Polarized Vector Bosons on the de Sitter Expanding Universe

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Abstract

The quantum theory of the vector field minimally coupled to the gravity of the de Sitter spacetime is built in a canonical manner starting with a new complete set of quantum modes of given momentum and helicity derived in the moving chart of conformal time. It is shown that the canonical quantization leads to new vector propagators which satisfy similar equations as the propagators derived by Tsamis and Woodard [*J.Math.Phys.* **48** (2007) 052306] but having a different structure.

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1   Introduction

In the quantum theory of fields on curved spacetimes the de Sitter (dS) expanding universe carrying fields variously coupled to gravity is of a special interest [1, 2]. The free scalar and Dirac fields minimally coupled to gravity were studied in static charts as well as in the (co)moving charts with proper or conformal times. The free massive vector field is considered only in static charts [3] with spherical coordinates and, therefore, the problem of its quantum modes in moving charts is still open.

In the simplest case of scalar fields the quantum theory leads to propagators which depend on the geodesic length [4, 5]. This fact suggests that the difficulties arising in the theory of vector fields can be avoided in an elegant manner considering directly two-point functions depending only on the geodesic length and its derivatives [6, 7]. By using this method one skips the principal steps of the traditional quantum theory based on the canonical quantization and, consequently, some delicate problems may remain unsolved. In other respects, the flat limit of the quantum field theory on curved spacetimes must recover the traditional theory where the canonical quantization is working. For this reason we emphasize that it deserves to construct the canonical quantum field theory on dS manifolds following the same steps as in special relativity.

To this aim we have then all the elements we need. First of all we can analytically solve the equations of the principal fields minimally coupled to the dS gravity. Moreover, the $SO(1,4)$ symmetry of the dS manifolds provides us with a large collection of operators commuting with those of the field equations [10, 11]. Thus we can choose complete sets of commuting operators which should determine suitable systems of fundamental solutions as common eigenfunctions of the operators of these sets. Normalizing the solutions with respect to a well-defined relativistic scalar product, we may then obtain the complete systems of fundamental solutions we need for canonically quantizing the free fields. On the dS spacetime the momentum operators commute with that of the field equation and, therefore, there are fundamental solutions representing eigenfunctions of the momentum operators. Such solutions form the momentum basis in which the Hamiltonian operator is not diagonal since it does not commute with the momentum operators. In this context, we proposed a new time-evolution picture [12] which allowed us to introduce the energy basis which completes the quantum theory of the scalar field [13] on the dS expanding universe. Within the same conjecture we developed the quantum theory of the Dirac [14, 16] and Maxwell [17] fields. What then remains is to study the Proca theory of the vector fields whose propagators or two-point functions in moving charts are of actual interest in cosmology [8, 9]. For this reason we present here, the canonical quantum theory of the vector field on dS moving charts as we proposed recently [15].

We start in the second section with a brief review of the Proca theory on the moving dS charts with conformal time, by introducing then the principal conserved operators generated by the Killing vectors. Furthermore, in section 3 we derive the complete set of fundamental solutions of the field equation determined by momentum and helicity. The next section is devoted to the canonical quantization of the vector field which leads to the new Green functions we look for. In section 5 we derive the principal one-particle operators in momentum representation.

The principal results presented here are the vector quantum modes of given momentum and helicity and the vector propagators of the canonical quantum theory on the dS spacetime. We must stress that these propagators are different from the maximally symmetric two-point functions derived before by Allen and Jacobson [6] but satisfy the equations proposed by Tsamis and Woodard [7].

2 Preliminaries

In a given local chart of coordinates x^μ ($\mu, \nu, ... = 0, 1, 2, 3$) and the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \,, \tag{1}$$

of a curved spacetime, (M, g), the Proca theory of the massive charged vector field A minimally coupled to gravity has the action

$$\mathcal{S}[A] = \int d^4x \sqrt{g}\, \mathcal{L} = \int d^4x \sqrt{g} \left[-\frac{1}{2}\, F^*_{\mu\nu}F^{\mu\nu} + m^2 A^*_\mu A^\mu \right], \tag{2}$$

where m is the mass, $g = |\det(g_{\mu\nu})|$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. From the resulted field equations,

$$\partial_\nu(\sqrt{g}\, g^{\nu\alpha} g^{\mu\beta} F_{\alpha\beta}) + m^2 \sqrt{g}\, A^\mu = 0 \,, \tag{3}$$

one deduces the Lorentz (or transversality) condition

$$\partial_\mu(\sqrt{g} A^\mu) = 0 \,, \tag{4}$$

which guarantees the uniqueness of the spin $s = 1$.

