Physics AUC, vol. 21-Sp.Issue, 207-215 (2011) **PHYSICS AUC**

Chaos and Stabilizing Mechanisms for Yang-Mills Mechanical Models

Radu Constantinescu, Carmen Ionescu
Dept. of Theoretical Physics,
University of Craiova, 13 A. I. Cuza Str., 200585, Craiova, Romania

Abstract

The paper intends to present how gauge field theories in mechanical context can be assimilated with problems of controlling the chaotic evolution of nonlinear dynamical systems. The free Yang-Mills model is used as a model and the ghost field appearing in the $sp(2)$ BRST approach are used as control parameters. Numerical computations evidence the existence of a critic value of the control parameter for the passage from the chaos to regular dynamics. The stabilizing mechanism is similar with Higgs mechanism and mass generation in Quark Gluon Plasma.

Key words: Yang-Mills model, Quark Gluon Plasma, Higgs mechanism
Pacs: 11.10.Ef, 05.45.-a

1 Introduction

The present lecture intends to present in an unitary frame two apparently distinct topics: the gauge field techniques as BRST is and techniques for controlling the chaotic behavior of nonlinear dynamical systems. The BRST approach consists in extending the space of generators of some gauge theory with nonphysical ghost-type variables. These variables usually disappear in the asymptotic states or when a gauge fixing procedure is applied. Sometimes, the ghosts can be still existing in the theory and can appear in the computation of specific Feynman diagrams. We try to promote the idea that there are also other reasons which can make useful the presence of the ghost variables in some evolutionary equations. More concretely, we shall use the ghosts as control parameters depending on whose values the dynamical evolution of a physical system could be chaotic or regular. By that, the BRST technique is at grade with other controlling techniques, as Higgs procedure in spontaneous symmetry breaking or mass generation in Quark Gluon Plasma. We shall shortly review these mechanisms looking for specificities and similarities.

The paper is structured in two distinct parts. In the first part of the paper, a monographic one, it is presented how a gauge field theory can be transformed in a mechanical model, described by a finite number of parameters called color factors. The procedure will be exemplified on the Yang-Mills fields. The transition towards a mechanical model allows identifying solutions of the field equations, and, more general, establishing the situation when the fields have regular or chaotic behavior.

In the second part of the lecture, the three important cases of stabilizing mechanisms based either on coupling of the gauge fields with other fields or on mass generation procedure, are discussed. The three cases are: *(i)* the Higgs mechanism; *(ii)* generation of the dynamical mass in the quark-gluon plasma and *(iii)* the BRST approach. In all the cases the dynamical equations have the form of the power-law equation and the numerical computations allow to identify a critical value of a control parameter k for which the passage between chaotic and regular dynamics is done.

2 The Stochasticity of the Free Yang-Mills fields

The action which describes the non-abelian Yang-Mills fields in Minkowski space-time have the form:

$$S = \int d^4x \ \left(-\frac{1}{4} F^m_{\mu\nu} F^{\mu\nu}_m \right) \tag{1}$$

where

$$F_m^{\mu\nu} = \partial^\mu A_m^\nu - \partial^\nu A_m^\mu - g\varepsilon_{mn}^r A_r^\mu A^{\nu n} \tag{2}$$

We will consider the case when the internal degrees of freedom respect an $SU(2)$ symmetry. With Noether' s theorem, one obtain the following conserved charges:

- the energy-momentum cuadrivector:

$$P_\alpha(t) = \int d^3\mathbf{x}\ j_{0\alpha}(x) = \int d^3\mathbf{x}\ \left(-F_{0m}^i F_{\alpha i}^m + g_{0\alpha}\frac{1}{2}F_{\sigma\beta}^m F_m^{\sigma\beta} \right),\ \alpha = 0,1,2,3 \tag{3}$$

- the spin vector with the components:

$$M_i(t) = \varepsilon_{ijk} M_{jk}(t) = \frac{1}{2}\varepsilon_{ijk}(F_{0j}^m A_k^m - F_{0k}^m A_j^m). \tag{4}$$

- the internal color charge:

$$N_m(t) = \int d^3\mathbf{x}\ j_m^0(\mathbf{x},t) = \int d^3\mathbf{x}\ (D_i)_m^n F_n^{0i}. \tag{5}$$

