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Linearized Gravity and Its Dual Formulations: Yes-Go Results on Their Consistent Couplings

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Abstract

Several new results are collected together from the perspective of studying consistent couplings of linearized gravity to other gauge theories. All three versions of linearized gravity (Pauli-Fierz model and massless tensor fields with the mixed symmetry $(k,1)$ and respectively (k,k)) are considered in various settings. In all cases interactions are obtained under some common hypotheses, specific to quantum field theory, from computations of the local cohomology of the BRST differential associated with each free model on certain spaces in the framework of the antifield-BRST method.

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1 Introduction

Tensor fields characterized by a mixed Young symmetry type (neither completely antisymmetric nor fully symmetric) [1, 2, 3, 4, 5, 6] attracted the attention lately on some important issues, like the dual formulation of field theories of spin two or higher [7, 8, 9, 10, 11, 12, 13], or the derivation of some exotic gravitational interactions [14, 15].

There exist in fact three different dual formulations of linearized gravity (LG) in D dimensions: the Pauli–Fierz description [16, 17], the version based on a massless tensor field with the mixed symmetry $(D-3,1)$ [3, 8, 18] and the formulation in terms of a massless tensor field with the mixed symmetry $(D-3,D-3)$ [19, 20]. The last two versions are obtained by dualizing on one and respectively on both indices the Pauli-Fierz field [7]. These dual formulations in terms of mixed symmetry tensor gauge fields have been systematically investigated from the perspective of M-theory [21, 22, 23].

There is a revived interest in the construction of dual gravity theories, which led to several new results, viz. a dual formulation of LG in first order tetrad formalism in arbitrary dimensions within the path integral framework [24] or a reformulation of nonlinear Einstein gravity in terms of the dual graviton together with the ordinary metric and a shift gauge field [25].

An important matter related to the dual formulations of LG is the study of their consistent interactions, among themselves as well as with other gauge theories. The most efficient approach to this problem is the cohomological one, based on the deformation of the solution to the master equation [26]. Related to the usual Pauli–Fierz formulation of LG, it was believed that its only consistent interactions produce the standard couplings from General Relativity. Nevertheless, it was proved that there are allowed different couplings, at least in the presence of p-form gauge fields, which break the PT invariance [29]. On the other hand, since the mixed symmetry tensor fields involved in dual formulations of LG allow no self-interactions, it was believed that they are also rigid under the introduction of couplings to other gauge theories. Nevertheless, some recent results prove the contrary. More precisely, it was shown that some theories with massless tensor fields exhibiting the mixed symmetry $(k,1)$ can be consistently coupled to a vector field $(k=3)$ [30], to an arbitrary p-form $(k=3)$ [31], to a topological BF model $(k=2)$ [32], and to a massless tensor field with the mixed symmetry of the Riemann tensor $(k=3)$ [33]. In addition, the study of cross-couplings among a collection of massless tensor fields with the mixed symmetry $(3,1)$ and a collection of massless tensor fields with the mixed symmetry of the Riemann tensor reveals a case of special interest [34]. In this paper we collect all these yes-go results on a systematic basis.

2 General method of constructing interacting gauge field theories

We begin with a "free" gauge theory, described by a Lagrangian action $S_0^{\mathrm{L}}[\Phi^{\alpha_0}]$, invariant under some gauge transformations $\delta_\epsilon \Phi^{\alpha_0} = \bar{Z}^{\alpha_0}_{\ \alpha_1} \epsilon^{\alpha_1}$, i.e. $\frac{\delta S_0^{\mathrm{L}}}{\delta \Phi^{\alpha_0}} \bar{Z}^{\alpha_0}_{\ \alpha_1} = 0$, and consider the problem of constructing consistent interactions among the fields Φ^{α_0} such that the couplings preserve the field spectrum and the original number of gauge symmetries. This matter is addressed by means of reformulating the problem of constructing consistent interactions as a deformation problem of the solution to the master equation corresponding to the "free" theory [26, 27, 28]. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If an interacting gauge theory can be consistently constructed, then the solution $\bar{S}$ to the master equation associated with the "free" theory, $(\bar{S},\bar{S}) = 0$, can be deformed into a solution S

$$\bar{S} \to S = \bar{S} + \lambda S_1 + \lambda^2 S_2 + \cdots = \bar{S} + \lambda \int d^D x \, a + \lambda^2 \int d^D x \, b + \cdots \tag{1}$$

of the master equation for the deformed theory

$$(S,S) = 0, \tag{2}$$

such that both the ghost and antifield spectra of the initial theory are preserved. The projection of equation (2) on the various orders in the coupling constant λ leads to the equivalent tower of equations

$$\begin{aligned} (\bar{S},\bar{S}) &= 0, && (3)\\ 2(S_1,\bar{S}) &= 0, && (4)\\ 2(S_2,\bar{S}) + (S_1,S_1) &= 0, && (5)\\ &\vdots \end{aligned}$$

Equation (3) is fulfilled by hypothesis. The next equation requires that the first-order deformation of the solution to the master equation, S_1, is a co-cycle of the "free" BRST differential s, $sS_1 = 0$. However, only cohomologically nontrivial solutions to (4) should be taken into account, since the BRST-exact ones can be eliminated by some (in general nonlinear) field redefinitions. This means that S_1 pertains to the ghost number zero cohomological space of s, $H^0(s)$, which is nonempty because it is isomorphic to the space of physical observables of the "free" theory. It has been shown (by of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations, namely (5), etc. However, the resulting interactions may be nonlocal and there might even appear obstructions if one insists on their locality. The analysis of these obstructions can be done with the help of cohomological techniques.

3 Nonstandard couplings of Pauli-Fierz model

3.1 Pauli-Fierz plus an Abelian vector field (one-form)

Our starting point is represented by a free Lagrangian action, written as the sum between the linearized Hilbert-Einstein action (also known as the Pauli-Fierz action [16, 17]) and Maxwell's action in $D > 2$ spacetime dimensions

$$\begin{aligned} S_0^{\mathrm{L}}[h_{\mu\nu}, A_\mu] &= \int d^D x \left[-\frac{1}{2} (\partial_\mu h_{\nu\rho}) \partial^\mu h^{\nu\rho} + (\partial_\mu h^{\mu\rho}) \partial^\nu h_{\nu\rho} - (\partial_\mu h) \partial_\nu h^{\nu\mu} + \frac{1}{2} (\partial_\mu h) \partial^\mu h - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \\ &\equiv \int d^D x \left(\mathcal{L}_0^{(\mathrm{PF})} + \mathcal{L}_0^{(\mathrm{vect})} \right). \end{aligned} \tag{6}$$

The restriction $D>2$ is required by the spin-two field action, which is known to reduce to a total derivative in $D=2$. Throughout the paper we work with the flat metric of 'mostly plus' signature, $\sigma_{\mu\nu}=(-+\ldots+)$. In the above h denotes the trace of the Pauli-Fierz field, $h=\sigma_{\mu\nu}h^{\mu\nu}$, and $F_{\mu\nu}$ represents the Abelian field-strength of the massless vector field ($F_{\mu\nu}\equiv\partial_{[\mu}A_{\nu]}$). The theory described by action (6) possesses an Abelian and irreducible generating set of gauge transformations

$$\delta_\epsilon h_{\mu\nu}=\partial_{(\mu}\epsilon_{\nu)},\qquad \delta_\epsilon A_\mu=\partial_\mu\epsilon, \tag{7}$$

with ϵ_μ and ϵ bosonic gauge parameters. The notation $[\mu\ldots\nu]$ (or $(\mu\ldots\nu)$) signifies antisymmetry (or symmetry) with respect to all indices between brackets without normalization factors (i.e., the independent terms appear only once and are not multiplied by overall numerical factors).

After applying the cohomological technique based on the local BRST cohomology of the starting free theory in ghost number zero and maximum form degree, we are able to prove the next theorem [29].

Theorem 3.1 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and at most two derivatives in the Lagrangian, there are two complementary types of consistent interactions between a graviton and an Abelian vector field.*

3.1.1 Type I solutions (standard General Relativity)

The fully interacting Lagrangian action in case I reads as

$$\begin{aligned}S^{\mathrm{L(I)}}\left[g_{\mu\nu},\bar{A}_\mu\right]=\int d^Dx\left[\frac{2}{\lambda^2}\sqrt{-g}\left(R-2\lambda^2\Lambda\right)-\frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\rho\lambda}\bar{F}_{\mu\rho}\bar{F}_{\nu\lambda}\right.\\ \left.+\lambda\left(q_1\delta_3^D\varepsilon^{\mu_1\mu_2\mu_3}\bar{A}_{\mu_1}\bar{F}_{\mu_2\mu_3}+q_2\delta_5^D\varepsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}\bar{A}_{\mu_1}\bar{F}_{\mu_2\mu_3}\bar{F}_{\mu_4\mu_5}\right)\right]\end{aligned} \tag{8}$$

and is invariant under the deformed gauge transformations

$$\delta_\epsilon^{(\mathrm{I})}g_{\mu\nu}=\lambda\bar{\epsilon}_{(\mu;\nu)},\qquad \delta_\epsilon^{(\mathrm{I})}\bar{A}_\mu=\partial_\mu\epsilon+\lambda\left(\partial_\mu\bar{\epsilon}^\nu\right)\bar{A}_\nu+\lambda\left(\partial_\nu\bar{A}_\mu\right)\bar{\epsilon}^\nu. \tag{9}$$

In the above $g_{\mu\nu}$ is the metric tensor $g_{\mu\nu}=\sigma_{\mu\nu}+\lambda h_{\mu\nu}$, $\bar{A}_\mu=e_\mu^aA_a$ represents the vector field with curved indices, and e_a^μ is the vielbein field (its inverse being denoted by e_μ^a). We use the common convention according to which Latin indices are flat and Greek ones are curved. Related to the vielbein components we use the partial gauge-fixing conditions $\sigma_{\mu[a}e_{b]}^\mu=0$ (see [35]). Regarding the right-hand side of (8), $\sqrt{-g}$ denotes the square root from the minus determinant of the metric tensor, R means the full scalar curvature, Λ the cosmological constant, $g^{\mu\nu}$ represent the elements of the inverse of the metric tensor, and $\bar{F}_{\mu\rho}$ the fully deformed field strength of the vector field, $\bar{F}_{\mu\nu}=\partial_{[\mu}\left(e_{\nu]}^aA_a\right)$. Finally, $q_{1,2}$ are two arbitrary real constants and $\varepsilon^{\mu_1\cdots\mu_D}$ means the Levi-Civita symbol with curved indices in D dimensions, $\varepsilon^{\mu_1\cdots\mu_D}=\sqrt{-g}e_{a_1}^{\mu_1}\cdots e_{a_D}^{\mu_D}\varepsilon^{a_1\ldots a_D}$. In the gauge transformations (9) the gauge parameters with curved indices, $\bar{\epsilon}^\nu$, follow from the condition $\delta_\epsilon\left(\sigma_{\mu[a}e_{b]}^\mu\right)=0$, with $\delta_\epsilon e_a^\mu=\bar{\epsilon}^\rho\partial_\rho e_a^\mu-e_a^\rho\partial_\rho\bar{\epsilon}^\mu+\epsilon_a{}^be_b^\mu$ and ϵ^{ab} the flat gauge parameters associated with Lorentz transformations. The notation $\bar{\epsilon}_{\mu;\nu}$ signifies the (full) covariant derivative of $\bar{\epsilon}_\mu$, such that the terms $\bar{\epsilon}_{(\mu;\nu)}$ implement diffeomorphisms as the gauge transformations of the metric tensor components.