The whole theory is invariant under the $U(1)$ internal symmetry transformations (generated by the identity operator I) and the isometries related to the Killing vectors of (M, g). For each isometry transformation $x \to x' = \phi_\xi(x)$ depending on the group parameter ξ there exists an associated Killing vector field, $K = \partial_\xi \phi_\xi|_{\xi=0}$ (which satisfy the Killing equation $K_{\mu;\nu} + K_{\nu;\mu} = 0$). Under such isometry the vector field transforms as $A \to A' = T_\xi A$, according to the operator-valued representation $\xi \to T_\xi$ of the isometry group defined by the well-known rule

$$\frac{\partial \phi_\xi^\nu(x)}{\partial x_\mu}\, (T_\xi A)_\nu\, [\phi(x)] = A_\mu(x) \,. \tag{5}$$

The corresponding generator, $X_K = i\, \partial_\xi T_\xi|_{\xi=0}$, has the action

$$(X_K A)_\mu = -i(K^\nu A_{\mu;\nu} + K^\nu_{\;;\mu} A_\nu) \,. \tag{6}$$

We say that these generators are conserved operators since they commute with the operator of the field equation [11]. Moreover, from the Noether theorem it results that any symmetry generator X gives rise to the time-independent quantity,

$$C[X] = -i \int_\Sigma d\sigma^\mu \sqrt{g}\, g^{\alpha\beta} \left[A_\alpha\, \overset{\leftrightarrow}{\partial_\mu}\, (X A_\beta) \right], \tag{7}$$

on a given space-like hypersurface $\Sigma \subset M$. Particularly, for the internal $U(1)$ symmetry we must take $X = I$.

We consider (M, g) to be the dS spacetime defined as a hyperboloid of radius $1/\omega$ [1] in a five-dimensional pseudo-Euclidean manifold, M^5, of coordinates z^A labeled by the indices $A, B, ... = 0, 1, 2, 3, 5$. The local charts of coordinates $\{x\}$ on M^5 can be easily introduced giving the specific functions $z^A(x)$. Here we consider only the moving chart $\{t, \mathbf{x}\}$ with the conformal time, t, Cartesian coordinates and the line element

$$ds^2 = \frac{1}{(\omega t)^2} \eta_{\mu\nu} dx^\mu dx^\nu = \frac{1}{(\omega t)^2} \left(dt^2 - d\mathbf{x} \cdot d\mathbf{x} \right), \tag{8}$$

where η is the the metric tensor of the Minkowski spacetime [1].

The isometry group of the dS manifold is just the group $SO(1,4)$ of the pseudo-orthogonal transformations in M^5. For this reason, the basis-generators of the $SO(1,4)$ algebra are associated to ten independent Killing vectors, $K_{(AB)} = -K_{(BA)}$, which give rise to the basis-generators $X_{(AB)}$ of the vector representation of the $SO(1,4)$ group carried by the space of the vector fields A. In what follows we focus only on the Hamiltonian (or energy) operator $H = \omega X_{(05)}$, the momentum components $P^i = \omega(X_{(5i)} - X_{(0i)})$ and those of the total angular momentum $J_i = \frac{1}{2}\varepsilon_{ijk}X_{(jk)}$ $(i, j, ... = 1, 2, 3)$ [14]. The action of these operators can be deduced from Eq. (6) using the concrete form of the corresponding Killing vectors whose components read [11]: $K^\mu_{(05)} = x^\mu$ and

$$K^0_{(5i)} - K^0_{(0i)} = 0, \qquad K^j_{(5i)} - K^j_{(0i)} = \frac{1}{\omega}\delta_{ij}, \tag{9}$$

$$K^0_{(ij)} = 0, \qquad K^k_{(ij)} = \delta_{ki}x^j - \delta_{kj}x^i. \tag{10}$$

The Hamiltonian and momentum operators do not have spin parts, acting as

$$(H\,A)_\mu(t, \mathbf{x}) = -i\omega(t\partial_t + x^i\partial_i + 1)A_\mu(t, \mathbf{x}), \tag{11}$$

$$(P^i A)_\mu(t, \mathbf{x}) = -i\partial_i\, A_\mu(t, \mathbf{x}), \tag{12}$$

while the action of the total angular momentum reads

$$(J_i\, A)_j(t, \mathbf{x}) = (L_i A)_j(t, \mathbf{x}) - i\varepsilon_{ijk}A_k(t, \mathbf{x}), \tag{13}$$

$$(J_i\, A)_0(t, \mathbf{x}) = (L_i A)_0(t, \mathbf{x}), \tag{14}$$

where $\mathbf{L} = \mathbf{x} \times \mathbf{P}$ is the usual angular momentum operator. In addition, we define the Pauli-Lubanski operator $W = \mathbf{P} \cdot \mathbf{J}$ whose action depends only on the spin parts,

$$(WA)_i(t, \mathbf{x}) = \varepsilon_{ijk}\partial_j A_k(t, \mathbf{x}), \quad (WA)_0(t, \mathbf{x}) = 0. \tag{15}$$

This operator will define the polarization in the canonical basis of the $so(3)$ algebra as in special relativity.