2.1 The Euler-Lagrange equations

The Euler–Lagrange equations for our free theory have the form

$$\partial_\mu F_m^{\mu\nu} + g\varepsilon_{mn}^r A_\mu^n F_r^{\mu\nu} = 0. \tag{6}$$

Using the gauge

$$A_m^0 = 0 \tag{7}$$

we arrive to the following equations of motions

$$\ddot{A}_m^i - \partial_j F_m^{ji} + g\varepsilon_{mnr} A_j^n F_{ji}^r = 0. \tag{8}$$

The system of equations (8) has nine degrees of freedom ($m, i = 1,2,3$) and four remaining integrals: H_{YM}, $M_i = \varepsilon_{ijk}\dot{A}_m^j A_m^k$ for the $SU(2)$ group.

In the sourceless case, one obtain the constraints:

$$M_i = 0;\ N_m = 0 \tag{9}$$

which lead to the equation:

$$\dot{A}_i^m (\partial_j A_i^m - \partial_i A_j^m) = 0 \tag{10}$$

The last equation has nontrivial solution in the following cases:

(i) homogeneous vector potentials $A_i^m(\mathbf{x},t)$

$$\partial_j A_i^m(\mathbf{x},t) = 0 \tag{11}$$

(ii) static vector potentials $A_i^m(\mathbf{x},t)$

$$\dot{A}_i^m(\mathbf{x},t) = 0 \tag{12}$$

(iii) irrotational vector potentials

$$\partial_j A_k^m(\mathbf{x},t) - \partial_k A_j^m(\mathbf{x},t) = 0 \tag{13}$$

2.2 Homogeneous fields in SU(2) context

For the case of homogeneous fields in $SU(2)$ context the general theory reduces to a simple Hamiltonian system with 9 degrees of freedom. A possible choice of the vector potential is:

$$A_i^m(t) = \frac{1}{g} f^{(m)}(t) O_i^{(m)} \tag{14}$$

where the brackets symbolize the fact that no summation over these indices is implied. The matrix with elements $O_i^{(m)}$ is a constant orthogonal matrix:

$$O_i^{(m)} O_j^{(m)} = \delta_{ij} \tag{15}$$

$$O_i^{(m)} O_i^{(n)} = \delta^{mn}. \tag{16}$$

Using (14) in equations of motion (8) we obtain the following system of equations in the unknown functions $f^{(m)}(t)$ of the form

$$\ddot{f}^{(m)} + f^{(m)}(\mathbf{f}^2 - f^{(m)2}) = 0. \tag{17}$$

For $m = 1$ (unidimensional case), $f^{(1)} \equiv f$, the previous system reduces to the equation:

$$\ddot{f} = 0 \tag{18}$$

In this case, the motion of the system is periodic. Starting from $m \geq 2$ the chaotic behavior could appear. In the bi-dimensional case ($m = 2$), we can use the notations:

$$f^{(1)} = x, \ f^{(2)} = y \tag{19}$$

The system is described by the following equations:

$$\ddot{x} + xy^2 = 0 \tag{20}$$

$$\ddot{y} + x^2 y = 0 \tag{21}$$

and by the Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}x^2 y^2. \tag{22}$$

As the figure from below suggests, the $2D$- system described by the equations (20)-(21) has a chaotic behavior.

3 Stabilizing mechanisms

It is well known the fact that the chaotic behavior is merely a rule as an exception in the case of the nonlinear dynamical systems. Despite that, there are mechanisms by which this chaotic behavior can be "stabilized", some of them based on the existence of a control parameter. In this section we shall present few of these stabilizing mechanism. More precisely, the following three important stabilizing mechanism of the chaotic behavior will be presented: *(i)* the Higgs mechanism; *(ii)* the dynamical mass generation in the QGP and *(iii)* stabilizing mechanism in extended spaces by coupling the real fields with ghost-types one.