3.1.2 Type II solutions (new, nonstandard results)

In this situation the coupled Lagrangian action 'lives' only in $D=3$ and is given by

$$S^{\mathrm{L(II)}}[h_{\mu\nu},A_\mu]=\int d^3x\left[\mathcal{L}_0^{(\mathrm{PF})}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-2\lambda\Lambda h-\lambda F^{\mu\nu}\varepsilon_{\mu\nu\rho}\partial^{[\theta}h^{\rho]}{}_\theta+2\lambda^2\left(\partial^{[\mu}h^{\rho]}{}_\mu\right)\partial_{[\nu}h_{\rho]}{}^\nu\right], \tag{10}$$

where $\mathcal{L}_0^{(\mathrm{PF})}$ is the Pauli-Fierz Lagrangian and Λ the cosmological constant, being invariant under a generating set of gauge transformations of the form

$$\delta_\epsilon^{(\mathrm{II})}h_{\mu\nu}=\partial_{(\mu}\epsilon_{\nu)},\qquad \delta_\epsilon^{(\mathrm{II})}A_\mu=\partial_\mu\epsilon+\lambda\varepsilon_{\mu\nu\rho}\partial^{[\nu}\epsilon^{\rho]}. \tag{11}$$

Action (10) can be set in a more suggestive form by introducing the deformed field strength of the vector field

$$F'_{\mu\nu} = F_{\mu\nu} + 2\lambda\varepsilon_{\mu\nu\rho}\partial^{[\theta}h^{\rho]}{}_{\theta}, \tag{12}$$

in terms of which we can write

$$S^{\mathrm{L(II)}}[h_{\mu\nu}, A_{\mu}] = \int d^3x \left(\mathcal{L}_0^{(\mathrm{PF})} - 2\lambda\Lambda h - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}\right). \tag{13}$$

Under this form, action (13) is manifestly invariant under the gauge transformations (11): its first two terms are known to be invariant under linearized diffeomorphisms and the third is gauge-invariant under (11) since the deformed field strength is so, $\delta_{\epsilon}^{(\mathrm{II})}F'_{\mu\nu} = 0$.

This result is new and will be generalized in the sequel to the case of couplings between a graviton and an arbitrary p-form. In conclusion, this case yields another possibility to establish nontrivial couplings between the Pauli-Fierz field and a vector field. It is *complementary to* case I (*General Relativity*) and is valid only in $D = 3$. The resulting Lagrangian action and gauge transformations are *not* series in the coupling constant. The Lagrangian contains pieces of maximum order two in the coupling constant, which are mixing-component terms (there is no interaction vertex at least cubic in the fields) and emphasize the deformation of the standard Abelian field strength of the vector field like in (12). Concerning the new gauge transformations, only those of the massless vector field are modified at order one in the coupling constant by adding to the original $U(1)$ gauge symmetry a term linear in the antisymmetric first-order derivatives of the Pauli-Fierz gauge parameters. As a consequence, the gauge algebra, defined by the commutators among the deformed gauge transformations, remains Abelian, just like for the free theory. We cannot stress enough that these two cases (I and II) cannot coexist, even in $D = 3$.

3.2 Pauli-Fierz plus an Abelian p-form gauge field

Next, we approach the generalization of the previous results to the case of consistent couplings between the Pauli-Fierz model and an arbitrary p-form gauge field [29], with $p > 1$. The starting point is given now by the sum between the Pauli-Fierz action and the Lagrangian action of an Abelian p-form with $p > 1$

$$S_0^{\mathrm{L}}[h_{\mu\nu}, A_{\mu_1\ldots\mu_p}] = \int d^Dx \left(\mathcal{L}_0^{(\mathrm{PF})} - \frac{1}{2\cdot(p+1)!}F_{\mu_1\ldots\mu_{p+1}}F^{\mu_1\ldots\mu_{p+1}}\right), \tag{14}$$

in $D \geq p+1$ spacetime dimensions, with $F_{\mu_1\ldots\mu_{p+1}}$ the Abelian field strength of the p-form gauge field $A_{\mu_1\ldots\mu_p}$, $F_{\mu_1\ldots\mu_{p+1}} = \partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}$. This action is known to be invariant under the gauge transformations

$$\delta_{\epsilon}h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}, \qquad \delta_{\epsilon}A_{\mu_1\ldots\mu_p} = \partial_{[\mu_1}\epsilon^{(p)}_{\mu_2\ldots\mu_p]}. \tag{15}$$

Unlike the Maxwell field ($p = 1$), the gauge transformations of the p-form for $p > 1$ are off-shell reducible of order $(p-1)$. This property has strong implications at the level of the BRST complex and of the BRST cohomology in the form sector: a whole tower of ghosts of ghosts and of antifields will be required in order to incorporate the reducibility, only the ghost of maximum pure ghost number, p, will enter $H(\gamma)$, and the local characteristic cohomology will be richer. In spite of these new cohomological ingredients, which complicate the analysis of deformations, the previous results for $p = 1$ can still be generalized.

Thus, after applying the cohomological tools and consequently computing the possible consistent interactions, two complementary cases are again revealed [29].

Theorem 3.2 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and at most two derivatives in the Lagrangian, there are two complementary types of consistent interactions between a graviton and an Abelian p-form gauge field.*

One describes the standard graviton-p-form interactions from General Relativity and leads to a Lagrangian action similar to (8) up to replacing $(1/4)\, g^{\mu\nu}g^{\rho\lambda}\bar{F}_{\mu\rho}\bar{F}_{\nu\lambda}$ with the expression $(2\cdot(p+1)!)^{-1}\times$

$\times g^{\mu_1\nu_1}\cdots g^{\mu_{p+1}\nu_{p+1}}\bar{F}_{\mu_1\ldots\mu_{p+1}}\bar{F}_{\nu_1\ldots\nu_{p+1}}$ and, if p is odd, also the terms containing the factors $\delta_3^D\varepsilon^{\mu_1\mu_2\mu_3}$ and respectively $\delta_5^D\varepsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}$ with some densities involving $\delta_{2p+1}^D\varepsilon^{\mu_1\ldots\mu_{2p+1}}$ and $\delta_{3p+2}^D\varepsilon^{\mu_1\ldots\mu_{3p+2}}$ respectively (if p is even, the terms proportional with either q_1 or q_2 must be suppressed). The other case emphasizes that *it is possible to construct some new deformations in $D = p+2$, describing a spin two-field coupled to a p-form and having (14) and (15) as a free limit, which are consistent to all orders in the coupling constant and are not subject to the rules of General Relativity.* Performing the necessary computations, we find the Lagrangian action

$$S^{\mathrm{L}}[h_{\mu\nu}, A_{\mu_1\ldots\mu_p}] = \int d^{p+2}x\left(\mathcal{L}_0^{(\mathrm{PF})} - 2\lambda\Lambda h - \frac{1}{2\cdot(p+1)!}F'_{\mu_1\ldots\mu_{p+1}}F'^{\mu_1\ldots\mu_{p+1}}\right), \tag{16}$$

where the field strength of the p-form is deformed as

$$F'_{\mu_1\ldots\mu_{p+1}} = F_{\mu_1\ldots\mu_{p+1}} + 2\,(-)^{p+1}\,\lambda y_3\varepsilon_{\mu_1\ldots\mu_{p+1}\rho}\partial^{[\theta}h^{\rho]}{}_{\theta}. \tag{17}$$

This action is fully invariant under the original Pauli-Fierz gauge transformations and

$$\bar{\delta}_\epsilon A_{\mu_1\ldots\mu_p} = \partial_{[\mu_1}\epsilon^{(p)}_{\mu_2\ldots\mu_p]} + \lambda y_3\varepsilon_{\mu_1\ldots\mu_p\nu\rho}\partial^{[\nu}\epsilon^{\rho]}. \tag{18}$$

The gauge algebra remains Abelian and the reducibility of (18) is not affected by these couplings: the associated functions and relations are the initial ones.

3.3 Side note: collection of Pauli-Fierz fields and a p-form gauge field

It is known that one cannot construct in a consistent manner multi-graviton theories, meaning that there are no consistent cross-couplings among different Hilbert-Einstein or Weyl gravitons, neither direct nor intermediated by other fields. There still remains open the question whether the couplings of the type revealed in the above, between several spin-two fields described in the free limit by Pauli-Fierz models and a p-form, still enforce this restriction. The answer is positive.

More precisely, we begin with a finite sum of Pauli-Fierz actions and a single Abelian p-form with $p \geq 1$ $(D \geq p+1)$

$$\begin{aligned} S_0^{\mathrm{L}}\left[h^A_{\mu\nu}, A_{\mu_1\ldots\mu_p}\right] \;=\; & \int d^Dx\left[-\frac{1}{2}\left(\partial_\mu h^A_{\nu\rho}\right)\partial^\mu h_A^{\nu\rho} + \left(\partial_\mu h_A^{\mu\rho}\right)\partial^\nu h^A_{\nu\rho} - \left(\partial_\mu h^A\right)\partial_\nu h_A^{\nu\mu}\right. \\ & \left. +\frac{1}{2}\left(\partial_\mu h^A\right)\partial^\mu h_A - \frac{1}{2\cdot(p+1)!}F_{\mu_1\ldots\mu_{p+1}}F^{\mu_1\ldots\mu_{p+1}}\right], \end{aligned} \tag{19}$$

where h_A is the trace of the Pauli-Fierz field $h_A^{\mu\nu}$ ($h_A = \sigma_{\mu\nu}h_A^{\mu\nu}$) and $A = \overline{1,n}$, for $n > 1$. The collection indices A, B, etc., are raised and lowered with a quadratic form k_{AB} that determines a positively-defined metric in the internal space. Action (19) is invariant under the gauge transformations

$$\delta_\epsilon h^A_{\mu\nu} = \partial_{(\mu}\epsilon^A_{\nu)}, \qquad \delta_\epsilon A_{\mu_1\ldots\mu_p} = \partial_{[\mu_1}\epsilon^{(p)}_{\mu_2\ldots\mu_p]}. \tag{20}$$

Under these considerations, the next theorem can be proved by means of the deformation theory combined with specific BRST cohomological computations [29].