Starting with the above results we can derive the time-independent quantities defined by Eq. (7) for any symmetry generator X and $\Sigma = \mathbb{R}^3$. After a little calculation we obtain the compact form

$$C[X] = -\eta^{\mu\nu} \langle A_\mu, (XA)_\nu \rangle, \tag{16}$$

with the new notation

$$\langle f, g \rangle = i \int d^3x\, f^*(t, \mathbf{x}) \overleftrightarrow{\partial}_t g(t, \mathbf{x}), \tag{17}$$

where $f \overleftrightarrow{\partial} g = f(\partial g) - g(\partial f)$. Defining now the relativistic scalar product of two vector fields as [1]

$$\langle A | A' \rangle = -\eta^{\mu\nu} \langle A_\mu, A'_\nu \rangle = -i\eta^{\mu\nu} \int d^3x\, A^*_\mu(t, \mathbf{x}) \overleftrightarrow{\partial}_t A'_\nu(t, \mathbf{x}), \tag{18}$$

we can write

$$C[X] = \langle A | XA \rangle. \tag{19}$$

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[1]We denote by ω the Hubble dS constant since H is reserved for the Hamiltonian operator

3 Polarized plane wave solutions

The specific feature of the quantum mechanics on dS manifolds is that the Hamiltonian operator (11) does not commute with the momentum operators (12). For this reason the particular solutions of the field equation may be eigenfunctions either of the momentum operators or of the Hamiltonian one. In what follows we derive a complete set of fundamental solutions as common eigenfunctions of the commuting operators P^i and W.

In the chart $\{t, \mathbf{x}\}$ the field equations take the form

$$\partial_t(\partial_i A_i) - \Delta A_0 + \frac{\mu^2}{t^2} A_0 = 0 \,, \tag{20}$$

$$\partial_t^2 A_k - \Delta A_k - \partial_k(\partial_t A_0) + \partial_k(\partial_i A_i) + \frac{\mu^2}{t^2} A_k = 0 \,, \tag{21}$$

where $\mu = m/\omega$, while the Lorentz condition reads

$$\partial_i A_i = \partial_t A_0 - \frac{2}{t} A_0 \,. \tag{22}$$

The solutions of these equations are vector fields which can be expanded as,

$$
\begin{aligned}
A &= A^{(+)} + A^{(-)} \\
&= \int d^3p \sum_\lambda \left\{ \mathrm{U}[\mathbf{p}, \lambda] a(\mathbf{p}, \lambda) + \mathrm{U}[\mathbf{p}, \lambda]^* b^*(\mathbf{p}, \lambda) \right\} \,,
\end{aligned}
\tag{23}
$$

in terms of wave functions in momentum representation, $a(\mathbf{p}, \lambda)$, and $b(\mathbf{p}, \lambda)$, which depend on the momentum $\mathbf{p} \in \mathbb{R}_p^3$ and the polarization $\lambda = 0, \pm 1$. Denoting $p = |\mathbf{p}|$ we assume that the vector fields $\mathrm{U}[\mathbf{p}, \lambda]$ satisfy the eigenvalue equations

$$P^i \mathrm{U}[\mathbf{p}, \lambda] = p_i \mathrm{U}[\mathbf{p}, \lambda] \,, \quad W \mathrm{U}[\mathbf{p}, \lambda] = p\,\lambda \mathrm{U}[\mathbf{p}, \lambda] \,, \tag{24}$$

and the orthonormalization relations

$$\langle \mathrm{U}[\mathbf{p}, \lambda]|\, \mathrm{U}[\mathbf{p}', \lambda'] \rangle = \delta_{\lambda\lambda'} \delta^3(\mathbf{p} - \mathbf{p}') \,. \tag{25}$$

The components of these vector fields, $\mathrm{U}[\mathbf{p}, \lambda]_\mu(x)$, form the desired system of fundamental plane wave solutions of positive frequencies depending on momentum and polarization. The corresponding fundamental solutions of negative frequencies are $\mathrm{U}[\mathbf{p}, \lambda]_\mu(x)^*$. We suppose that these solutions are of the form

$$\mathrm{U}[\mathbf{p}, \lambda]_i(x) = \begin{cases} \alpha(t, p)\, e_i(\mathbf{n}_p, \lambda)\, e^{i\mathbf{p}\cdot\mathbf{x}} & \text{for} \quad \lambda = \pm 1 \\ \beta(t, p)\, e_i(\mathbf{n}_p, \lambda)\, e^{i\mathbf{p}\cdot\mathbf{x}} & \text{for} \quad \lambda = 0 \end{cases} \tag{26}$$