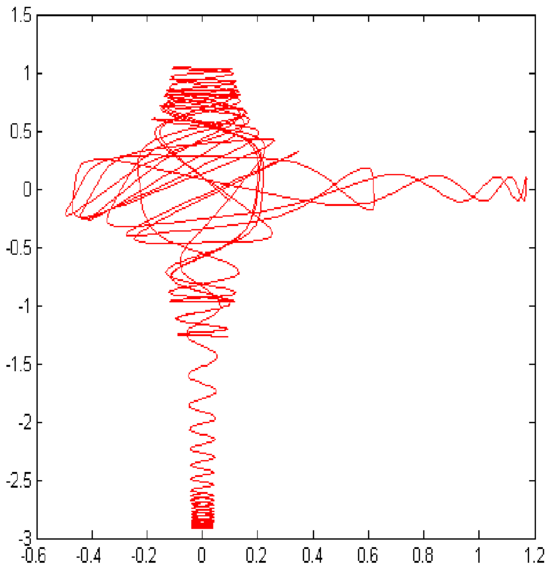

Figure 1: Chaotic behavior of the $2D$-system described by the equations (20) and (21).

3.1 The spontaneous breaking of the symmetry by Higgs mechanism

By introducing extra Higgs scalars with sufficiently large expectation values, deterministic behavior ("ordered phase") can be reach [1], [2], [3]. The mechanism consists in a symmetry breaking using in the $SU(2)$ case the iso-doublet scalar Higgs field:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} iB_1 + B_2 \\ \sigma - iB_3 - v \end{pmatrix}. \tag{23}$$

where the parameter v is related to the vacuum expectation value [4]:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -v \end{pmatrix}. \tag{24}$$

For the Yang-Mills field considered in the previous section the Higgs mechanism is applied starting from the Yang-Mills-Higgs Hamiltonian:

$$\mathcal{H} = -\frac{1}{4} F^m_{\mu\nu} F^{\mu\nu}_m + \frac{1}{2}(\dot{\sigma}^2 + \dot{B_a}^2) + \frac{g^2}{4} A^i_m A^m_i (B^2_a + (\sigma - v)^2) +$$
$$+ \frac{\lambda^2}{2}(B^2_a + (\sigma - v)^2 - v^2)^2 \tag{25}$$

where λ is a coupling constant which describe the strength of the quartic self-interaction of the Higgs field.

The generalization of the Gauss law from the non-abelian fields to our case will have the form:

$$\varepsilon_{mnr} A^n_i \dot{A}^r_i - \frac{1}{2}(\varepsilon_{mnr} B^n \dot{B}^r + \dot{\sigma} B_m - (\sigma - v)\dot{B}_m) = 0. \tag{26}$$

A possible configuration of the Higgs field which satisfy the previous constraint would be

$$\sigma = B_1 = B_2 = B_3 = 0 \tag{27}$$
$$\phi = \langle \phi \rangle = v \neq 0. \tag{28}$$

We will consider the bidimensional case $x(t) = gA_1^1(t), y(t) = gA_2^2(t)$. In this case, the Hamiltonian of the system will have the form

$$\mathcal{H} \equiv \mu^4 = \frac{1}{2g^2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2g^2}x^2y^2 + \frac{v^2}{4}(x^2 + y^2). \tag{29}$$

Performing the scale transformations $t \to \alpha t,\ x \to \beta x,\ y \to \beta y$.we obtain that the motion governed by the Hamiltonian (29) is controlled by a single dimensionless parameter

$$k = \frac{g^2}{4}\left(\frac{v}{\mu}\right)^4 \tag{30}$$

The equations of motion will have the form:

$$\ddot{x} = -4kxy^2 - 2kx \tag{31}$$
$$\ddot{y} = -4kx^2y - 2ky \tag{32}$$

The spontaneous breaking of the gauge symmetry by the Higgs mechanism has a stabilizing effect on the nonabelian gauge field dynamics. One can see in the next pictures that there is a critical value of the control parameter $k_c = 0,4$. So that, for $k > k_c$ some traces of regularity appear.