Theorem 3.3 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance, a positive-definite metric tensor in the inner space of collection indices and at most two derivatives in the Lagrangian, there are no cross-couplings among different spin-two fields intermediated by a p-form gauge field.*

In fact, only one of the spin-two fields gets coupled to the p-form exactly like in the previous subsection, while the other spin-two fields remain free.

4 Nontrivial couplings of dual formulations of LG

4.1 Massless tensor field with the mixed symmetry $(3,1)$ plus an Abelian vector field (one-form)

In this situation we start from the Lagrangian action

$$
\begin{aligned}
& S_0\left[t_{\lambda\mu\nu|\alpha}, A_\mu\right] = \int d^D x \left\{ \frac{1}{2}\left[\left(\partial^\rho t^{\lambda\mu\nu|\alpha}\right)\left(\partial_\rho t_{\lambda\mu\nu|\alpha}\right) - \left(\partial_\alpha t^{\lambda\mu\nu|\alpha}\right)\left(\partial^\beta t_{\lambda\mu\nu|\beta}\right)\right] \right. \\
& -\frac{3}{2}\left[\left(\partial_\lambda t^{\lambda\mu\nu|\alpha}\right)\left(\partial^\rho t_{\rho\mu\nu|\alpha}\right) + \left(\partial^\rho t^{\lambda\mu}\right)\left(\partial_\rho t_{\lambda\mu}\right)\right] + 3\left(\partial_\alpha t^{\lambda\mu\nu|\alpha}\right)\left(\partial_\lambda t_{\mu\nu}\right) \\
& \left. +3\left(\partial_\rho t^{\rho\mu}\right)\left(\partial^\lambda t_{\lambda\mu}\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right\} \equiv S_0^{\mathrm{t}}\left[t_{\lambda\mu\nu|\alpha}\right] + S_0^{\mathrm{A}}\left[A_\mu\right], \qquad (21)
\end{aligned}
$$

in $D \geq 5$ spacetime dimensions. The massless tensor field $t_{\lambda\mu\nu|\alpha}$ has the mixed symmetry $(3,1)$ and hence transforms according to an irreducible representation of $GL(D,\mathbb{R})$ corresponding to a 4-cell Young diagram with two columns and three rows. It is thus completely antisymmetric in its first three indices and satisfies the identity $t_{[\lambda\mu\nu|\alpha]} \equiv 0$. The field strength of the vector field A_μ is defined in the standard manner by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The trace of $t_{\lambda\mu\nu|\alpha}$ is defined by $t_{\lambda\mu} = \sigma^{\nu\alpha} t_{\lambda\mu\nu|\alpha}$ and it is obviously an antisymmetric tensor. A generating set of gauge transformations for action (21) can be taken of the form

$$
\delta_{\epsilon,\chi} t_{\lambda\mu\nu|\alpha} = -3\partial_{[\lambda}\epsilon_{\mu\nu\alpha]} + 4\partial_{[\lambda}\epsilon_{\mu\nu]\alpha} + \partial_{[\lambda}\chi_{\mu\nu]|\alpha}, \qquad \delta_\epsilon A_\mu = \partial_\mu \epsilon, \qquad (22)
$$

where the gauge parameters $\epsilon_{\lambda\mu\nu}$ determine a completely antisymmetric tensor, the other set of gauge parameters displays the mixed symmetry $(2,1)$, such that they are antisymmetric in the first two indices and satisfy the identity $\chi_{[\mu\nu|\alpha]} \equiv 0$, and the gauge parameter ϵ is a scalar. The generating set of gauge transformations (22) is off-shell, second-stage reducible, the accompanying gauge algebra being obviously Abelian. More precisely, the gauge transformations (22) are off-shell, second-stage reducible. This is because: 1. 1. If in the first relation (22) we make the transformations

$$
\epsilon_{\mu\nu\alpha} \rightarrow \epsilon_{\mu\nu\alpha}^{(\omega,\psi)} = -\frac{1}{2}\partial_{[\mu}\omega_{\nu\alpha]}, \qquad \chi_{\mu\nu|\alpha} \rightarrow \chi_{\mu\nu|\alpha}^{(\omega,\psi)} = \partial_{[\mu}\psi_{\nu]\alpha} + 2\partial_\alpha \omega_{\mu\nu} - \partial_{[\mu}\omega_{\nu]\alpha}, \qquad (23)
$$

with $\omega_{\nu\alpha}$ antisymmetric and $\psi_{\nu\alpha}$ symmetric (but otherwise arbitrary), then the gauge variation of the tensor field identically vanishes $\delta_{\epsilon^{(\omega,\psi)},\chi^{(\omega,\psi)}} t_{\lambda\mu\nu|\alpha} \equiv 0$. 2. If in (23) we perform the changes

$$
\omega_{\nu\alpha} \rightarrow \omega_{\nu\alpha}^{(\theta)} = \partial_{[\mu}\theta_{\nu]}, \qquad \psi_{\nu\alpha} \rightarrow \psi_{\nu\alpha}^{(\theta)} = -3\partial_{(\mu}\theta_{\nu)}, \qquad (24)
$$

with θ_ν an arbitrary vector field, where $(\mu\nu\cdots)$ signifies symmetrization with respect to the indices between parentheses without normalization factors, then the transformed gauge parameters (23) identically vanish $\epsilon_{\mu\nu\alpha}^{\left(\omega^{(\theta)},\psi^{(\theta)}\right)} \equiv 0$, $\chi_{\mu\nu|\alpha}^{\left(\omega^{(\theta)},\psi^{(\theta)}\right)} \equiv 0$. 3. There is no nonvanishing local transformation of θ_ν that simultaneously annihilates $\omega_{\nu\alpha}^{(\theta)}$ and $\psi_{\nu\alpha}^{(\theta)}$ of the form (24) and hence no further local reducibility identity.

After applying the entire cohomological procedure of finding all consistent interactions that can be added to this free model, we are able to prove the next theorem [30].

Theorem 4.1 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and at most two derivatives in the Lagrangian, there appear consistent couplings between a massless tensor field with the mixed symmetry* $(3,1)$ *and an Abelian vector field, but only in* $D = 5$.

These couplings 'live' only in a five-dimensional spacetime, the corresponding Lagrangian action being written as

$$
\bar{S}_0\left[t_{\lambda\mu\nu|\alpha}, A_\mu\right] = S_0^{\mathrm{t}}\left[t_{\lambda\mu\nu|\alpha}\right] - \frac{1}{4}\int d^5 x\, \bar{F}_{\mu\nu}\bar{F}^{\mu\nu}, \qquad (25)
$$

in terms of the deformed field strength

$$\bar{F}^{\mu\nu} = F^{\mu\nu} + \frac{4\lambda}{3}\varepsilon^{\mu\nu\alpha\beta\gamma}\partial_{[\rho}t_{\alpha\beta\gamma]|}{}^{\rho}, \tag{26}$$

where $S_0^{\rm t}\left[t_{\lambda\mu\nu|\alpha}\right]$ is the Lagrangian action of the massless tensor field $t_{\lambda\mu\nu|\alpha}$ appearing in (21) in $D = 5$. We observe that the action (25) contains only mixing-component terms of order one and two in the coupling constant. The deformed gauge transformations of the above action read as

$$\bar{\delta}_{\epsilon,\chi}t_{\lambda\mu\nu|\alpha} = -3\partial_{[\lambda}\epsilon_{\mu\nu\alpha]} + 4\partial_{[\lambda}\epsilon_{\mu\nu]\alpha} + \partial_{[\lambda}\chi_{\mu\nu]|\alpha}, \qquad \bar{\delta}_{\epsilon,\chi}A_{\mu} = \partial_{\mu}\epsilon + 4\lambda\varepsilon_{\mu\alpha\beta\gamma\delta}\partial^{\alpha}\epsilon^{\beta\gamma\delta}. \tag{27}$$

It is interesting to note that only the gauge transformations of the vector field are modified during the deformation process. This is enforced at order one in the coupling constant by a term linear in the antisymmetrized first-order derivatives of some gauge parameters from the $(3,1)$ sector. The gauge algebra and the reducibility structure of the coupled model are not modified during the deformation procedure, being the same like in the case of the starting free action (21) with the gauge transformations (22). It is easy to see from (25) and (27) that if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions (we must set $\lambda = 0$ in these formulas).

It is important to stress that the problem of obtaining consistent interactions strongly depends on the spacetime dimension. For instance, if one starts with action (21) in $D > 5$, then *no* term can be added to either the original Lagrangian or its gauge transformations.

4.2 Massless tensor field with the mixed symmetry $(3,1)$ plus an Abelian p-form gauge field

In this situation the starting point is given by a free model describing a massless tensor field $t_{\lambda\mu\nu|\alpha}$ and an Abelian p-form

$$S_0\left[t_{\lambda\mu\nu|\alpha}, A_{\mu_1\ldots\mu_p}\right] = S_0^{\rm t}\left[t_{\lambda\mu\nu|\alpha}\right] + S_0^{\rm A}\left[A_{\mu_1\ldots\mu_p}\right], \tag{28}$$

where $S_0^{\rm A}\left[A_{\mu_1\ldots\mu_p}\right]$ and $S_0^{\rm t}\left[t_{\lambda\mu\nu|\alpha}\right]$ follow respectively from formulas (21) and (14). The spacetime dimension is subject to the inequality $D \geq \max(5, p+1)$, which ensures that the number of physical degrees of freedom of this free model is nonnegative. The Abelian p-form field strength is defined in the usual manner as before. Action (28) is invariant under a generating set of gauge transformations given by the first relation in (22) for the field $t_{\lambda\mu\nu|\alpha}$ and respectively by the latter formula in (15) for the p-form gauge field. The gauge symmetries of $S_0^{\rm t}\left[t_{\lambda\mu\nu|\alpha}\right]$ are reducible of order two, while the gauge transformations of $A_{\mu_1\ldots\mu_p}$ are reducible of order $(p-1)$, such that the overall reducibility order will be equal to $\max(2, p-1)$.