and

$$\mathrm{U}[\mathbf{p}, \lambda]_0(x) = \begin{cases} 0 & \text{for} \quad \lambda = \pm 1 \\ \gamma(t, p)\, e^{i\mathbf{p}\cdot\mathbf{x}} & \text{for} \quad \lambda = 0 \end{cases} \tag{27}$$

where $\mathbf{n}_p = \mathbf{p}/p$ and $e_i(\mathbf{n}_p, \lambda)$ are the polarization vectors of the helicity basis. For $\lambda = 0$ these vectors are longitudinal, i.e. $\mathbf{e}(\mathbf{n}_p, 0) = \mathbf{n}_p$, while the vectors with $\lambda = \pm 1$ are transversal such that $\mathbf{p} \cdot \mathbf{e}(\mathbf{n}_p, \pm 1) = 0$. In general, they have c-number components which must satisfy [18, 19]

$$
\begin{aligned}
\mathbf{e}(\mathbf{n}_p, \lambda)^* \cdot \mathbf{e}(\mathbf{n}_p, \lambda') &= \delta_{\lambda\lambda'} \,, & (28) \\
\mathbf{e}(\mathbf{n}_p, \lambda)^* \wedge \mathbf{e}(\mathbf{n}_p, \lambda) &= i\lambda\, \mathbf{n}_p \,, & (29) \\
\sum_\lambda e_i(\mathbf{n}_p, \lambda)^*\, e_j(\mathbf{n}_p, \lambda) &= \delta_{ij} \,. & (30)
\end{aligned}
$$

Introducing now the functions (26) and (27) in Eqs. (20), (21) and (22) we obtain the system

$$\left(\frac{d^2}{dt^2} + p^2 + \frac{\mu^2}{t^2}\right)\alpha(t,p) = 0 \tag{31}$$

$$\left(\frac{d^2}{dt^2} - \frac{2}{t}\frac{d}{dt} + p^2 + \frac{\mu^2 + 2}{t^2}\right)\gamma(t,p) = 0 \tag{32}$$

$$\beta(t,p) = -\frac{i}{p}\left(\frac{d}{dt} - \frac{2}{t}\right)\gamma(t,p) \tag{33}$$

which can be solved in terms of Bessel functions. The solutions read

$$\alpha(t,p) = N_1 e^{-\frac{1}{2}\pi k}(-t)^{\frac{1}{2}} H_{ik}^{(1)}(-pt), \tag{34}$$

$$\gamma(t,p) = N_2 e^{-\frac{1}{2}\pi k}(-t)^{\frac{3}{2}} H_{ik}^{(1)}(-pt), \tag{35}$$

$$\beta(t,p) = iN_2 e^{-\frac{1}{2}\pi k}\left[\frac{1}{p}\left(ik + \frac{1}{2}\right)(-t)^{\frac{1}{2}} H_{ik}^{(1)}(-pt)\right.$$
$$\left. -(-t)^{\frac{3}{2}} H_{ik+1}^{(1)}(-pt)\right], \tag{36}$$

where $H^{(1)}$ are Hankel functions, N_1 and N_2 are normalization factors and $k = \sqrt{\mu^2 - \frac{1}{4}}$ provided $m > \omega/2$. By using then the formulas given in the Appendix A, we find that the orthonormalization condition (25) is accomplished only if we take (up to phase factors)

$$N_1 = \frac{\sqrt{\pi}}{2}\frac{1}{(2\pi)^{3/2}}, \quad N_2 = \frac{\sqrt{\pi}}{2}\frac{1}{(2\pi)^{3/2}}\frac{\omega p}{m}. \tag{37}$$

With this normalization our solutions satisfy the identity

$$i\sum_\lambda \int d^3p\, U[\mathbf{p},\lambda]_i^*(t,\mathbf{x}) \overleftrightarrow{\partial_t} U[\mathbf{p},\lambda]_j(t,\mathbf{x}')$$

$$= \left(-\delta_{ij} + \frac{\omega^2 t^2}{m^2}\partial_i\partial_j\right)\delta^3(\mathbf{x} - \mathbf{x}'), \tag{38}$$

which plays the role of a completeness condition. A similar equation can be derived for the components $(0,0)$ but for $(0,i)$ and $(i,0)$ there are imaginary parts that can not be evaluated. However, this is not an impediment since the Lorentz condition reduces the number of canonical variable and, therefore, Eq. (38) is enough for testing the completeness.

We derived thus the complete system of orthonormalized fundamental solutions which are common eigenfunctions of the complete set of commuting operators $\{P^i, W\}$. The solution of positive and negative frequencies correspond to the eigenvalues $\{p^i, p\lambda\}$ and respectively $\{-p^i, -p\lambda\}$. We note that the massless limit makes sense only if we take $\beta(t,p) = \gamma(t,p) = 0$ which leads to the Coulomb gauge of the Maxwell free field [17].