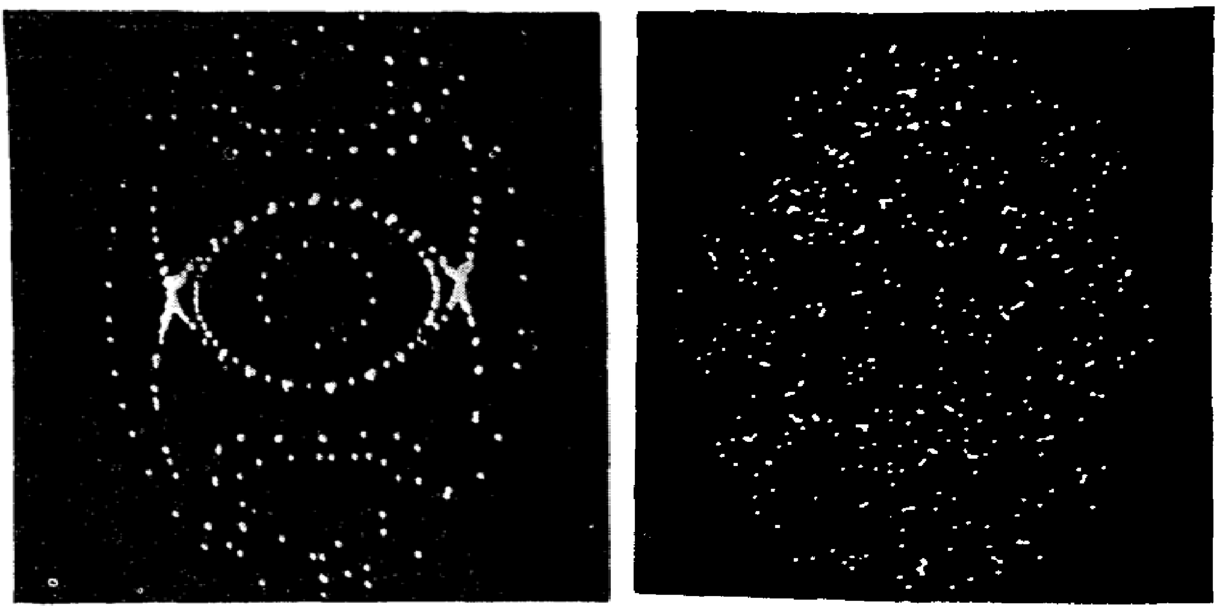

Figure 2: (*left*) $k = 4,84$ ($k_c = 0,4$), (*right*) $k = 0,35$ ($k_c = 0,4$).

3.2 Generation of the dynamical mass in the Quark -Gluon Plasma

Quark-Gluon Plasma (QGP) is a hot gas of quarks and gluons. The long wavelength excitations of a QGP are collective excitations described by nonlinear equations generalizing the classical Yang-Mills equations. The collective dynamics in the high temperature deconfined plasma is entirely described by the equations for the gauge fields:

$$igf_{mnr}A^{\nu n}F^r_{\nu\mu} = j^{ind}_{\mu m}(x) \tag{33}$$

The induced current describes the response of the plasma to the color gauge fields. It is proportional with the fluctuations in the phase-space color densities of quarks and gluons:

$$j^{ind}_m(t) = -\omega_p^2 \sum_{i=1}^{3} A^i_m(t)A^m_i(t) = -\omega_p^2 \mathbf{A}_m \tag{34}$$

This "thermal mass term" has a strong effect on the dynamics.

It is possible a choice for the non-abelian gauge fields similar as for the free Yang-Mills:

$$A_i^a(t) = \mathcal{A}_i^a h_i(t) \tag{35}$$

(no sum after i) with $\mathcal{A}_i^a$ constant functions and $h_i(t)$ arbitrary functions. We will consider the $SU(2)$ case and we assume that $\mathcal{A}_i^a = \delta_i^a$. The equations of motion for the "color factors" become:

$$\begin{aligned} \ddot{h}_1 + \omega_p^2 h_1 + g^2 h_1(h_2^2 + h_3^2) &= 0 \\ \ddot{h}_2 + \omega_p^2 h_2 + g^2 h_2(h_1^2 + h_3^2) &= 0 \\ \ddot{h}_3 + \omega_p^2 h_3 + g^2 h_3(h_1^2 + h_2^2) &= 0 \end{aligned} \tag{36}$$

The associated energy densities an integral of motion and acts as an effective Hamiltonian for the color factors:

$$H = \frac{1}{2}\sum_i \left(\dot{h}_i^{\,2} + \omega_p^2 h_i^2\right) + \frac{g^2}{2}\left(h_1^2 h_2^2 + h_1^2 h_3^2 + h_2^2 h_3^2\right) \tag{37}$$

