum effects in quantum problems). By a series of transformations the equation (10) is written as [27]

$$-\rho_i \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho_i}{\partial t} + 2\left[c_i(t) - \int \frac{\partial v_i}{\partial t}\, dx\right] \frac{\partial \rho_i}{\partial x}$$

$$+\frac{1}{4}\frac{\partial^3\rho_i}{\partial x^3}+\beta\rho_i\frac{\partial}{\partial x}a+2\beta a\frac{\partial\rho_i}{\partial x}=0 \tag{11}$$

where c_i are arbitrary integration constants with respect to x, eventually time dependent. Although (11) seams to be more complicated then the initial equation (10), it can be solved in two special situations, namely

- motion with constant velocities $v_1 = v_2 = v_0$
- motion with stationary profile current velocity, when all the quantities $\rho_i(x,t),\ v_i(x,t),\ a(x,t)$ are depending on x and t through the combination $\xi = x - u_0 t$.

Both cases will be analyzed in the following.

4 Motion with constant velocity ($v_1 = v_2 = v_0$)

In this case from the continuity equations (8) one sees that both $\rho_1(x,t)$ and $\rho_2(x,t)$ depend on $\xi = x - v_0 t$. We assume that also $a(x,t)$ depends only on ξ. Then the a equation gives

$$a=\mu(\rho_1+\rho_2), \qquad \mu=\frac{\gamma}{v_0} \tag{12}$$

and the equations (11) writes

$$\frac{1}{4}\frac{d^3\rho_i}{d\xi^3}-E_i\frac{d\rho_i}{d\xi}+2\beta a\frac{d\rho_i}{d\xi}+\beta\rho_i\frac{da}{d\xi}=0, \tag{13}$$

where by E_i we denoted $-(2c_i - v_0^2)$. We shall discuss firstly the situation $E_1 = E_2$, the discussion of the more general case $E_1 \neq E_2$ being postponed for the next section. Then the equations (13) becomes ($\beta\mu \to \mu$)

$$\frac{1}{4}\frac{d^3\rho_i}{d\xi^3}-E\frac{d\rho_i}{d\xi}+\mu\rho_i\frac{d}{d\xi}(\rho_1+\rho_2)+2\mu(\rho_1+\rho_2)\frac{d\rho_i}{d\xi}=0. \tag{14}$$

These are exactly as the equations obtained in the case of Manakov's model [28] and extensively discussed by us in [29], [30]. In the following we shall present several periodic and traveling wave solutions of (14).

Introducing the quantity $z_+ = \rho_1+\rho_2 \quad (\xi \to 2\xi)$ by adding the two equations (14) we get ($2\xi \to \xi$)

$$\frac{d^3z_+}{d\xi^3}-E\frac{dz_+}{d\xi}+\frac{3}{2}\mu\frac{d}{d\xi}z_+^2=0, \tag{15}$$

which integrated twice gives

$$\frac{1}{4}\left(\frac{dz_+}{d\xi}\right)^2=-\mu z_+^3+Ez_+^2+Az_++B=P_3(z_+). \tag{16}$$

Subtracting the two equations (14) and denoting $z_- = \rho_1 - \rho_2$ we get

$$\frac{d^3z_-}{d\xi^3}-E\frac{dz_-}{d\xi}+\mu z_-\frac{dz_+}{d\xi}+2\mu z_+\frac{dz_-}{d\xi}=0, \tag{17}$$

a linear differential equation in z_- once $z_+(\xi)$ is known. A special solution is

$$z_-=(p_1^2-p_2^2)z_+ \qquad p_1^2+p_2^2=1 \tag{18}$$

which together with the definition of z_+ gives

$$\rho_1=p_1^2z_+, \qquad \rho_2=p_2^2z_+. \tag{19}$$

But for constant velocities, as is easily seen from (10) the densities ρ_i have to satisfy the additional conditions

$$\frac{1}{2}\frac{1}{\sqrt{\rho_i}}\frac{\partial^2\sqrt{\rho_i}}{\partial x^2} + \mu z_+(\xi) = \lambda_i, \tag{20}$$

which for the previous solutions (19) lead us to the following constraint ($\lambda_1 = \lambda_2 = \lambda$)

$$\frac{d^2 z_+}{d\xi^2} - \frac{1}{2z_+}\left(\frac{dz_+}{d\xi}\right)^2 + \mu z_+ - \lambda z_+ = 0. \tag{21}$$

It is easily seen that this is satisfied if $\lambda = \frac{E}{2} \qquad B = 0.$

Now let us assume that the third order polynomial $P_3(z_+)$ has three distinct roots.

$$P_3(z_+) = -\mu(z_+ - z_1)(z_+ - z_2)(z_+ - z_3). \tag{22}$$

The restriction $B = 0$ means that one of the roots z_2 or z_3 is zero. We are interested in positive solutions of (16) for which $P_3(z_+)$ is also positive. The periodic solutions of (16) can be expressed through Jacobi elliptic functions and taking into account the positivity requirement mentioned before we identify two acceptable situations

$$z_1 > 0, \qquad z_2 = 0, \qquad z_3 < 0$$

$$z_+ = z_1 \, \mathsf{cn}^2 u \tag{23}$$

$$u = \frac{2\sqrt{\mu}}{g}\xi, \qquad k^2 = \frac{z_1}{z_1 + |z_3|}, \qquad g = \frac{2}{\sqrt{z_1 + |z_3|}}$$

$$z_3 = 0, \quad 0 < z_2 < z_3$$

$$z_+ = z_1 - (z_1 - z_2)\mathsf{sn}^2 u \tag{24}$$

$$u = \frac{2\sqrt{\mu}}{g}\xi, \qquad k^2 = \frac{z_1 - z_2}{z_1}, \qquad g = \frac{2}{\sqrt{z_1}}.$$

Solitary wave solutions are obtained in the limiting case $k = 1$ when $\mathsf{cn}\ u \to \mathsf{sech}\ u, \quad \mathsf{sn}\ u \to \mathsf{tanh}\ u,$ and both solutions (23) and (24) become a bright soliton

$$z_+ \to z_1 \frac{1}{\mathsf{cosh}^2 u} \qquad u = \frac{2\sqrt{\mu}}{g}, \qquad g = \frac{2}{\sqrt{z_1}}. \tag{25}$$

In conclusion, in this case of equal velocities $v_1 = v_2 = v_0$, and equal "energies" $E_1 = E_2 = E$, the solutions of the Ψ equations are bright solitons. It is clear that in this case no energy transfer between the two components takes place.