Although the cohomological structure in the case of a p-form with $p > 1$ is clearly richer than in the presence of a vector field, nevertheless the cohomology of the tensor fields with the mixed symmetry $(3,1)$ is dominant. Just like in the previous situation of a vector field, we can prove the next theorem [30].

Theorem 4.2 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and at most two derivatives in the Lagrangian, there appear consistent couplings between a massless tensor field with the mixed symmetry* $(3,1)$ *and an Abelian p-form gauge field, but only in* $D = p + 4$.

Regarding the Lagrangian structure of this coupled model, we mention that the associated Lagrangian action takes the form

$$\bar{S}_0\left[t_{\lambda\mu\nu|\alpha}, A_{\mu_1\ldots\mu_p}\right] = S_0^{\rm t}\left[t_{\lambda\mu\nu|\alpha}\right] - \frac{1}{2\cdot(p+1)!}\int d^{p+4}x\bar{F}_{\mu_1\ldots\mu_{p+1}}\bar{F}^{\mu_1\ldots\mu_{p+1}}, \tag{29}$$

in terms of the deformed field strength

$$\bar{F}^{\mu_1\ldots\mu_{p+1}} = F^{\mu_1\ldots\mu_{p+1}} + (-)^{p+1}\frac{4\lambda}{3}\varepsilon^{\mu_1\ldots\mu_{p+1}\alpha\beta\gamma}\partial_{[\rho}t_{\alpha\beta\gamma]|}{}^{\rho}, \tag{30}$$

where $S_0^{\mathrm{t}}\left[t_{\lambda\mu\nu|\alpha}\right]$ is the free action for the tensor field with the mixed symmetry $(3,1)$ evolving on a spacetime of dimension $D=p+4$. The deformed gauge transformations reduce to the first relation in (22) in $D=p+4$ for the field $t_{\lambda\mu\nu|\alpha}$ and respectively to

$$\bar{\delta}_{\epsilon,\chi}A_{\mu_1\ldots\mu_p}=\partial_{[\mu_1}\epsilon^{(p)}_{\mu_2\ldots\mu_p]}+4\lambda\varepsilon_{\mu_1\ldots\mu_p\alpha\beta\gamma\delta}\partial^{\alpha}\epsilon^{\beta\gamma\delta}, \tag{31}$$

such that only the gauge symmetries of the p-form are modified.

The couplings reduce to mixing-component terms, like in the vector case and, again, only the gauge transformations of the p-form gauge field are modified during the deformation process at order one in the coupling constant by a term linear in the antisymmetrized first-order derivatives of some gauge parameters from the $(3,1)$ sector. The gauge algebra and the reducibility structure of the coupled model are not modified during the deformation procedure. Meanwhile if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions.

All the results of this subsection apply to the case of a collection of massless tensor fields with the mixed symmetry $(3,1)$ plus an Abelian p-form gauge field, described by the Lagrangian action

$$S_0\left[t^A_{\lambda\mu\nu|\alpha},A_{\mu_1\ldots\mu_p}\right]=\sum_{A=1}^{n}S_0^{\mathrm{t}}\left[t^A_{\lambda\mu\nu|\alpha}\right]+S_0^{\mathrm{A}}\left[A_{\mu_1\ldots\mu_p}\right],\qquad n>1, \tag{32}$$

where each $S_0^{\mathrm{t}}\left[t^A_{\lambda\mu\nu|\alpha}\right]$ reads as in (21) modulo the replacement $t^A_{\lambda\mu\nu|\alpha}\rightarrow t_{\lambda\mu\nu|\alpha}$, and subject to the gauge transformations

$$\delta_{\epsilon,\chi}t^A_{\lambda\mu\nu|\alpha}=-3\partial_{[\lambda}\epsilon^A_{\mu\nu\alpha]}+4\partial_{[\lambda}\epsilon^A_{\mu\nu]\alpha}+\partial_{[\lambda}\chi^A_{\mu\nu]|\alpha} \tag{33}$$

and respectively the latter formula in (15). The uppercase indices A, B, etc. stand for the collection indices of the fields with the mixed symmetry $(3,1)$ and are assumed to take discrete values: 1, 2, $\ldots$, n. They are lowered with a symmetric, constant, and invertible matrix, of elements k_{AB}, and are raised with the help of the elements k^{AB} of its inverse. In this situation we obtain the following (rather disappointing, although expected) result [31], which expresses the fact that there can be only a single massless tensor field with the mixed symmetry $(3,1)$ in a given universe, just like in the case of the multi-graviton problem.

Theorem 4.3 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance, a positive-definite metric tensor in the inner space of collection indices and at most two derivatives in the Lagrangian, there are no cross-couplings among different tensor fields with the mixed symmetry* $(3,1)$ *intermediated by a p-form gauge field.*

In fact, only one of the mixed symmetry-type fields gets coupled to the p-form like in (29), while the others remain free.

4.3 Massless tensor field with the mixed symmetry $(2,1)$ plus an Abelian BF model with a maximal field spectrum in $D=5$

The starting point is a free theory in $D=5$, whose Lagrangian action is written as the sum between the Lagrangian action of an Abelian BF model with a maximal field spectrum (a single scalar field φ, two types of one-forms H^{μ} and V_{μ}, two kinds of two-forms $B^{\mu\nu}$ and $\phi_{\mu\nu}$, and one three-form $K^{\mu\nu\rho}$) and the Lagrangian action of a free, massless tensor field with the mixed symmetry $(2,1)$ $t_{\mu\nu|\alpha}$ (meaning it is antisymmetric in its first two indices $t_{\mu\nu|\alpha}=-t_{\nu\mu|\alpha}$ and fulfills the identity $t_{[\mu\nu|\alpha]}\equiv 0$)

$$\begin{aligned}S_0^{\mathrm{L}}\left[\Phi^{\alpha_0}\right] &= \int d^5x\left[H^{\mu}\partial_{\mu}\varphi+\tfrac{1}{2}B^{\mu\nu}\partial_{[\mu}V_{\nu]}+\tfrac{1}{3}K^{\mu\nu\rho}\partial_{[\mu}\phi_{\nu\rho]}\right.\\ &\qquad \left.-\tfrac{1}{12}\left(F_{\mu\nu\rho|\alpha}F^{\mu\nu\rho|\alpha}-3F_{\mu\nu}F^{\mu\nu}\right)\right]\\ &\equiv \int d^5x\left(\mathcal{L}_0^{\mathrm{BF}}+\mathcal{L}_0^{\mathrm{t}}\right),\end{aligned} \tag{34}$$

where we used the notations

$$\Phi^{\alpha_0} = \left(\varphi, H^{\mu}, V_{\mu}, B^{\mu\nu}, \phi_{\mu\nu}, K^{\mu\nu\rho}, t_{\mu\nu|\alpha}\right), \tag{35}$$

$$F_{\mu\nu\rho|\alpha} = \partial_{[\mu}t_{\nu\rho]|\alpha}, \qquad F_{\mu\nu} = \sigma^{\rho\alpha}F_{\mu\nu\rho|\alpha}. \tag{36}$$

We work with the same conventions like in the previous (sub)sections.

Action (34) is found invariant under the gauge transformations

$$\delta_{\Omega}\varphi = 0, \qquad \delta_{\Omega}H^{\mu} = 2\partial_{\nu}\epsilon^{\mu\nu}, \tag{37}$$

$$\delta_{\Omega}V_{\mu} = \partial_{\mu}\epsilon, \qquad \delta_{\Omega}B^{\mu\nu} = -3\partial_{\rho}\epsilon^{\mu\nu\rho}, \tag{38}$$

$$\delta_{\Omega}\phi_{\mu\nu} = \partial_{[\mu}\xi_{\nu]}, \qquad \delta_{\Omega}K^{\mu\nu\rho} = 4\partial_{\lambda}\xi^{\mu\nu\rho\lambda}, \tag{39}$$

$$\delta_{\Omega}t_{\mu\nu|\alpha} = \partial_{[\mu}\theta_{\nu]\alpha} + \partial_{[\mu}\chi_{\nu]\alpha} - 2\partial_{\alpha}\chi_{\mu\nu}, \tag{40}$$

where all the gauge parameters are bosonic, with $\epsilon^{\mu\nu}$, $\epsilon^{\mu\nu\rho}$, $\xi^{\mu\nu\rho\lambda}$, and $\chi_{\mu\nu}$ completely antisymmetric and $\theta_{\mu\nu}$ symmetric. By Ω we denoted collectively all the gauge parameters as

$$\Omega^{\alpha_1} \equiv \left(\epsilon^{\mu\nu}, \epsilon, \epsilon^{\mu\nu\rho}, \xi_{\mu}, \xi^{\mu\nu\rho\lambda}, \theta_{\mu\nu}, \chi_{\mu\nu}\right). \tag{41}$$

The gauge transformations given by (37)–(40) are off-shell reducible of order three (the reducibility relations hold everywhere in the space of field history, and not only on the stationary surface of field equations). We observe that the free theory under study is a usual linear gauge theory (its field equations are linear in the fields), whose generating set of gauge transformations is third-order reducible, such that we can define in a consistent manner its Cauchy order, which is found to be equal to five.

After completing the cohomological computations based on the antifield-BRST differential of the free model, we are led to the following results regarding its consistent interactions [32].