For simplicity reasons and by doing a scale transformation, we can consider:

$$\tau \equiv \omega_p t \tag{38}$$

$$x(\tau) \equiv \frac{g}{\omega_p} h_1(t),\;\; y(\tau) \equiv \frac{g}{\omega_p} h_2(t) \equiv \frac{g}{\omega_p} h_3(t) \tag{39}$$

The mechanical equations become:

$$\begin{aligned} \ddot{x}(\tau) + x(\tau) + 2x(\tau)y^2(\tau) &= 0, \\ \ddot{y}(\tau) + y(\tau) + [x^2(\tau) + y^2(\tau)]y(\tau) &= 0. \end{aligned} \tag{40}$$

The "Hamiltonian" becomes similar with those of a system of two nonlinearly coupled harmonic oscillators:

$$H = \frac{\omega_p^4}{g^2}\frac{1}{2}\left(\dot{x}^2 + y^2 + x^2 + y^2 + x^2 y^2\right) \tag{41}$$

The equations (40) are similar with (31)-(32) and the Hamiltonian (37) can be seen as similar with (29). There is not a control parameter now, the chaos being stabilized through a mass term $\frac{\omega_p^2 h_i^2}{2}$.

3.3 The stabilizing mechanism in the frame of the BRST formalism

In the BRST approach of the gauge fields along with the real gauge fields some ghost-type variables have to be considered. In the usual interpretation the ghosts are variables without physical significance. We shall try to use these non-physical variables giving them the sense of control parameters which help to adjust the chaotic behavior. Coming back to the Yang-Mills model, the extended phase space in the $SU(2)$ gauge group is generated by the real fields and by the ghost-type variables [5]:

$$Q^A \equiv \{Q^{(2)m}, m = 1,2,3\};\;\; P_A \equiv \{P_m^{(1)}, m = 1,2,3\} \tag{42}$$

As the real fields are bosonic, $\varepsilon(A_m^\mu) = 0$, the ghost fields have to be fermions, $\varepsilon(Q^A) = \varepsilon(P_A) = 1$.

The gauge fixed action obtained using the BRST technique is [6]:

$$S_Y = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^m F_m^{\mu\nu} + p_{0m}(\partial^\mu A_\mu^m) + (\partial_\mu P_m^{(1)})(D^\mu)_n^m Q^{(2)n}\right) \tag{43}$$

The action is invariant both to the Lorentz space-time transformations and to the BRST transformations which are:

$$sA_\mu^m = (D_\mu)_n^m Q^{(2)n} \tag{44}$$

$$sQ^{(2)m} = \frac{1}{2} g\varepsilon_{nr}^m Q^{(2)r} Q^{(2)n} \tag{45}$$

3.3.1 The BRST conservative charges

The invariance leads, on the basis of the Noether theorem, to the existence of the following conservative charges:

$$P_\alpha(t) = \int d^3\mathbf{x}\ (-F^{0i}_m F^m_{\alpha i} + \delta^0_\alpha(\frac{1}{4}F^m_{\sigma\beta}F^{\sigma\beta}_m + p_{0m}(\partial^\sigma A^m_\sigma) + (\partial_\sigma P^{(1)}_m)(D^\sigma)^m_n Q^{(2)n}). \tag{46}$$

$$M_{ik}(t) = \int d^3\mathbf{x}\ (-\frac{1}{2}F^{0j}_m((\partial_i A^m_j)x_k - (\partial_k A^m_j)x_i) + \frac{1}{2}(F^m_{0i}A^m_k - F^m_{0k}A^m_i). \tag{47}$$

$$N_m = \int d^3\mathbf{x}\ \left((D_i)^n_m F^{0i}_n + g\varepsilon^{r}_{mn}\dot{P}^{(1)}_r Q^{(2)n}\right) = \int d^3\mathbf{x}\ \left(g\varepsilon^n_{mr}(\dot{A}^i_n A^r_i + \dot{P}^{(1)}_r Q^{(2)n}\right). \tag{48}$$