4 Quantization and propagators

The quantization can be done in canonical manner transforming the wave functions a and b of the fields (23) into field operators (such that $b^* \to b^\dagger$) [19]. These operators must fulfill the standard commutation relations in the momentum representation among them the non-vanishing ones are

$$[a(\mathbf{p},\lambda), a^\dagger(\mathbf{p}',\lambda')] = [b(\mathbf{p},\lambda), b^\dagger(\mathbf{p}',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\mathbf{p} - \mathbf{p}'). \tag{39}$$

The field operators act on the Fock space supposed to have an unique vacuum state $|0\rangle$ accomplishing

$$a(\mathbf{p},\lambda)|0\rangle = 0, \quad \langle 0| a^\dagger(\mathbf{k},\lambda) = 0, \tag{40}$$

and similarly for b and $b^\dagger$. The sectors with a given number of particles have to be constructed using the standard methods, obtaining thus the generalized momentum basis of the Fock space.

4.1 Commutator functions

The Green functions of the vector field are related to the partial commutator functions (of positive or negative frequencies) defined as

$$D_{\mu\nu}^{(\pm)}(x, x') = i[A_\mu^{(\pm)}(x), A_\nu^{(\pm)\,\dagger}(x')]\,, \tag{41}$$

and the total one, $D_{\mu\nu} = D_{\mu\nu}^{(+)} + D_{\mu\nu}^{(-)}$, which is a real function since $[D_{\mu\nu}^{(\pm)}]^* = D_{\mu\nu}^{(\mp)}$. All these functions are solutions of the field equations and obey the Lorentz condition in both the sets of variables, x and x'. The properties of the commutator functions can be deduced focusing only on the functions of positive frequencies,

$$D_{\mu\nu}^{(+)}(x, x') = i \sum_\lambda \int d^3p\, \mathrm{U}[\mathbf{p}, \lambda]_\mu(x) \mathrm{U}[\mathbf{p}, \lambda]_\nu(x')^*\,, \tag{42}$$

derived from Eqs. (23) and (30). According to Eqs. (26) and (27) we obtain the mode integral expansions

$$
\begin{aligned}
D_{ij}^{(+)}(x, x') \;=\; & i \int d^3p \left[\left(\delta_{ij} - \frac{p^i p^j}{p^2} \right) \alpha(t, p)\alpha(t', p)^* \right.\\
& \left. + \frac{p^i p^j}{p^2}\, \beta(t, p)\beta(t', p)^* \right] e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\,,
\end{aligned}
\tag{43}
$$

$$D_{i0}^{(+)}(x, x') \;=\; i \int d^3p\, \frac{p^i}{p} \beta(t, p)\gamma(t', p)^* e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\,, \tag{44}$$

$$D_{0i}^{(+)}(x, x') \;=\; i \int d^3p\, \frac{p^i}{p} \gamma(t, p)\beta(t', p)^* e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\,, \tag{45}$$

$$D_{00}^{(+)}(x, x') \;=\; i \int d^3p\, \gamma(t, p)\gamma(t', p)^* e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\,, \tag{46}$$

which show that $D_{\mu\nu}^{(\pm)}(x, x') = D_{\mu\nu}^{(\pm)}(t, t', \mathbf{x} - \mathbf{x}')$.

The above equations help us to deduce what happens at equal times, $t' = t$. Indeed, bearing in mind that $D_{\mu\nu}$ are real functions and using the identities given in Appendix A we find

$$D_{ij}(x, x')|_{t'=t} = 0\,, \qquad D_{00}(x, x')|_{t'=t} = 0\,, \tag{47}$$

$$D_{i0}(x, x')|_{t'=t} = D_{0i}(x, x')|_{t'=t} = \frac{\omega^2 t^2}{m^2}\, \partial_i\, \delta^3(\mathbf{x} - \mathbf{x}')\,, \tag{48}$$

and

$$\frac{1}{2}(\partial_t - \partial_{t'})D_{ij}(x, x')\Big|_{t'=t} = \left(-\delta_{ij} + \frac{\omega^2 t^2}{m^2}\, \partial_i \partial_j \right) \delta^3(\mathbf{x} - \mathbf{x}')\,, \tag{49}$$

$$\frac{1}{2}(\partial_t - \partial_{t'})D_{00}(x, x')\Big|_{t'=t} = \frac{\omega^2 t^2}{m^2}\, \Delta_x\, \delta^3(\mathbf{x} - \mathbf{x}')\,, \tag{50}$$

$$\frac{1}{2}(\partial_t - \partial_{t'})D_{i0}(x, x')\Big|_{t'=t} = \frac{1}{2}(\partial_t - \partial_{t'})D_{0i}(x, x')\Big|_{t'=t} = 0\,. \tag{51}$$