Theorem 4.4 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and preservation of the number of derivatives on each field with respect to the free Lagrangian, there appear consistent couplings between dual linearized gravity in $D = 5$ and a topological BF model.*

More precisely, the Lagrangian action of the interacting theory is

$$\begin{aligned} S^{\mathrm{L}}\left[\Phi^{\alpha_0}\right] = & \int d^5x \left\{H_{\mu}\partial^{\mu}\varphi + \tfrac{1}{2}B^{\mu\nu}\partial_{[\mu}V_{\nu]} + \tfrac{1}{3}K^{\mu\nu\rho}\partial_{[\mu}\phi_{\nu\rho]}\right. \\ & +\lambda\left[W_1V_{\mu}H^{\mu} + W_2B_{\mu\nu}\phi^{\mu\nu} - W_3\phi_{[\mu\nu}V_{\rho]}K^{\mu\nu\rho} + \bar{M}(\varphi)\right. \\ & \left.+\varepsilon^{\alpha\beta\gamma\delta\varepsilon}\left(9W_4V_{\alpha}\tilde{K}_{\beta\gamma}\tilde{K}_{\delta\varepsilon} + \tfrac{1}{4}W_5V_{\alpha}\phi_{\beta\gamma}\phi_{\delta\varepsilon} + W_6B_{\alpha\beta}K_{\gamma\delta\varepsilon}\right)\right] \\ & -\tfrac{1}{12}\left(F_{\mu\nu\rho|\alpha}F^{\mu\nu\rho|\alpha} - 3F_{\mu\nu}F^{\mu\nu}\right) \\ & \left.+\lambda\left(k_1\phi^{\mu\nu} - \tfrac{k_2}{20}\tilde{K}^{\mu\nu}\right)\left[F_{\mu\nu} + \tfrac{3\lambda}{2}\left(k_1\phi_{\mu\nu} - \tfrac{k_2}{20}\tilde{K}_{\mu\nu}\right)\right]\right\}, \end{aligned} \tag{42}$$

where Φ^{α_0} is the field spectrum (35). The above action is parameterized by six functions depending only on the undifferentiated scalar field, $(W_a(\varphi))_{a=\overline{1,6}}$ and by two real constants. We cannot stress enough that these functions and constants are *not* arbitrary. The consistency of interactions at all orders in the coupling constant λ requires that they satisfy the equations

$$\frac{d\bar{M}(\varphi)}{d\varphi}W_1(\varphi) = 0, \qquad W_1(\varphi)W_2(\varphi) = 0, \tag{43}$$

$$W_1(\varphi)\frac{dW_2(\varphi)}{d\varphi} - 3W_2(\varphi)W_3(\varphi) + 6W_5(\varphi)W_6(\varphi) = 0, \tag{44}$$

$$W_2(\varphi)W_3(\varphi) + W_5(\varphi)W_6(\varphi) = 0, \tag{45}$$

$$W_1(\varphi)\frac{dW_6(\varphi)}{d\varphi}+3W_3(\varphi)W_6(\varphi)-6W_2(\varphi)W_4(\varphi)=0, \tag{46}$$

$$W_1(\varphi)W_6(\varphi)=0,\qquad W_2(\varphi)W_4(\varphi)+W_3(\varphi)W_6(\varphi)=0, \tag{47}$$

$$W_2(\varphi)W_5(\varphi)=0,\qquad W_4(\varphi)W_6(\varphi)=0, \tag{48}$$

$$k_1W_3+\tfrac{k_2}{60}W_5=0,\qquad k_1W_4+\tfrac{k_2}{2\cdot 5!}W_3=0, \tag{49}$$

$$k_1W_6+\tfrac{k_2}{5!}W_2=0. \tag{50}$$

The concrete form of the gauge transformations of action (42) can be written in the form

$$\bar{\delta}_\Omega\varphi=-\lambda W_1\epsilon, \tag{51}$$

$$\begin{aligned}
\bar{\delta}_\Omega H^\mu \;=\;& 2D_\nu\epsilon^{\mu\nu}+\lambda\left(\frac{dW_1}{d\varphi}H^\mu-3\frac{dW_3}{d\varphi}K^{\mu\nu\rho}\phi_{\nu\rho}\right)\epsilon\\
&-3\lambda\frac{dW_2}{d\varphi}\phi_{\nu\rho}\epsilon^{\mu\nu\rho}+2\lambda\left(\frac{dW_2}{d\varphi}B^{\mu\nu}-3\frac{dW_3}{d\varphi}K^{\mu\nu\rho}V_\rho\right)\xi_\nu\\
&+12\lambda\frac{dW_3}{d\varphi}V_\nu\phi_{\rho\lambda}\xi^{\mu\nu\rho\lambda}+2\lambda\frac{dW_6}{d\varphi}B^{\mu\nu}\varepsilon_{\nu\alpha\beta\gamma\delta}\xi^{\alpha\beta\gamma\delta}\\
&+3\lambda K^{\mu\nu\rho}\left(4\frac{dW_4}{d\varphi}V_\nu\varepsilon_{\rho\alpha\beta\gamma\delta}\xi^{\alpha\beta\gamma\delta}-\frac{dW_6}{d\varphi}\varepsilon_{\nu\rho\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma}\right)\\
&+\lambda\varepsilon^{\mu\nu\rho\lambda\sigma}\left[\tfrac{1}{4}\frac{dW_4}{d\varphi}\varepsilon_{\nu\rho\alpha\beta\gamma}K^{\alpha\beta\gamma}\varepsilon_{\lambda\sigma\alpha'\beta'\gamma'}K^{\alpha'\beta'\gamma'}\epsilon\right.\\
&\left.-\frac{dW_5}{d\varphi}\phi_{\nu\rho}\left(V_\lambda\xi_\sigma-\tfrac{1}{4}\phi_{\lambda\sigma}\epsilon\right)\right],
\end{aligned}\tag{52}$$

$$\bar{\delta}_\Omega V_\mu=\partial_\mu\epsilon-2\lambda W_2\xi_\mu-2\lambda\varepsilon_{\mu\nu\rho\lambda\sigma}W_6\xi^{\nu\rho\lambda\sigma}, \tag{53}$$

$$\begin{aligned}
\bar{\delta}_\Omega B^{\mu\nu} \;=\;& -3\partial_\rho\epsilon^{\mu\nu\rho}-2\lambda W_1\epsilon^{\mu\nu}+6\lambda W_3\left(2\phi_{\rho\lambda}\xi^{\mu\nu\rho\lambda}+K^{\mu\nu\rho}\xi_\rho\right)\\
&+\lambda\left(12W_4K^{\mu\nu\rho}\varepsilon_{\rho\alpha\beta\gamma\delta}\xi^{\alpha\beta\gamma\delta}-W_5\varepsilon^{\mu\nu\rho\lambda\sigma}\phi_{\rho\lambda}\xi_\sigma\right),
\end{aligned}\tag{54}$$

$$\begin{aligned}
\bar{\delta}_\Omega\phi_{\mu\nu} \;=\;& D^{(-)}_{[\mu}\xi_{\nu]}+3\lambda\left(W_3\phi_{\mu\nu}\epsilon-2W_4V_{[\mu}\varepsilon_{\nu]\alpha\beta\gamma\delta}\xi^{\alpha\beta\gamma\delta}\right)\\
&+3\lambda\varepsilon_{\mu\nu\rho\lambda\sigma}\left(2W_4K^{\rho\lambda\sigma}\epsilon+W_6\epsilon^{\rho\lambda\sigma}-\tfrac{k_2}{180}\partial^{[\rho}\chi^{\lambda\sigma]}\right),
\end{aligned}\tag{55}$$

$$\begin{aligned}
\bar{\delta}_\Omega K^{\mu\nu\rho} \;=\;& 4D^{(+)}_\lambda\xi^{\mu\nu\rho\lambda}-3\lambda\left(W_2\epsilon^{\mu\nu\rho}+W_3K^{\mu\nu\rho}\epsilon\right)\\
&-\lambda\varepsilon^{\mu\nu\rho\lambda\sigma}W_5\left(V_\lambda\xi_\sigma-\tfrac{1}{2}\phi_{\lambda\sigma}\epsilon\right)-2\lambda k_1\partial^{[\mu}\chi^{\nu\rho]},
\end{aligned}\tag{56}$$

$$\bar{\delta}_\Omega t_{\mu\nu|\alpha}=\partial_{[\mu}\theta_{\nu]\alpha}+\partial_{[\mu}\chi_{\nu]\alpha}-2\partial_\alpha\chi_{\mu\nu}+\lambda k_1\sigma_{\alpha[\mu}\xi_{\nu]}-\tfrac{\lambda k_2}{5!}\sigma_{\alpha[\mu}\varepsilon_{\nu]\beta\gamma\delta\varepsilon}\xi^{\beta\gamma\delta\varepsilon}, \tag{57}$$

where, in addition, we used the notations

$$D_\nu=\partial_\nu-\lambda\frac{dW_1}{d\varphi}V_\nu,\qquad D^{(\pm)}_\nu=\partial_\nu\pm3\lambda W_3V_\nu. \tag{58}$$

We observe that the cross-interaction terms,

$$\lambda\left(k_1\phi^{\mu\nu}-\tfrac{k_2}{20}\tilde{K}^{\mu\nu}\right)F_{\mu\nu},$$

are only of order one in the deformation parameter and couple the tensor field $t_{\lambda\mu|\alpha}$ to the two-form $\phi_{\mu\nu}$ and to the three-form $K^{\mu\nu\rho}$ from the BF sector. Also, it is interesting to see that the interaction components

$$\tfrac{3\lambda^2}{2}\left(k_1\phi^{\mu\nu}-\tfrac{k_2}{20}\tilde{K}^{\mu\nu}\right)\left(k_1\phi_{\mu\nu}-\tfrac{k_2}{20}\tilde{K}_{\mu\nu}\right),$$

which describe self-interactions in the BF sector, are strictly due to the presence of the tensor $t_{\lambda\mu|\alpha}$ (in its absence $k_1 = k_2 = 0$, so they would vanish). The gauge transformations of the BF fields $\phi_{\mu\nu}$ and $K^{\mu\nu\rho}$ are deformed in such a way to include gauge parameters from the $(2,1)$ sector. Related to the other BF fields, φ, H^μ, V_μ, and $B^{\mu\nu}$, their gauge transformations are also modified with respect to the free theory, but only with terms specific to the BF sector. A remarkable feature is that the gauge transformations of the tensor $t_{\lambda\mu|\alpha}$ are modified by shift terms in some of the gauge parameters from the BF sector.

Regarding the gauge structure of the interacting theory, we mention that the gauge algebra corresponding to the interacting theory is open (the commutators among the deformed gauge transformations only close on-shell), by contrast to the free theory, where the gauge algebra is Abelian, while the reducibility relations associated with the interacting model only hold on-shell, by contrast to those corresponding to the free theory, which hold off-shell.

Finally, a word of caution. The existence of cross-couplings is entirely determined by the existence of solutions to the consistency equations (43)–(50). Such solutions in fact exist and can be synthesized as follows [32].

