Physics AUC, vol. 21-Sp.Issue, 203-206 (2011) **PHYSICS AUC**

Lagrangians and Hamiltonian Structures and Foliations

Paul Popescu, Marcela Popescu
Department of Applied Mathematics, University of Craiova
paul_p_popescu@ yahoo.com, marcelacpopescu@ yahoo.com

Abstract

The purpose of this note is to find a new criteria by which a foliation is Riemannian. We construct an integration operator and prove that the existence of a positively admissible and transverse Hamiltonian implies that the foliation is Riemanian.

Following E. Ghys' Appendix E of [6], Miernowski and Mozgawa formulated in [4, Theorem 3.2] the following question: is any Finslerian foliation (see [3, 4, 7]) a Riemannian foliation? A partial result of the problem is given in [3], for a Finslerian foliation on a compact manifold. Using a different method, the general form of the result is proved in a Lagrangian setting in [7, Theorem 4]: *a foliation that allows a positively allowed transverse Lagrangian (in particular a transverse Finslerian) is a Riemannian foliation.* The main idea in the proof is averaging the transverse vertical Hessian of the Lagrangian, using a measure that in the Finslerian case is the Bussemann-Hausdorff measure (see [8, Section 5.1]). A similar idea is used also in the present paper, applied to transverse Lagrangians on transverse vector bundles. Moreover, we give some new criteria for a foliation to be a Riemannian one, as follows. We prove a dual Hamiltonian result: *the existence of a transverse, allowed and positively Hamiltonian implies that the foliation is Riemanian* (Proposition 7). Taking into account the fact that the Legendre duality does not assure the allowance of the dual Hamiltonian in the general case (it works only in the Finsler-Cartan case), we are compelled to make a direct proof, using similar techniques.

All the objects considered are of class C^∞. We use notations and general statements on vector bundles and Lagrangians from [5]. Let $E \xrightarrow{p} M$ be a vector bundle. A *positively allowed Lagrangian* on E is a differentiable map $L : E_* = E\backslash\{0\} \to I\!R$, where $\{0\}$ is the image of the null section, such that the following two conditions hold: 1) L is positively defined (i.e. its basic Hessian is positively defined) and $L(x,y) \geq 0 = L(x,0)$, $(\forall)x \in M$ and $y \in E_x = p^{-1}(x)$; 2) the Lagrangian L has the property that there is a smooth function $\varphi : M \to (0,\infty)$, such that for every $x \in M$ there is $y \in E_x$ such that $L(x,y) = \varphi(x)$. If a positively Lagrangian F is 2–homogeneous (i.e. $F(x,\lambda y) = \lambda^2 F(x,y)$, $(\forall)\lambda > 0$), one say that F is a *Finslerian*; it is also a positively allowed Lagrangian, since one can take $\varphi \equiv 1$, or any positive constant.

Proposition 1 *There is an$\mathcal{F}(M)$–linear integration operator $\Phi_L : \mathcal{F}(E_*) \to \mathcal{F}(M)$.*

Proof. The ideea is averaging the vertical Hessian of L, using a measure that in the Finsler case is the Bussemann-Hausdorff measure (see [8, Section 5.1]). Let us consider a local trivialisation map $p^{-1}(U) \to U \times I\!R^k$ and local coordinates (x^i, y^a) on $p^{-1}(U)$, where (x^i) are local of coordinates on an open subset $U \subset M$. Let us consider in $x \in U$ the compact subset $B_x = \{(y^a) \in \mathbb{R}^k\colon \frac{\varphi(x^i)}{2} \leq L(x^i, y^a) \leq \varphi(x^i)\} \subset \mathbb{R}^k$. Let us denote by $vol(B_x)$ the euclidean volume of B_x, according to the usual euclidean structure of $\mathbb{R}^k$. Let us suppose that the change rule of coordinates (x^i, y^a), on the intersection of two domains that correspond to U and U' is $x^{i'} = x^{i'}(x^i)$, $y^{a'} = y^a g_a^{a'}(x^i)$. Let us denote by $J(x) = \left(g_a^{a'}(x^i)\right)$ and consider B'_x, corresponding to the new coordinates. Then it is easy to see that $B'_x = J(x) B_x$, $vol(B'_x) = vol(B_x) \det J(x)$. Taking a differentiable function $f : E_* \to \mathbb{R}$, then $\int\limits_{B'_x} f(x, y^{a'})dv' = (\det J(x)) \int\limits_{Bx} f(x, y^a)dv$, thus

$$\Phi_L(f)(x) = \frac{\int\limits_{B_x} f(x^i, y^a)dv}{vol(B_x)} \tag{1}$$

does not depend on local coordinates; $\Phi_L(f) : M \to I\!R$ is differentiable and Φ_L is $\mathcal{F}(M)$–linear. □