The mode integrals (43)-(46) can be solved in terms of a scalar function and some simple operators. This is just the commutator function of positive frequencies of the scalar field *conformally* coupled to the dS gravity [1], defined by the integral

$$\mathcal{D}^{(+)}(x, x') = \frac{i\pi\omega^2}{4} \frac{e^{-\pi k}}{(2\pi)^3} (tt')^{3/2} \int d^3p\, e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}} H_{ik}^{(1)}(-pt) H_{ik}^{(1)}(-pt')^* \tag{52}$$

which can be solved as [4, 5],

$$\mathcal{D}^{(+)}(x, x') = \frac{im^2}{16\pi} e^{-\pi k}\mathrm{sech}(\pi k)\, {}_2F_1\left(\frac{3}{2} + ik, \frac{3}{2} - ik; 2; 1 - \frac{y}{4} \right)\,, \tag{53}$$

where the quantity

$$y(x, x') = \frac{-(t - t' - i\epsilon)^2 + (\mathbf{x} - \mathbf{x}')^2}{tt'} \tag{54}$$

is related to the geodesic length between x and x'. We note that the function $\mathcal{D}^{(+)}$ satisfies the equation

$$\left(\partial_t^2 - \frac{2}{t} \partial_t - \Delta_x + \frac{\mu^2 + 2}{t^2} \right) \mathcal{D}^{(+)}(x, x') = 0 \tag{55}$$

(and similarly for x'). With its help and using Eqs. (34), (35) and (33) we can write:

$$D_{ij}^{(+)}(t, t', \mathbf{x}) = \frac{1}{\omega^2 tt'} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta} \right) \mathcal{D}^{(+)}(t, t', \mathbf{x})$$
$$+ \frac{1}{m^2} \frac{\partial_i \partial_j}{\Delta} \left(\partial_t - \frac{2}{t} \right) \left(\partial_{t'} - \frac{2}{t'} \right) \mathcal{D}^{(+)}(t, t', \mathbf{x}), \tag{56}$$

$$D_{i0}^{(+)}(t, t', \mathbf{x}) = -\frac{1}{m^2} \partial_i \left(\partial_t - \frac{2}{t} \right) \mathcal{D}^{(+)}(t, t', \mathbf{x}) \tag{57}$$

$$D_{0j}^{(+)}(t, t', \mathbf{x}) = \frac{1}{m^2} \partial_i \left(\partial_{t'} - \frac{2}{t'} \right) \mathcal{D}^{(+)}(t, t', \mathbf{x}) \tag{58}$$

$$D_{00}^{(+)}(t, t', \mathbf{x}) = -\frac{1}{m^2} \Delta \mathcal{D}^{(+)}(t, t', \mathbf{x}). \tag{59}$$

Similar formulas can be derived for $D_{\mu\nu}^{(-)}$ and $D_{\mu\nu}$ which are related to the scalar functions $\mathcal{D}^{(-)} = [\mathcal{D}^{(+)}]^*$ and $\mathcal{D} = \mathcal{D}^{(+)} + \mathcal{D}^{(-)}$ respectively.

We succeeded thus to express all the commutator functions of the vector field in terms of some operators acting on scalar functions of y. A similar result was obtained recently for the Dirac field whose anti-commutator functions resulted from the canonical quantization are given by differential matrix-operators acting on specific scalar functions depending on the geodesic length [20]. Moreover, in both these cases there are factors depending on powers of t and t' which show that the entire commutator or anti-commutator functions are no longer genuine functions of y (or of the geodesic length) and its derivatives. Notice that, in addition, the non-differential operator Δ^{-1} of Eq. (56) affects the dependence on spaces variables too.

4.2   Green functions

According to the standard procedure, we now define the retarded (R), advanced (A), and (causal) Feynman (F) propagators,

$$\tilde{D}_{\mu\nu}^R(x, x') = \theta(t - t') D_{\mu\nu}(x, x'), \tag{60}$$

$$\tilde{D}_{\mu\nu}^A(x, x') = -\theta(t' - t) D_{\mu\nu}(x, x'), \tag{61}$$

$$\tilde{D}_{\mu\nu}^F(x, x') = i \langle 0 | T [A_\mu(x) A_\nu^\dagger(x')] | 0 \rangle$$
$$= \theta(t - t') D_{\mu\nu}^{(+)}(x, x') - \theta(t' - t) D_{\mu\nu}^{(-)}(x, x'). \tag{62}$$

The corresponding Green functions,

$$G_{\mu\nu}^{R/A/F}(x.x') = \frac{\omega^2 t^2}{m^2} \delta_\mu^0 \delta_\nu^0 \, \delta^4(x - x') + \tilde{D}_{\mu\nu}^{R/A/F}(x, x'), \tag{63}$$

satisfy the equation

$$\eta^{\alpha\beta} \partial_\alpha \left[\partial_\beta G_{\mu\nu}(x, x') - \partial_\mu G_{\beta\nu}(x, x') \right] + \frac{m^2}{\omega^2 t^2} G_{\mu\nu}(x, x') = \eta_{\mu\nu} \delta^4(x - x'). \tag{64}$$

