Lagrangians and Hamiltonian Structures and Foliations

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Abstract

The purpose of this note is to find a new criteria by which a foliation is Riemannian. We construct an integration operator and prove that the existence of a positively admissible and transverse Hamiltonian implies that the foliation is Riemannian.

Following E. Ghys’ Appendix E of [6], Miernowski and Mozgawa formulated in [4, Theorem 3.2] the following question: is any Finslerian foliation (see [3, 4, 7]) a Riemannian foliation? A partial result of the problem is given in [3], for a Finslerian foliation on a compact manifold. Using a different method, the general form of the result is proved in a Lagrangian setting in [7, Theorem 4]: a foliation that allows a positively allowed transverse Lagrangian (in particular a transverse Finslerian) is a Riemannian foliation. The main idea in the proof is averaging the transverse vertical Hessian of the Lagrangian, using a measure that in the Finslerian case is the Bussemann-Hausdorff measure (see [8, Section 5.1]). A similar idea is used also in the present paper, applied to transverse Lagrangians on transverse vector bundles. Moreover, we give some new criteria for a foliation to be a Riemannian one, as follows. We prove a dual Hamiltonian result: the existence of a transverse, allowed and positively Hamiltonian implies that the foliation is Riemannian (Proposition 7). Taking into account the fact that the Legendre duality does not assure the allowance of the dual Hamiltonian in the general case (it works only in the Finsler-Cartan case), we are compelled to make a direct proof, using similar techniques.

All the objects considered are of class $C^\infty$. We use notations and general statements on vector bundles and Lagrangians from [5]. Let $E \overset{p}{\rightarrow} M$ be a vector bundle. A positively allowed Lagrangian on $E$ is a differentiable map $L : E_\ast = E \setminus \{0\} \rightarrow \mathbb{R}$, where $\{0\}$ is the image of the null section, such that the following two conditions hold: 1) $L$ is positively defined (i.e. its basic Hessian is positively defined) and $L(x, y) \geq 0 = L(x, 0)$, $(\forall)x \in M$ and $y \in E_x = p^{-1}(x)$; 2) the Lagrangian $L$ has the property that there is a smooth function $\varphi : M \rightarrow (0, \infty)$, such that for every $x \in M$ there is $y \in E_x$ such that $L(x, y) = \varphi(x)$. If a positively Lagrangian $F$ is 2–homogeneous (i.e. $F(x, \lambda y) = \lambda^2 F(x, y)$, $(\forall)\lambda > 0$), one say that $F$ is a Finslerian; it is also a positively allowed Lagrangian, since one can take $\varphi \equiv 1$, or any positive constant.

**Proposition 1** There is an $\mathcal{F}(M)$–linear integration operator $\Phi_L : \mathcal{F}(E_\ast) \rightarrow \mathcal{F}(M)$.

**Proof.** The idea is averaging the vertical Hessian of $L$, using a measure that in the Finsler case is the Bussemann-Hausdorff measure (see [8, Section 5.1]). Let us consider a local trivialisation map $p^{-1}(U) \rightarrow U \times \mathbb{R}^k$ and local coordinates $(x^i, y^a)$ on $p^{-1}(U)$, where $(x^i)$ are local coordinates on an open subset $U \subset M$. Let us consider in $x \in U$ the compact subset $B_x = \{(y^a) \in \mathbb{R}^k : \frac{\varphi(x^i)}{x^i} \leq L(x^i, y^a) \leq \varphi(x^i)\} \subset \mathbb{R}^k$. Let us denote by $\text{vol}(B_x)$ the euclidian volume of $B_x$, according to the usual euclidian structure of $\mathbb{R}^k$. Let us suppose that the change rule of coordinates $(x^i, y^a)$, on the intersection of two domains that correspond to $U$ and $U'$ is $x'^i = x'^i(x^i)$, $y'^a = y^a y^a_{\mu} g^a_{\mu}(x^i)$. Let us denote by $J(x) = \left( g^a_{\mu}(x^i) \right)$ and consider $B'_x$, corresponding to the new coordinates. Then it is easy to see that $B'_x = J(x)B_x$, $\text{vol}(B'_x) = \text{vol}(B_x) \det J(x)$. Taking a differentiable function $f : E_\ast \rightarrow \mathbb{R}$,
The integration operator can be adapted to the transverse structure, as follows.

A positively allowed Lagrangian, asking in the second condition that \( U \subset L \) is a Riemannian foliation. A transverse Lagrangian on the foliate vector bundle \( F \) be considered when the foliated vector bundle is transversely parallelizable; if \( (\text{canonical}) \) transverse Lagrangian on \( L \) then there is an \( (\text{canonical}) \) transverse Lagrangian. Then there is an transverse bilinear form \( b \) on \( E \) which is transverse iff for any two transverse (local) sections \( s_1, s_2 \), then \( b(s_1, s_2) \) is a (local) basic function. A transverse bilinear form \( b \) gives rise to a (canonical) transverse Lagrangian on \( E \), given by the quadratic form defined by \( b \). A special case can be considered when the foliated vector bundle is transversely parallelizable: if \( \nu F \) is parallelizable, then \( F \) is a Riemannian foliation. A transverse Lagrangian on the foliate vector bundle \( E \) is a Lagrangian \( L : E_\ast \to \mathbb{R} \) such that for every foliated section \( s : U \to E \), the function \( x \to L(x, s(x)) \) is basic on \( U \). The definition of a positively allowed transverse Lagrangian is analogous to the definition of a positively allowed Lagrangian, asking in the second condition that \( \varphi \in \mathcal{F}(M) \) be a basic function. The integration operator can be adapted to the transverse structure, as follows.