I. The real constants k_1 and k_2 are arbitrary ($k_1^2+k_2^2>0$), functions $\bar{M}$ and W_2 are some arbitrary, real, smooth functions of the undifferentiated scalar field, and

$$W_1(\varphi) = W_3(\varphi) = W_4(\varphi) = W_5(\varphi) = 0, \tag{59}$$

$$W_6(\varphi) = -\frac{k_2}{5!k_1} W_2(\varphi). \tag{60}$$

The above formulas allow one to infer directly the solution in the general case $k_2 = 0$. This class of solutions can be equivalently reformulated as: the real constants k_1 and k_2 are arbitrary ($k_1^2 + k_2^2 > 0$), functions $\bar{M}$ and W_6 are some arbitrary, real, smooth functions of the undifferentiated scalar field, and

$$W_1(\varphi) = W_3(\varphi) = W_4(\varphi) = W_5(\varphi) = 0, \tag{61}$$

$$W_2(\varphi) = -\frac{5!k_1}{k_2} W_6(\varphi). \tag{62}$$

The last formulas are useful at writing down the solution in the particular case $k_1 = 0$.

II. The real constants k_1 and k_2 are arbitrary ($k_1^2+k_2^2>0$), functions $\bar{M}$ and W_5 are some arbitrary, real, smooth functions of the undifferentiated scalar field, and

$$W_1(\varphi) = W_2(\varphi) = W_6(\varphi) = 0, \tag{63}$$

$$W_3(\varphi) = -\frac{k_2}{60k_1} W_5(\varphi), \qquad W_4(\varphi) = \left(\frac{k_2}{5!k_1}\right)^2 W_5(\varphi). \tag{64}$$

The above formulas allow one to infer directly the solution in the general case $k_2 = 0$. This class of solutions can be equivalently reformulated as: the real constants k_1 and k_2 are arbitrary ($k_1^2 + k_2^2 > 0$), functions $\bar{M}$ and W_4 are some arbitrary, real, smooth functions of the undifferentiated scalar field, and

$$W_1(\varphi) = W_2(\varphi) = W_6(\varphi) = 0, \tag{65}$$

$$W_3(\varphi) = -2\cdot 5!\frac{k_1}{k_2} W_4(\varphi), \qquad W_5(\varphi) = \left(\frac{5!k_1}{k_2}\right)^2 W_4(\varphi). \tag{66}$$

The last formulas are useful at writing down the solution in the particular case $k_1 = 0$.

III. The real constants k_1 and k_2 are arbitrary ($k_1^2+k_2^2>0$), functions W_1 and W_5 are some arbitrary, real, smooth functions of the undifferentiated scalar field, and

$$W_2(\varphi) = W_6(\varphi) = \bar{M}(\varphi) = 0, \tag{67}$$

$$W_3(\varphi) = -\frac{k_2}{60k_1} W_5(\varphi), \qquad W_4(\varphi) = \left(\frac{k_2}{5!k_1}\right)^2 W_5(\varphi). \tag{68}$$

The above formulas allow one to infer directly the solution in the general case $k_2 = 0$. This class of solutions can be equivalently reformulated as: the real constants k_1 and k_2 are arbitrary ($k_1^2 + k_2^2 > 0$), functions W_1 and W_4 are some arbitrary, real, smooth functions of the undifferentiated scalar field, and

$$W_2(\varphi) = W_6(\varphi) = \bar{M}(\varphi) = 0, \tag{69}$$

$$W_3(\varphi) = -2 \cdot 5! \frac{k_1}{k_2} W_4(\varphi), \qquad W_5(\varphi) = \left(\frac{5!k_1}{k_2}\right)^2 W_4(\varphi). \tag{70}$$

The last formulas are useful at writing down the solution in the particular case $k_1 = 0$.

For all classes of solutions the emerging interacting theories display the following common features:

1. there appear nontrivial cross-couplings between the BF fields and the tensor field with the mixed symmetry $(2,1)$;
2. the gauge transformations are modified with respect to those of the free theory and the gauge algebras become open (only close on-shell);
3. the first-order reducibility functions are changed during the deformation process and the first-order reducibility relations take place on-shell.

Nevertheless, there appear the following differences between the above classes of solutions at the level of the higher-order reducibility:

a) for class I the second-order reducibility functions are modified with respect to the free ones and the corresponding reducibility relations take place on-shell. The third-order reducibility functions remain those from the free case and hence the associated reducibility relations hold off-shell;

b) for class II both the second- and third-order reducibility functions remain those from the free case and hence the associated reducibility relations hold off-shell;

c) for class III all the second- and third-order reducibility functions are deformed and the corresponding reducibility relations only close on-shell.

4.4 Massless tensor fields with the mixed symmetries $(3,1)$ and $(2,2)$

We begin with the free Lagrangian action

$$S_0\left[t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}\right] = S_0^{\mathrm{t}}\left[t_{\lambda\mu\nu|\alpha}\right] + S_0^{\mathrm{r}}\left[r_{\mu\nu|\alpha\beta}\right], \tag{71}$$

in $D \geq 5$ spacetime dimensions, where $S_0^{\mathrm{t}}\left[t_{\lambda\mu\nu|\alpha}\right]$ follows from (21) and

$$\begin{aligned} S_0^{\mathrm{r}}\left[r_{\mu\nu|\alpha\beta}\right] = & \int d^D x \left[\frac{1}{8}\left(\partial^\lambda r^{\mu\nu|\alpha\beta}\right)\left(\partial_\lambda r_{\mu\nu|\alpha\beta}\right) - \frac{1}{2}\left(\partial_\mu r^{\mu\nu|\alpha\beta}\right)\left(\partial^\lambda r_{\lambda\nu|\alpha\beta}\right)\right. \\ & - \left(\partial_\mu r^{\mu\nu|\alpha\beta}\right)\left(\partial_\beta r_{\nu\alpha}\right) - \frac{1}{2}\left(\partial^\lambda r^{\nu\beta}\right)\left(\partial_\lambda r_{\nu\beta}\right) \\ & \left. + \left(\partial_\nu r^{\nu\beta}\right)\left(\partial^\lambda r_{\lambda\beta}\right) - \frac{1}{2}\left(\partial_\nu r^{\nu\beta}\right)\left(\partial_\beta r\right) + \frac{1}{8}\left(\partial^\lambda r\right)\left(\partial_\lambda r\right)\right]. \end{aligned} \tag{72}$$

The massless tensor field $t_{\lambda\mu\nu|\alpha}$ has the mixed symmetry $(3,1)$. The trace of $t_{\lambda\mu\nu|\alpha}$ is defined by $t_{\lambda\mu} = \sigma^{\nu\alpha} t_{\lambda\mu\nu|\alpha}$ and it is obviously an antisymmetric tensor. The massless tensor field $r_{\mu\nu|\alpha\beta}$ of degree four has the mixed symmetry of the linearized Riemann tensor, and hence transforms according to an irreducible representation of $GL(D,\mathbb{R})$, corresponding to the rectangular Young diagram $(2,2)$ with two columns and two rows. Thus, it is separately antisymmetric in the pairs $\{\mu,\nu\}$ and $\{\alpha,\beta\}$, is symmetric under the interchange of these pairs ($\{\mu,\nu\} \longleftrightarrow \{\alpha,\beta\}$), and satisfies the identity $r_{[\mu\nu|\alpha]\beta} \equiv 0$ associated with the above diagram. The notation $r_{\nu\beta}$ signifies the trace of the original tensor field, $r_{\nu\beta} = \sigma^{\mu\alpha} r_{\mu\nu|\alpha\beta}$, which is symmetric, $r_{\nu\beta} = r_{\beta\nu}$, while r denotes its double trace, $r = \sigma^{\nu\beta} r_{\nu\beta} \equiv r^{\mu\nu}{}_{|\mu\nu}$, which is a scalar.

A generating set of gauge transformations for action (71) can be taken of the form given by the first relation in (22) for the field $t_{\lambda\mu\nu|\alpha}$ and respectively by

$$\delta_\xi r_{\mu\nu|\alpha\beta} = \partial_\mu \xi_{\alpha\beta|\nu} - \partial_\nu \xi_{\alpha\beta|\mu} + \partial_\alpha \xi_{\mu\nu|\beta} - \partial_\beta \xi_{\mu\nu|\alpha}. \tag{73}$$

The gauge parameters $\epsilon_{\lambda\mu\nu}$ determine a completely antisymmetric tensor, while the gauge parameters $\chi_{\mu\nu|\alpha}$ and $\xi_{\mu\nu|\alpha}$ display the mixed symmetry $(2,1)$, such that they are antisymmetric in the first two indices and satisfy the identities $\chi_{[\mu\nu|\alpha]} \equiv 0$ and $\xi_{[\mu\nu|\alpha]} \equiv 0$. This generating set of gauge transformations is off-shell, second-stage reducible, the accompanying gauge algebra being obviously Abelian.

By means of cohomological arguments, we can prove the next theorem [33].

Theorem 4.5 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and at most two derivatives in the Lagrangian, there appear consistent couplings between massless tensor fields with the mixed symmetry* $(3,1)$ *and respectively* $(2,2)$*, but only in* $D = 6$*.*

These consistent couplings ‘live’ in a six-dimensional spacetime. The Lagrangian action can be organized as

$$\begin{aligned} \bar{S}_0\left[t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}\right] = {} & S_0\left[t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}\right] + \lambda \int d^6x \, \Big[r \\ & -2 t_{\lambda\mu\nu|\rho} \varepsilon^{\lambda\mu\nu\alpha\beta\gamma} \left(\partial_\sigma \partial_\alpha r_{\beta\gamma|}{}^{\sigma\rho} - \frac{1}{2} \delta^\rho{}_\gamma \partial^\tau \partial_\alpha r_{\beta\tau} \right) \\ & -\lambda \left(5 r^{\lambda\rho|[\alpha\beta,\gamma]} r_{\lambda\rho|[\alpha\beta,\gamma]} - 6 r_{\lambda\rho|}{}^{[\alpha\beta,\rho]} r^{\lambda\sigma|}{}_{[\alpha\beta,\sigma]} \right) \Big], \end{aligned} \tag{74}$$

where $S_0\left[t_{\lambda\mu\nu|\alpha}, r_{\mu\nu|\alpha\beta}\right]$ is the Lagrangian action appearing in (71) in $D = 6$. We observe that action (74) contains only mixing-component terms of order one and two in the coupling constant. The deformed gauge transformations of the coupled action can be set in the form

$$\begin{aligned} \bar{\delta}_{\epsilon,\chi,\xi} t_{\lambda\mu\nu|\alpha} = {} & 3\partial_\alpha \epsilon_{\lambda\mu\nu} + \partial_{[\lambda} \epsilon_{\mu\nu]\alpha} + \partial_{[\lambda} \chi_{\mu\nu]|\alpha} \\ & -2\lambda \varepsilon_{\lambda\mu\nu\rho\beta\gamma} \left(\partial^\rho \xi^{\beta\gamma|}{}_\alpha - \frac{1}{4} \delta^\gamma{}_\alpha \partial^{[\rho} \xi^{\beta\tau]|}{}_\tau \right), \end{aligned} \tag{75}$$

$$\bar{\delta}_\xi r_{\mu\nu|\alpha\beta} = \partial_\mu \xi_{\alpha\beta|\nu} - \partial_\nu \xi_{\alpha\beta|\mu} + \partial_\alpha \xi_{\mu\nu|\beta} - \partial_\beta \xi_{\mu\nu|\alpha} = \delta_\xi r_{\mu\nu|\alpha\beta}. \tag{76}$$