This can be proved by using the identities $\partial_t^2[\theta(t) f(t)] = \delta(t) \partial_t f(t) - \delta(t) f(t) \partial_t$ and $\partial_t[\delta(t) f(t)] = -\delta(t) f(t) \partial_t$, the artifice $\partial_t f(t - t') = \frac{1}{2}(\partial_t - \partial_{t'}) f(t - t')$ and Eqs. (47)-(51). In addition, the Lorentz condition yields

$$\eta^{\alpha\beta} \partial_\alpha \left[\frac{1}{\omega^2 t^2} G_{\beta\mu}(x, x') \right] = \frac{1}{m^2} \partial_\mu \delta^4(x - x'). \tag{65}$$

In the flat limit these propagators and Green functions become the well-known ones of special relativity [18] as we briefly present in Appendix B.

Different Green functions can be defined by changing the gauge. The transverse ones, obeying the exact Lorentz condition,

$$\partial_\mu \left[\sqrt{g(x)} g^{\mu\nu}(x) G_{\nu\sigma}^{tr}(x,x') \right] = \partial_\mu' \left[\sqrt{g(x')} g^{\mu\nu}(x') G_{\sigma\nu}^{tr}(x,x') \right] = 0 \,, \qquad (66)$$

can be defined as

$$G_{\mu\nu}^{tr}(x,x') = G_{\mu\nu}(x,x') + \frac{1}{m^2} \partial_\mu \partial_\nu' G_0(x,x') \,, \qquad (67)$$

where G_0 is the massless scalar Green function which satisfies

$$\partial_\mu \left[\sqrt{g(x)} g^{\mu\nu}(x) \partial_\nu G_0(x,x') \right] = \delta^4(x-x') \,. \qquad (68)$$

According to Eqs. (64) and (4.2) we find that the equation of the transverse Green functions,

$$\sqrt{g(x)} g^{\alpha\beta}(x) \partial_\alpha \left[\partial_\beta G_{\mu\nu}^{tr}(x,x') - \partial_\mu G_{\beta\nu}^{tr}(x,x') \right] + m^2 \sqrt{g(x)}\, G_{\mu\nu}^{tr}(x,x')$$
$$= g_{\mu\nu}(x) \delta^4(x-x') + \sqrt{g(x)}\, \partial_\mu \partial_\nu' G_0(x,x') \,, \qquad (69)$$

coincides to that of Ref. [7] where one uses the same Lorentz condition (66). In the flat limit this equation takes the usual form (82).

Finally, we must stress that the transverse Green functions obtained here are different from those derived in [7] even though all of them satisfy the same equation (69). This is because the canonical quantization we use generates propagators that do not depend only on y and its derivatives as those of Ref. [7] which are forced to do this by definition.

5 Conclusions

We presented here the quantum theory of the massive charged vector field minimally coupled to the gravity of the dS expanding universe. Our approach is based on a complete set of commuting operators which determines in turn the fundamental solutions of given momentum and polarization. These form a complete set of orthonormalized solutions with respect to the relativistic scalar product. Under such circumstances, the method of canonical quantization was used for constructing the Fock space.

It is worth pointing out that our approach has good massless an flat limits. In the massless limit this leads to the quantum theory of the free Maxwell field in Coulomb gauge on the dS expanding universe we proposed recently [17]. The flat limit of our theory recovers all the well-know results of the Proca theory on Minkowski backgrounds. For this reason, our Green functions (63) have the same structure (78) and similar properties to those of the corresponding Green functions on the flat spacetime. However, despite of these similarities there are major differences. Apart from the analytical expressions, the principal difference is that on dS spacetimes the vector Green functions can not be related to the scalar propagators such as done in Eq. (77)and, consequently, there are no propagator equations similar to Eq. (79).

In other respects, it is obvious that our transverse Green functions (67) differ from the maximally symmetric two-point functions proposed by Allen and Jacobson [6] but satisfy the same equation and Lorentz condition as the propagators constructed axiomatically by Tsamis and Woodard [7]. The difference is that our propagators are no longer functions only of y and its derivatives while those of Ref. [7] have this property. This suggests that our transverse Green functions we obtained using the canonical quantization are new solutions of the transverse equation (69).

The Proca theory on the dS expanding universe we presented here completes our previous works [13, 14, 16, 17] opening thus the perspective to a realistic quantum theory on the dS expanding universe involving scalar, vector and spinor fields. This theory must be based on trustworthy results given by the canonical quantization and perturbation theory in the reduction formalism. In this framework one could deal with more sophisticated methods but now preserving the minimum requirements of rigor and consistency.