The phase $\theta(x,t)$ is easily calculated writing $\theta_i(x,t) = v_0 x + \gamma_i(t)$; then using (9) we get

$$\theta_i = v_0 x - \left(\frac{1}{2}v_0^2 - \frac{E}{2}\right)t + \delta_i \tag{26}$$

5 Motion with stationary-profile current velocity

In the case when all the functions depend only on $\xi = x - u_0 t$ integrating the continuity equation (8) we get

$$v_i(x,t) = u_0 + \frac{A_i}{\rho_i}, \tag{27}$$

with A_i some integration constants. It is easily seen that the equations of motion keep the same form as (14) with $E_i = -(2c_i + u_0^2)$. We shall discuss again the case $E_1 = E_2 = E$ and the same equations as before are obtained, but without any restrictions on the roots of the polynomials $P_3(z_+)$. The periodic solution is given by ($0 < z_2 < z_1$)

$$z_+ = z_1 - (z_1 - z_2)\mathsf{sn}^2 u \tag{28}$$

$$u = \frac{2\sqrt{\mu}}{g}\xi, \qquad k^2 = \frac{z_1 - z_2}{z_1 - z_3}, \qquad g = \frac{2}{\sqrt{z_1 - z_2}}$$

which in the degenerate case becomes

$$z_+ = z_1 - (z_1 - z_2)\tanh^2 u, \tag{29}$$

describing a bright soliton with nonvanishing values at infinity (shifted-bright soliton). The expression of the phase takes a more complicated form containing incomplete elliptic integral of third kind [29].

For $E_1 \neq E_2$ it is convenient to use a direct method to solve the coupled system of equations (13). We seek after solutions of the form

$$\rho_i = A_i + B_i \mathsf{sn}\ u, \qquad u = 2\lambda\xi, \tag{30}$$

with $A_i,\ B_i,\ \lambda$ constants to be determined. Introducing into (13) we get [29]

$$B_1 + B_2 = -\frac{4\lambda^2 k^2}{\gamma} \tag{31}$$

$$-[4\lambda^2(1+k^2) + E_i]B_i + \gamma(B_1 + B_2)A_i + 2\gamma(A_1 + a_2)B_i = 0.$$

Writing

$$B_i = -\frac{-4\lambda^2 k^2}{\gamma}b, \qquad A_i = \frac{4\lambda^2 k^2}{\gamma}a_i \tag{32}$$

the first equation (31) gives

$$b_1 + b_2 = 1. \tag{33}$$

It is convenient to write

$$E_1 = 4\lambda^2 k^2(e + \delta), \qquad E_2 = 4\lambda^2 k^2 (e - \delta) \tag{34}$$

and without any loss of generality to consider $\delta > 0$. We get [29]

$$\begin{aligned} a_1 &= \frac{1}{3}\left(e + \frac{1+k^2}{k^2} + \delta + 4\delta(1 - b_1)\right) b_1 \\ a_2 &= \frac{1}{3}\left(e + \frac{1+k^2}{k^2} - \delta - 4\delta(1 - b_2)\right) b_2. \end{aligned} \tag{35}$$

As is expected this result verify the symmetry condition $1 \leftrightarrow 2$ if $\delta \leftrightarrow -\delta$.

Several restrictions result from the positiveness of ρ_i. If both b_i are positive quantities smaller than unity this requirement implies

$$a_i > b_i > 0. \tag{36}$$

Introducing the notation

$$\mu = \frac{1}{3}\left(e + \frac{1+k^2}{k^2} - \delta(1 + 4b)\right) \tag{37}$$

the condition (36) is satisfied if $\mu > 1$. In the limiting case $k^2 = 1$ the solutions are [29]

$$\begin{aligned} \rho_1 &= \frac{4\lambda^2}{\gamma} b(\mu + 2\delta - \tanh^2 u) \\ \rho_2 &= \frac{4\lambda^2}{\gamma}(1 - b)(\mu - \tanh^2 u) \end{aligned} \tag{38}$$

representing shifted bright solitons.

6 Conclusions

A special case of a vector soliton problem is considered, namely the interaction of three nonlinear waves, where two of them correspond to bright solitons (anomalous dispersion) and one to a dark one (normal dispersion). Using a multiple scales analysis a two-component one-dimensional Zakharov-Yajima-Oikawa system is obtained in the case of long wave-short wave resonance. The system corresponding to the bright solitons is discussed further using a Madelung fluid description. Periodic solutions expressed through Jacobi elliptic functions and stationary solutions obtained when $k^2 = 1$ are presented in two simplifying conditions, namely for constant velocity and for motion with stationary profile. This two-component Zakharov - Yajima - Oikawa system is completely integrable and many-solitons solutions can be found using different methods, as the bi-linear method of Hirota [24]. The Madelung fluid description could be useful to find various solutions of a generalized ZYO system, containing additional nonlinear terms.

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