It is interesting to note that only the gauge transformations of the tensor field $(3,1)$ are modified during the deformation process. This is enforced at order one in the coupling constant by a term linear in the first-order derivatives of the gauge parameters from the $(2,2)$ sector. Regarding the reducibility, only the first-order reducibility functions are modified at order one in the coupling constant, the others coinciding with the original ones

$$\epsilon_{\mu\nu\alpha} \rightarrow \bar{\epsilon}^{(\omega,\varphi)}_{\mu\nu\alpha} = -\frac{1}{2} \partial_{[\mu} \omega_{\nu\alpha]} + \lambda \varepsilon_{\mu\nu\alpha\lambda\beta\gamma} \partial^\lambda \varphi^{\beta\gamma}, \tag{77}$$

$$\chi_{\mu\nu|\alpha} \rightarrow \chi^{(\omega,\psi)}_{\mu\nu|\alpha} = 2\partial_\alpha \omega_{\mu\nu} - \partial_{[\mu} \omega_{\nu]\alpha} + \partial_{[\mu} \psi_{\nu]\alpha}, \tag{78}$$

$$\xi_{\mu\nu|\alpha} \rightarrow \xi^{(\varphi)}_{\mu\nu|\alpha} = 2\partial_\alpha \varphi_{\mu\nu} - \partial_{[\mu} \varphi_{\nu]\alpha}. \tag{79}$$

Consequently, the first-order reducibility relations for $t_{\lambda\mu\nu|\alpha}$ become

$$\bar{\delta}_{\bar{\epsilon}^{(\omega,\varphi)},\chi^{(\omega,\psi)},\xi^{(\varphi)}} t_{\lambda\mu\nu|\alpha} \equiv 0, \tag{80}$$

while those for $r_{\mu\nu|\alpha\beta}$ are not changed with respect to the free theory. Moreover, the gauge algebra of the coupled model is unchanged by the deformation procedure, being the same Abelian one like for the starting free theory and also the second-order reducibility functions remain the same, and hence the second-order reducibility relations are exactly the initial ones. If we impose the PT-invariance at the level of the coupled model, then we obtain no interactions.

4.5 Collections of massless tensor fields with the mixed symmetries $(3,1)$ and $(2,2)$

In this situation we start from a free theory in $D \geq 5$ that describes two finite collections of massless tensor fields with the mixed symmetries $(3,1)$ and respectively $(2,2)$

$$S_0\left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}\right] = S^{\mathrm{t}}_0\left[t^A_{\lambda\mu\nu|\kappa}\right] + S^{\mathrm{r}}_0\left[r^a_{\mu\nu|\kappa\beta}\right], \tag{81}$$

where

$$\begin{aligned} S^{\mathrm{t}}_0\left[t^A_{\lambda\mu\nu|\kappa}\right] = & \int\left\{\frac{1}{2}\left[\left(\partial^\rho t^{\lambda\mu\nu|\kappa}_A\right)\left(\partial_\rho t^A_{\lambda\mu\nu|\kappa}\right) - \left(\partial_\kappa t^{\lambda\mu\nu|\kappa}_A\right)\left(\partial^\beta t^A_{\lambda\mu\nu|\beta}\right)\right]\right. \\ & -\frac{3}{2}\left[\left(\partial_\lambda t^{\lambda\mu\nu|\kappa}_A\right)\left(\partial^\rho t^A_{\rho\mu\nu|\kappa}\right) + \left(\partial^\rho t^{\lambda\mu}_A\right)\left(\partial_\rho t^A_{\lambda\mu}\right)\right] \\ & \left.+3\left[\left(\partial_\kappa t^{\lambda\mu\nu|\kappa}_A\right)\left(\partial_\lambda t^A_{\mu\nu}\right) + \left(\partial_\rho t^{\rho\mu}_A\right)\left(\partial^\lambda t^A_{\lambda\mu}\right)\right]\right\} d^D x, \end{aligned} \tag{82}$$

$$\begin{aligned} S^{\mathrm{r}}_0\left[r^a_{\mu\nu|\kappa\beta}\right] = & \int\left\{-\frac{1}{2}\left[\left(\partial_\mu r^{\mu\nu|\kappa\beta}_a\right)\left(\partial^\lambda r^a_{\lambda\nu|\kappa\beta}\right) + \left(\partial^\lambda r^{\nu\beta}_a\right)\left(\partial_\lambda r^a_{\nu\beta}\right)\right.\right. \\ & \left.+\left(\partial_\nu r^{\nu\beta}_a\right)\left(\partial_\beta r^a\right)\right] + \frac{1}{8}\left[\left(\partial^\lambda r^{\mu\nu|\kappa\beta}_a\right)\left(\partial_\lambda r^a_{\mu\nu|\kappa\beta}\right) + \left(\partial^\lambda r_a\right)\left(\partial_\lambda r^a\right)\right] \\ & \left.-\left(\partial_\mu r^{\mu\nu|\kappa\beta}_a\right)\left(\partial_\beta r^a_{\nu\kappa}\right) + \left(\partial_\nu r^{\nu\beta}_a\right)\left(\partial^\lambda r^a_{\lambda\beta}\right)\right\} d^D x. \end{aligned} \tag{83}$$

The uppercase indices A, B, etc. stand for the collection indices of the fields with the mixed symmetry $(3,1)$ and are assumed to take discrete values: 1, 2, …, N. They are lowered with a symmetric, constant, and invertible matrix, of elements k_{AB}, and are raised with the help of the elements k^{AB} of its inverse. This means that $t^{\lambda\mu\nu|\kappa}_A = k_{AB}t^{B\lambda\mu\nu|\kappa}$ and $t^A_{\lambda\mu\nu|\kappa} = k^{AB}t_{B\lambda\mu\nu|\kappa}$. Each field $t^A_{\lambda\mu\nu|\kappa}$ is completely antisymmetric in its first three (Lorentz) indices and satisfies the identity $t^A_{[\lambda\mu\nu|\kappa]} \equiv 0$. The notation $t^A_{\lambda\mu}$ signifies the trace of $t^A_{\lambda\mu\nu|\kappa}$, defined by $t^A_{\lambda\mu} = \sigma^{\nu\kappa}t^A_{\lambda\mu\nu|\kappa}$. The trace components define an antisymmetric tensor, $t^A_{\lambda\mu} = -t^A_{\mu\lambda}$. The lowercase indices a, b, etc. stand for the collection indices of the fields with the mixed symmetry $(2,2)$ and are assumed to take the discrete values 1, 2, …, n. They are lowered with a symmetric, constant, and invertible matrix, of elements k_{ab}, and are raised with the help of the elements k^{ab} of its inverse, such that $r^{\mu\nu|\kappa\beta}_a = k_{ab}r^{b\mu\nu|\kappa\beta}$ and $r^a_{\lambda\nu|\kappa\beta} = k^{ab}r_{b\lambda\nu|\kappa\beta}$. Each tensor field $r^a_{\mu\nu|\kappa\beta}$ is separately antisymmetric in the pairs $\{\mu,\nu\}$ and $\{\kappa,\beta\}$, is symmetric under their permutation ($\{\mu,\nu\} \longleftrightarrow \{\kappa,\beta\}$), and satisfies the identity $r^a_{[\mu\nu|\kappa]\beta} \equiv 0$. The notations $r^a_{\nu\beta}$ signify the traces of $r^a_{\mu\nu|\kappa\beta}$, $r^a_{\nu\beta} = \sigma^{\mu\kappa}r^a_{\mu\nu|\kappa\beta}$, which are symmetric, $r^a_{\nu\beta} = r^a_{\beta\nu}$, while r^a represent their double traces, $r^a = \sigma^{\nu\beta}r^a_{\nu\beta}$, which are scalars.

A generating set of gauge transformations of action (81) can be taken as

$$\delta_{\epsilon,\chi}t^A_{\lambda\mu\nu|\kappa} = 3\epsilon^A_{\lambda\mu\nu,\kappa} + \partial_{[\lambda}\epsilon^A_{\mu\nu]\kappa} + \partial_{[\lambda}\chi^A_{\mu\nu]|\kappa}, \tag{84}$$

$$\delta_\xi r^a_{\mu\nu|\kappa\beta} = \xi^a_{\kappa\beta|[\nu,\mu]} + \xi^a_{\mu\nu|[\beta,\kappa]}, \tag{85}$$

where we used the standard notation $f_{,\mu} = \partial f/\partial x^\mu$. All the gauge parameters are bosonic, with $\epsilon^A_{\lambda\mu\nu}$ completely antisymmetric and $\chi^A_{\mu\nu|\kappa}$ together with $\xi^a_{\mu\nu|\kappa}$ defining two collections of tensor fields with the mixed symmetry $(2,1)$. The former gauge transformations, (84), are off-shell, second-order reducible in the space of all field histories, the associated gauge algebra being Abelian, while the gauge symmetries (85) are off-shell, first-order reducible, the corresponding algebra being also Abelian. It follows that the free theory (81) is a linear gauge theory with the Cauchy order equal to four. The simplest gauge invariant quantities are precisely the curvature tensors

$$K_A^{\lambda\mu\nu\xi|\kappa\beta} = t_A^{[\mu\nu\xi,\lambda]|[\beta,\kappa]}, \quad F^a_{\mu\nu\lambda|\kappa\beta\gamma} = r^a_{[\mu\nu,\lambda]|[\kappa\beta,\gamma]}, \tag{86}$$

and their space-time derivatives. It is easy to check that they display the mixed symmetry $(4,2)$ and $(3,3)$ respectively.