Acknowledgments

Appendix A: Properties of some Bessel functions

Let us consider the Hankel functions $H_\nu^{(1,2)}(s)$ in the special case when $\nu = ik$ and denote

$$Z(s) = e^{-\pi k/2} H_{ik}^{(1)}(s)\,, \quad Z^*(s) = e^{\pi k/2} H_{ik}^{(2)}(s)\,. \tag{70}$$

Then, by using the Wronskian W of the Bessel functions [22] we find that

$$Z^*(s) \overset{\leftrightarrow}{\partial_s} Z(s) = W[H_{ik}^{(2)}, H_{ik}^{(1)}] = \frac{4i}{\pi s}\,. \tag{71}$$

Starting with this property and Eq. (22) we find the useful identities

$$\alpha(t,p)^* \overset{\leftrightarrow}{\partial_t} \alpha(t,p) = \frac{4i}{\pi}|N_1|^2 = \frac{i}{(2\pi)^3}\,, \tag{72}$$

$$\gamma(t,p)^* \overset{\leftrightarrow}{\partial_t} \gamma(t,p) = \frac{4i}{\pi}t^2|N_2|^2 = \frac{i}{(2\pi)^3}\frac{\omega^2 t^2}{m^2}\,p^2\,, \tag{73}$$

$$\beta(t,p)^* \overset{\leftrightarrow}{\partial_t} \beta(t,p) = \left(1 + \frac{\mu^2}{p^2 t^2}\right)\gamma(t,p)^* \overset{\leftrightarrow}{\partial_t} \gamma(t,p)$$

$$= \frac{4i}{\pi}\left(t^2 + \frac{\mu^2}{p^2}\right)|N_2|^2 = \frac{i}{(2\pi)^3}\left(1 + \frac{\omega^2 t^2}{m^2}\,p^2\right)\,, \tag{74}$$

which helped us to find the normalization factors (37) according to Eqs. (25) and (18). In addition, from Eq. (22) we deduce

$$\Re\left[\gamma(t,p)^*\beta(t,p)\right] = \frac{1}{(2\pi)^3}\frac{\omega^2 t^2}{m^2}\frac{p}{2}\,. \tag{75}$$

Appendix B: The flat limit

In the flat limit, when $\omega \to 0$ and $\omega t \to -1$, Eq. (64) becomes the standard equation of the vector Green functions of special relativity,

$$\eta^{\alpha\beta}\partial_\alpha\left[\partial_\beta G_{\mu\nu}(x) - \partial_\mu G_{\beta\nu}(x)\right] + m^2 G_{\mu\nu}(x) = \eta_{\mu\nu}\delta^4(x)\,, \tag{76}$$

whose solutions [18],

$$G_{\mu\nu}(x) = \left(\eta_{\mu\nu} + \frac{1}{m^2}\partial_\mu\,\partial_\nu\right)\tilde{D}(x) \tag{77}$$

depend on the scalar propagator $\tilde{D}$ which obeys $(\partial^2 + m^2)\tilde{D}(x) = \delta^4(x)$. These Green functions can be written as [18]

$$G_{\mu\nu}(x) = \frac{1}{m^2}\delta_\mu^0\delta_\nu^0\,\delta^4(x) + \tilde{D}_{\mu\nu}(x)\,, \tag{78}$$

where the vector propagators $\tilde{D}_{\mu\nu}$ are defined as in Eqs. (60)-(62). According to Eq. (77), one can replace Eq. (76) by the well-known one,

$$(\partial^2 + m^2)G_{\mu\nu}(x) = \left(\eta_{\mu\nu} + \frac{1}{m^2}\partial_\mu\,\partial_\nu\right)\delta^4(x)\,, \tag{79}$$

and show that the Lorentz condition yields,

$$\partial^\mu G_{\mu\nu}(x) = \frac{1}{m^2}\,\partial_\nu\delta^4(x)\,. \tag{80}$$

The transverse propagator,

$$G_{\mu\nu}^{tr}(x) = G_{\mu\nu}(x) - \frac{1}{m^2}\partial_\mu\partial_\nu\tilde{D}_0(x), \qquad (81)$$

depend on the massless scalar propagator $\tilde{D}_0$. It satisfies the exact Lorentz condition, $\partial^\mu G_{\mu\nu}^{tr}(x) = 0$, and the equation

$$\eta^{\alpha\beta}\partial_\alpha\left[\partial_\beta G_{\mu\nu}^{tr}(x) - \partial_\mu G_{\beta\nu}^{tr}(x)\right] + m^2 G_{\mu\nu}^{tr}(x) = \eta_{\mu\nu}\delta^4(x) - \partial_\mu\partial_\nu\tilde{D}_0(x). \qquad (82)$$

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