The next step is to consider the foliate case. This case worths to be studied as a special case, when the Lagrangian is not regular. Let $\mathcal{F}$ be a given foliation on the base M of the vector bundle $E \xrightarrow{p} M$. One say that the vector bundle is *foliated* if there is a vectorial atlas of local trivialisations on E such that all the components of the structural matrices are basic functions. Notice that a canonical foliation $\mathcal{F}_E$ on E is induced. Let us consider now a local base (s_a) of the module of sections on the restriction of an open $U \subset U_0 \subset M$, where U_0 is an open domain of a vectorial chart of the atlas. A section s on U is called *foliated* if $s = f^a s_a$ and f^a are basic functions. For example every section s_a is foliated. It is easy to see that a section s on $U \cup U'$, foliated on U and U' is also foliated on $U \cap U'$, thus the definition of a foliated section can be extended to every section on an open $U \subset M$, particulary on M.

Examples of foliated vector bundles are the transverse bundle of the foliation itself, denoted by $\nu\mathcal{F}$, and the various tensor bundles constructed using $\nu\mathcal{F}$. Analogously, given a foliated vector bundle, its tensor bundles give rise to foliated vector bundles. For example, one can consider the transverse vector bundle of bilinear forms on the fibers of E; a *transverse bilinear form* b on E is a section in this vector bundle; a bilinear forms b on the fibers of E is transverse iff for any two transverse (local) sections s_1, s_2, then $b(s_1, s_2)$ is a (local) basic function. A transverse bilinear form b gives rise to a (canonical) transverse Lagrangian on E, given by the quadratic form defined by b. A special case can be considered when the foliated vector bundle is transversely parallelizable; if $\nu\mathcal{F}$ is parallelizable, then $\mathcal{F}$ is a Riemannian foliation. A transverse Lagrangian on the foliate vector bundle E is a Lagrangian $L : E_* \to I\!R$, such that for every foliated section $s : U \to E$, the function $x \to L(x, s(x))$ is basic on U. The definition of a *positively allowed transverse Lagrangian* is analogous to the definition of a positively allowed Lagrangian, asking in the second condition that $\varphi \in \mathcal{F}(M)$ be a basic function. The integration operator can be adapted to the transverse structure, as follows.

Proposition 2 *Let $E \xrightarrow{p} M$ be a transverse vector bundle and $L : E_* \to I\!R$ be a positively allowed transverse Lagrangian. Then there is an$\mathcal{F}(M)$–linear integration operator $\bar{\Phi}_L : \mathcal{F}(E_*) \to \mathcal{F}(M)$ that sends basic functions to basic functions.*

Nevertheless, we use in the sequel only basic functions. Let us consider a foliation $\mathcal{F}$ on M and two arbitary transverse vector bundles $E_1 \xrightarrow{p_1} M$ and $E_2 \xrightarrow{p_2} M$. Let us consider the induced vector bundle $p_0 = p_2^* p_1 : p_1^* E_1 \to E_2$; this is a foliated vector bundle, according to the canonical foliation $\mathcal{F}_{E_2}$ on E_2.

Proposition 3 *Let us suppose that the vector bundle E_2 allows a positively allowed transverse Lagrangian L and the foliated vector bundle $p_0 : p_2^* E_1 \to E_{2*}$ allows a transverse Riemannian metric b on fibers. Then the foliated bundle E_1 allows a transverse Riemannian metric.*

We remark that the condition that b be Riemannian in every point can be replaced with the condition that b be Riemannian in the points of its non-void support of a strict positive measure, of every fiber of $E_1 \to M$. We use this weeker condition below in the proof of Proposition **??**, but of sake of simplicity we do not insist on this. We consider now some special cases of Proposition 3. First, let us consider the case when the first transverse bundle $E_1 \xrightarrow{p_1} M$ is the transverse vector bundle $\nu F \xrightarrow{p_F} M$ of the foliation $\mathcal{F}$ and the second transverse bundle $E_1 \xrightarrow{p_1} M$ is an arbitary transverse vector bundle $E \xrightarrow{p} M$. Then Proposition 3 have the following form.