In order to determine all consistent interactions that can be added to this free model, we apply the general procedure based on the deformation of the generator of the antifield-BRST symmetry and find the following results [34].

Theorem 4.6 *Under the assumptions of analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance and at most two derivatives in the Lagrangian, there appear consistent cross-couplings between two collections of massless tensor fields with the mixed symmetry* $(3,1)$ *and respectively* $(2,2)$*, but only in* $D=6$*.*

As a consequence, we deduce the coupled Lagrangian action

$$\begin{aligned}
&\bar{S}_0\left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}\right] = S_0\left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}\right] \\
&+\lambda\int\left[c_a r^a - 2f^A_a \varepsilon^{\lambda\mu\nu\kappa\beta\gamma} t_{A\lambda\mu\nu|\rho}\left(\partial_\sigma\partial_\kappa r^{a\ \ \ \ \sigma\rho}_{\beta\gamma|} - \frac{1}{2}\delta^\rho_\gamma \partial^\tau\partial_\kappa r^a_{\beta\tau}\right)\right. \\
&\left.-\lambda f^a_A f^A_b\left(5r_a^{\lambda\rho|[\kappa\beta,\gamma]} r^b_{\lambda\rho|[\kappa\beta,\gamma]} - 6r_{a\lambda\rho|}^{\ \ \ \ \ [\kappa\beta,\rho]} r^{b\lambda\sigma|}_{\ \ \ \ \ [\kappa\beta,\sigma]}\right)\right] d^6x,
\end{aligned} \tag{87}$$

where $S_0\left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}\right]$ is the free action (81) in $D=6$ space-time dimensions. We observe that action (87) contains only mixing-component terms of order one and two in the coupling constant. Apparently, it seems that (87) contains non-trivial couplings between different tensor fields with the mixed symmetry of the Riemann tensor

$$-\lambda^2 f^a_A f^A_b\left(5r_a^{\lambda\rho|[\kappa\beta,\gamma]} r^b_{\lambda\rho|[\kappa\beta,\gamma]} - 6r_{a\lambda\rho|}^{\ \ \ \ \ [\kappa\beta,\rho]} r^{b\lambda\sigma|}_{\ \ \ \ \ [\kappa\beta,\sigma]}\right), \quad a \neq b. \tag{88}$$

The appearance of these cross-couplings is dictated by the properties of the matrix M of elements $M^a_b = f^a_A f^A_b$.

Let us analyze the properties of the quadratic matrix M. It is more convenient to work with the symmetric matrix $\hat{M} = (M_{ab})$, of elements $M_{ab} = f^A_a f^B_b k_{AB}$. From (83) and (87) we observe that there appear effective cross-couplings among different fields from the collection $\left\{r^a_{\mu\nu|\kappa\beta}\right\}_{a=\overline{1,n}}$ if and only if the symmetric matrices $\hat{M} = (M_{ab})$ and $\hat{k} = (k_{ab})$ are simultaneously diagonalizable. We recall $\hat{k}$ is the quadratic form defined by the kinetic terms of action (83), or, in other words, the metric tensor in the inner space of collection indices $a = \overline{1,n}$. This means that there exists an orthogonal matrix $\hat{O} = (O^a{}_b)$ that diagonalizes simultaneously [36] $\hat{M}$ and $\hat{k}$, i.e.

$$O^c{}_a O^d{}_b k_{cd} = k_a \delta_{ab}, \quad O^c{}_a O^d{}_b M_{cd} = m_a \delta_{ab}, \tag{89}$$

where k_a represent the eigenvalues of the matrix $\hat{k}$ and m_a those of $\hat{M}$. Indeed, if there exists a matrix $\hat{O}$ that satisfies the conditions (89), then action (87) can be brought to the form

$$\begin{aligned}
&\bar{S}_0\left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}\right] = \bar{S}'_0\left[t^A_{\lambda\mu\nu|\kappa}, r'^a_{\mu\nu|\kappa\beta}\right] = S^{\mathrm{t}}_0\left[t^A_{\lambda\mu\nu|\kappa}\right] \\
&+\int\sum_{a=1}^n k_a\left\{-\frac{1}{2}\left[\left(\partial_\mu r'^{a\mu\nu|\kappa\beta}\right)\left(\partial^\lambda r'^a_{\lambda\nu|\kappa\beta}\right) + \left(\partial^\lambda r'^{a\nu\beta}\right)\left(\partial_\lambda r'^a_{\nu\beta}\right)\right.\right. \\
&\left.+\left(\partial_\nu r'^{a\nu\beta}\right)\left(\partial_\beta r'^a\right)\right] + \frac{1}{8}\left[\left(\partial^\lambda r'^{a\mu\nu|\kappa\beta}\right)\left(\partial_\lambda r'^a_{\mu\nu|\kappa\beta}\right) + \left(\partial^\lambda r'^a\right)\left(\partial_\lambda r'^a\right)\right] \\
&\left.-\left(\partial_\mu r'^{a\mu\nu|\kappa\beta}\right)\left(\partial_\beta r'^a_{\nu\kappa}\right) + \left(\partial_\nu r'^{a\nu\beta}\right)\left(\partial^\lambda r'^a_{\lambda\beta}\right)\right\} d^6x
\end{aligned}$$

$$+\lambda \int \left[c'_a r'^a - 2 f'^A_a \varepsilon^{\lambda\mu\nu\kappa\beta\gamma} t_{A\lambda\mu\nu|\rho} \left(\partial_\sigma \partial_\kappa r'^{a}_{\beta\gamma|}{}^{\sigma\rho} - \frac{1}{2} \delta^\rho_\gamma \partial^\tau \partial_\kappa r'^a_{\beta\tau} \right) \right.$$

$$\left. -\lambda \sum_{a=1}^{n} m_a \left(5 r'^{a\lambda\rho|[\kappa\beta,\gamma]} r'^a_{\lambda\rho|[\kappa\beta,\gamma]} - 6 r'^a_{\lambda\rho|}{}^{[\kappa\beta,\rho]} r'^{a\lambda\sigma|}{}_{[\kappa\beta,\sigma]} \right) \right] d^6x, \tag{90}$$

where we made the transformations

$$r^a_{\mu\nu|\kappa\beta} \to r'^a_{\mu\nu|\kappa\beta} = \bar{O}^a{}_b r^b_{\mu\nu|\kappa\beta}, \tag{91}$$

and used the notations

$$c'_a = c_b O^b{}_a, \quad f'^A_a = f^A_b O^b{}_a. \tag{92}$$

The quantities $\bar{O}^a{}_b$ from (91) denote the elements of the inverse of $\hat{O}$. These considerations allow us to conclude that:

1. If the matrix $\hat{k}$ is positive-definite, then the symmetric matrices $\hat{M} = (M_{ab})$ and $\hat{k} = (k_{ab})$ are simultaneously diagonalizable and hence there appear no cross-couplings among different fields from the collection $\left\{ r^a_{\mu\nu|\kappa\beta} \right\}_{a=\overline{1,n}}$. Taking $\hat{k}$ to be positive-definite might be essential for the physical consistency of the theory (absence of negative-energy excitations or stability of the Minkowski vacuum);
2. If the matrix $\hat{k}$ is indefinite, then the matrices $\hat{M}$ and $\hat{k}$ cannot be diagonalized simultaneously (because then the matrix $\hat{C} = \hat{k}^{-1}\hat{M}$ is not normal [36]) and therefore there appear cross-couplings among different fields from the collection $\left\{ r^a_{\mu\nu|\kappa\beta} \right\}_{a=\overline{1,n}}$.

The gauge transformations of the deformed Lagrangian action, (87) are given by

$$\begin{aligned} \bar{\delta}_{\epsilon,\chi,\xi} t^A_{\lambda\mu\nu|\kappa} &= 3\partial_\kappa \epsilon^A_{\lambda\mu\nu} + \partial_{[\lambda}\epsilon^A_{\mu\nu]\kappa} + \partial_{[\lambda}\chi^A_{\mu\nu]|\kappa} \\ &\quad -2\lambda f^A_a \varepsilon_{\lambda\mu\nu\rho\beta\gamma} \left(\partial^\rho \xi^{a\beta\gamma|}{}_\kappa - \frac{1}{4} \delta^\gamma_\kappa \partial^{[\rho} \xi^{a\,\beta\tau]|}{}_\tau \right), \end{aligned} \tag{93}$$

$$\bar{\delta}_\xi r^a_{\mu\nu|\kappa\beta} = \partial_\mu \xi^a_{\kappa\beta|\nu} - \partial_\nu \xi^a_{\kappa\beta|\mu} + \partial_\kappa \xi^a_{\mu\nu|\beta} - \partial_\beta \xi^a_{\mu\nu|\kappa} = \delta_\xi r^a_{\mu\nu|\kappa\beta}. \tag{94}$$

It is interesting to note that only the gauge transformations of the tensor fields $(3,1)$ are modified during the deformation process. This is enforced at order one in the coupling constant by terms linear in the first-order derivatives of the gauge parameters from the $(2,2)$ sector. Only the first-order reducibility functions are modified at order one in the coupling constant, the others coinciding with the original ones. Consequently, the first-order reducibility relations corresponding to the fields $t^A_{\lambda\mu\nu|\kappa}$ take place off-shell, like the free ones, while the first-order reducibility relations associated with the fields $r^a_{\mu\nu|\kappa\beta}$ remain the original ones. The gauge algebra of the coupled model is unchanged by the deformation procedure, being the same Abelian one like for the starting free theory. Along the same line, the second-order reducibility functions remain the same, and hence the second-order reducibility relations are exactly the initial ones. It is easy to see from (87) and (93)–(94) that if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions at all.

5 Conclusion

There are three main conclusions of this paper. First, the spin-two field, described in the free limit by the Pauli-Fierz model, allows for new, consistent interactions that do not fall under the general prescriptions of General Relativity, like those with massless p-form gauge fields. Nevertheless, these new couplings still forbid the existence of more than one dual formulations in a given world, agreeing thus with the impossibility of multi-graviton theories. Second, some of the dual formulations of linearized gravity can be coupled consistently to other gauge theories or one to each other, breaking thus the common belief that they are rather rigid to couplings. Third, topological BF models, which are known to describe gravity theories (sometimes in the presence of extra constraints) can be coupled to some dual formulations of linearized gravity.

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