The energy density of the system will be:

$$H = -\frac{1}{2}F^m_{0i}F^{0i}_m + \frac{1}{4}F^m_{jk}F^{jk}_m + p_{0m}(\partial^\sigma A^m_\sigma) + (\partial_\sigma P^{(1)}_m)(D^\sigma)^m_n Q^{(2)n} \tag{49}$$

The analogue of the Gauss' law is now:

$$g\varepsilon^r_{mn}A^i_r\dot{A}^m_i = -g\varepsilon^m_{nr}\dot{P}^{(1)}_m Q^{(2)r} \tag{50}$$

3.3.2 Evolutionary Equations for Ghosts

In a homogeneous configuration of nonabelian gauge fields, when the term from the left hand side vanishes, the Gauss law takes the form:

$$g\varepsilon^m_{nr}\dot{P}^{(1)}_m Q^{(2)r} = 0 \tag{51}$$

The solution of these equations is:

$$\dot{P}^{(1)}_r = 0 \tag{52}$$

If we consider that the ghosts can be seen as real fields, their equations of motion would be:

$$\partial_\mu(D^\mu)^n_m P^{(1)}_n = 0 \tag{53}$$

which is equivalent with

$$\partial^0 P^{(1)}_m = g\varepsilon^r_{mn}A^{0n}(t)P^{(1)}_r = 0 \tag{54}$$

$$\partial^i P^{(1)}_m = g\varepsilon^r_{mn}A^{in}(t)P^{(1)}_r \tag{55}$$

We have to note that the previous equations mix the bosons $A^{\mu m}$ with the fermions $P^{(1)}_m$. These fact will disappear in the next section, where the same equations will be transformed in pure bosonic ones.

3.3.3 The BRST Mechanical Model

Let us now consider an extension of the Yang-Mills mechanical model presented in the section two from before. The extension will consist in considering the ghost field as real dynamical variables with the evolution defined in the extended phase space. Such an approach will impose to add to the representation (14) of the gauge fields some similar representations for the ghosts (42):

$$\begin{aligned} P^{(1)}_a(\mathbf{r},t) &= h(t)u^{(1)}_a(\mathbf{r}) \\ Q^{(2)a}_a(\mathbf{r},t) &= h(t)w^{(2)a}(\mathbf{r}) \end{aligned} \tag{56}$$

The temporal functions $h(t)$ are pure bosonic and take the role of the color factors, while the whole fermionic character of the ghosts is fully transferred to the functions $u_a^{(1)}(\mathbf{r})$ and $w^{(2)a}(\mathbf{r})$. These functions are chosen so that the multiplication of two functions suppress the fermionic character and, moreover, the following relations are observed:

$$(\partial_j u_a^{(1)}) w^{(2)a} = 1, (\partial_j u_a^{(1)})(\partial^j w^{(2)a}) = 0.$$

In fact, we shall not develop here this approach of ghosts as dynamical fields. Rather, we shall consider that the ghosts take the role of some supplementary parameters used in order to control the stochastic behavior. We shall consider a temporal average of the equations (56) defined as:

$$\begin{aligned} \left\langle P_a^{(1)}(\mathbf{r},t)\right\rangle &= \langle h(t)\rangle\, u_a^{(1)}(\mathbf{r}) \equiv v\ u_a^{(1)}(\mathbf{r}) \\ \left\langle Q_a^{(2)a}(\mathbf{r},t)\right\rangle &= \langle h(t)\rangle\, w^{(2)a}(\mathbf{r}) \equiv v\ w^{(2)a}(\mathbf{r}) \end{aligned} \tag{57}$$

Moreover, in order to keep down the number of the parameters, we shall consider that:

$$\left\langle P_1^{(1)}\right\rangle = \left\langle P_2^{(1)}\right\rangle = \left\langle P_3^{(1)}\right\rangle = \left\langle Q^{(2)1}\right\rangle = \left\langle Q^{(2)2}\right\rangle = \left\langle Q^{(2)3}\right\rangle \tilde{\ } v \tag{58}$$

This choice transforms the Hamiltonian (49) into:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{x^2y^2}{2} + 2g^2v^2(x^2 + y^2) \tag{59}$$