**Proposition 2** Let \( E \xrightarrow{\nu} M \) be a transverse vector bundle and \( L : E_\ast \to \mathbb{R} \) be a positively allowed transverse Lagrangian. Then there is an \( \mathcal{F}(M) \)--linear integration operator \( \Phi_L : \mathcal{F}(E_\ast) \to \mathcal{F}(M) \) that sends basic functions to basic functions.

Nevertheless, we use in the sequel only basic functions. Let us consider a foliation \( \mathcal{F} \) on \( M \) and two arbitrary transverse vector bundles \( E_1 \xrightarrow{\nu_1} M \) and \( E_2 \xrightarrow{\nu_2} M \). Let us consider the induced vector bundle \( p_0 = p_0 \circ p_1 : p_1^* E_1 \to E_2; \) this is a foliated vector bundle, according to the canonical foliation \( \mathcal{F}_{E_2} \) on \( E_2 \).

**Proposition 3** Let us suppose that the vector bundle \( E_2 \) allows a positively allowed transverse Lagrangian \( L \) and the foliated vector bundle \( p_0 : p_0^* E_1 \to E_2 \ast \) allows a transverse Riemannian metric \( b \) on fibers. Then the foliated bundle \( E_1 \) allows a transverse Riemannian metric.

We remark that the condition that \( b \) be Riemannian in every point can be replaced with the condition that \( b \) be Riemannian in the points of its non-void support of a strict positive measure, of every fiber of \( E_1 \to M \). We use this weaker condition below in the proof of Proposition 2, but of sake of simplicity we do not insist on this. We consider now some special cases of Proposition 3. First, let us consider the case when the first transverse bundle \( E_1 \xrightarrow{\nu_1} M \) is the transverse vector bundle \( \nu F \xrightarrow{\nu F} M \) of the foliation \( \mathcal{F} \) and the second transverse bundle \( E_1 \xrightarrow{\nu_1} M \) is an arbitrary transverse vector bundle \( E \xrightarrow{\nu} M \). Then Proposition 3 have the following form.

**Proposition 4** Let us suppose that the vector bundle \( E \) allows a positively allowed transverse Lagrangian \( L \) and the foliated vector bundle \( p_0 : p_0^* \nu F \to E \) allows a transverse Riemannian metric \( b \) on fibers. Then the foliation \( \mathcal{F} \) is Riemannian.

\[ \text{then } \int_{B_x} f(x, y^a) dv' = (\det J(x)) \int_{B_x} f(x, y^a) dv, \]
In the second place, let us take $E_1 = E_2 = E$ and $p_1 = p_2 = p$. The vertical Hessian $H$ of a Lagrangian $L : E_* \to \mathbb{R}$ can be regarded as a bilinear form on the fibers of the vector bundle $p^*E \to E$. If the vector bundle $E$ is foliated and $L$ is a positively allowed transverse Lagrangian, then $H$ is strict positively defined and we obtain the following result.

**Proposition 5** Let us suppose that the vector bundle $E$ is foliated and $L$ is a positively allowed transverse Lagrangian on $E$. Then the vector bundle $E$ allows a transverse Riemannian metric $b$ on fibers.

In particular, let us consider the case when $E = \nu F$ is the transverse vector bundle of the foliation $\mathcal{F}$.

**Proposition 6** Let us suppose that there is a transverse allowed Lagrangian $L : \nu F_* \to \mathbb{R}$. Then the foliation $\mathcal{F}$ is a Riemannian foliation.

The above result is proved in [7, Theorem 4] and it gives a positive answer to an old question rised by E. Ghys, as stated in introduction. The particular case, when $L$ is homogeneous is known as the Finslerian case. An other particular case of Proposition 6, i.e. the case of a Hamiltonian foliation. A Lagrangian $H : (E^*)_* = E^*_* \to \mathbb{R}$ on the dual bundle $E^* \overset{\pi}{\to} M$ is called a Hamiltonian on the vector bundle $E \overset{p}{\to} M$. If $H$ is homogeneous, it is known as a Cartan map (the dual notion of a Finsler map). Let us suppose that $E \overset{\pi}{\to} M$ is a foliated vector bundle. Then its dual bundle $E^* \overset{\pi}{\to} M$ is a foliated vector bundle as well.

A positively allowed transverse Hamiltonian is, by definition, a positively allowed transverse Lagrangian on $E^*$. Let us remark that a dual positively transverse Hamiltonian can be associated with a positively transverse Lagrangian, via a Legendre map. But if the Hamiltonian is allowed (according to a basic function $\varphi'$), it seems no simple to prove the existenve of a basic function $\varphi$ for its dual Lagrangian $L$, such that $L$ be allowed. However, as in the case of a Lagrangian, we prove that a positively allowed transverse Hamiltonian gives rise also to a Riemannian foliation. It can be regarded as well as a particular case of Proposition 5, since a transverse Riemannian metric on $\nu^*F = (\nu F)^*$ is equivalent with a transverse Riemannian metric on $\nu F$.

**Proposition 7** Let us suppose that there is a transverse, positively and allowed Hamiltonian $H : \nu^*F_* \to \mathbb{R}$. Then the foliation $\mathcal{F}$ is Riemannian.

**References**