Proposition 4 *Let us suppose that the vector bundle E allows a positively allowed transverse Lagrangian L and the foliated vector bundle $p_0 : p^* \nu F \to E$ allows a transverse Riemannian metric b on fibers. Then the foliation $\mathcal{F}$ is Riemannian.*

In the second place, let us take $E_1 = E_2 = E$ and $p_1 = p_2 = p$. The vertical Hessian H of a Lagrangian $L : E_* \to I\!R$ can be regarded as a bilinear form on the fibers of the vector bundle $p^*E \to E$. If the vector bundle E is foliated and L is a positively allowed transverse Lagrangian, then H is strict positively defined and we obtain the following result.

Proposition 5 *Let us suppose that the vector bundle E is foliated and L is a positively allowed transverse Lagrangian on E. Then the vector bundle E allows a transverse Riemannian metric b on fibers.*

In particular, let us consider the case when $E = \nu\mathcal{F}$ is the transverse vector bundle of the foliation $\mathcal{F}$.

Proposition 6 *Let us suppose that there is a transverse allowed Lagrangian $L : \nu F_* \to I\!R$. Then the foliation $\mathcal{F}$ is a Riemanian foliation.*

The above result is proved in [7, Theorem 4] and it gives a positive answer to an old question rised by E. Ghys, as stated in introduction. The particular case, when L is homogeneous is known as the Finslerian case. An other particular case of Proposition 5, gives the dual case of Proposition 6, i.e. the case of a Hamiltonian foliation. A Lagrangian $H : (E^*)_* = E^*_* \to I\!R$ on the dual bundle $E^* \xrightarrow{\pi} M$ is called a *Hamiltonian* on the vector bundle $E \xrightarrow{p} M$. If H is homogeneous, it is known as a Cartan map (the dual notion of a Finsler map). Let us suppose that $E \xrightarrow{p} M$ is a foliated vector bundle. Then its dual bundle $E^* \xrightarrow{\pi} M$ is a foliated vector bundle as well.

A *positively allowed transverse Hamiltonian* is, by definition, a positively allowed transverse Lagrangian on E^*. Let us remark that a dual positively transverse Hamiltonian can be associated with a positively transverse Lagrangian, via a Legendre map. But if the Hamiltonian is allowed (according to a basic function φ'), it seems no simple to prove the existenve of a basic function φ for its dual Lagrangian L, such that L be allowed. However, as in the case of a Lagrangian, we prove that a positively allowed transverse Hamiltonian gives rise also to a Riemannian foliation. It can be regarded as well as a particular case of Proposition 5, since a transverse Riemannian metric on $\nu^*F = (\nu F)^*$ is equivalent with a transverse Riemannian metric on νF.

Proposition 7 *Let us suppose that there is a transverse, positively and allowed Hamiltonian $H : \nu^*F_* \to I\!R$. Then the foliation $\mathcal{F}$ is Riemanian.*

References

[1] Cruceanu V., *Sur la théorie des sous-fibrés vectoriels*, C.R. Acad. Sc. Paris, Ser. I, 302 (20) (1986) 705-708.

[2] Giachetta G., Mangiarotti L., Sardanashvily G., *New Lagrangian and Hamiltonian. Methods in Field Theory*, World Scientific, Singapore, 1997.

[3] Józefowicz M., Wolak R., *Finsler foliations of compact manifolds are Riemannian*, Differential Geometry and its Applications, 26 (2) (2008) 224–226.

[4] Miernowski A., Mozgawa W., *Lift of the Finsler foliation to its normal bundle*, Differential Geometry and its Applications, 24 (2006) 209–214.

[5] Miron R., Anastasiei, M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[6] Molino P., *Riemannian foliations*, Progress in Mathematics, Vol. 73, Birhäuser, Boston, 1988.

[7] Popescu M., Popescu P., *Lagrangians adapted to submersions and foliations*, Differential Geometry and its Applications, 27 (2) (2009) 171-178.

[8] Shen Z., *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[9] Tarquini C., *Feuilletages de type fini compact*, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 209–214.