By applying the scale transformation we shall be able to have an only one dimensionless parameter which will control the dynamics:

$$x \rightarrow \alpha x,\ y \rightarrow \alpha y,\ t \rightarrow \beta t \tag{60}$$

$$H \rightarrow \mu^4 H \tag{61}$$

We define the parameter

$$k = \frac{g^2v^4}{2\mu^4} \tag{62}$$

The corresponding equation of motion are:

$$\ddot{x} = -2kxy^2 - 8kx \tag{63}$$

$$\ddot{y} = -2kx^2y - 8ky \tag{64}$$

The equations (63)-(64) have a similar form with the equations (31)-(32) obtained when the Higgs mechanism was considered. The ghost fields generated now similar control parameter as in the case of the Higgs mechanism. For different values of k the system could pass from more chaos to a quite periodic trajectory.

3.4 Concluding remarks

The main aim of our paper whose to bring a new light on the BRST formalism usually meant as a formalism which help to overcome the mathematical inconsistencies of the gauge theories. There are at least other two possible approaches: (i) the ghosts seen as true dynamical variables when their equations of motion have to be considered [7] and (ii) the approach in which the ghosts are seen as parameters which can be used in order to control the chaotic behavior of real physical systems. The second context is used in this paper and the procedure is compared with other two control procedures: Higgs mechanism and mass generation in QGP.

As a common feature of all cases presented in the previous sections, the dynamics of the attached "mechanical" models is described by nonlinear power law equations:

$$\ddot{w} + aw^{n-1} + bw^n + cw^{2n-1} = 0 \tag{65}$$

By rescaling the equation (65) the coefficients a, b and c can be expressed by a unique parameter k which can be used as a dimensionless control parameter. In the particular model we had considered the parameter k is connected with the conserved energy. There is a critical value, k_c such as for $k > k_c$ the system have a regular dynamics and for $k < k_c$ it pass to a "disordered" phase.

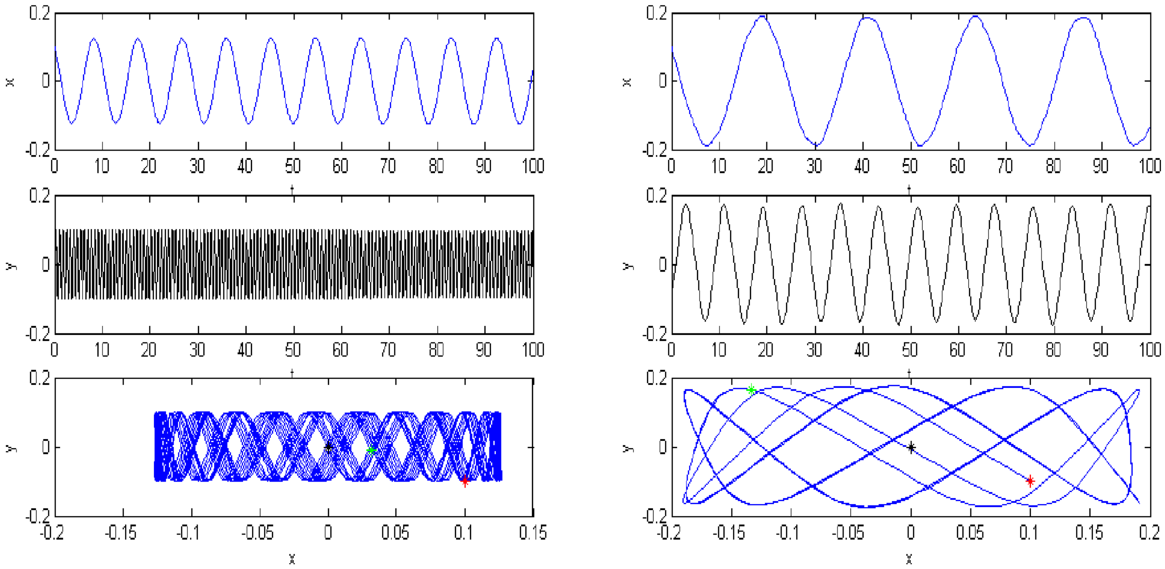

Figure 3: (*left*) k=5: chaos , (*right*) k=100: periodic orbits.

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