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Some Aspects Concerning the Dynamics Given by Pfaff Forms

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Abstract

The aim of the paper is to extend Lagrangian dynamics to Pfaff form dynamics, where a Pfaff form is a differential form on a tangent bundle, non necessary closed. Considering the action of a Pfaff form on curves, given by a second order Lagrangian linear in accelerations, we obtain the equations of the first and second variations, using variational methods. In the non-singular case, considered mainly in the paper, the generalized Euler-Lagrange equation is a third order differential equation. As examples, we find that the solutions of the differential equations of motion of a charge in a field and the Euler equations for the rotational dynamics of a rigid body about its center of mass can be obtained as particular solutions of suitable Pfaff forms, with non-negative second variations.

1 Introduction

The Euler-Lagrange equation of a first order Lagrangian is a well-known and widely used variational equation which arises from many problems from mathematics, mechanics, physics and other scientific fields. Its solutions are the critical curves of the action defined by the Lagrangians on curves; in the case when the Lagrangian comes from a Riemannian, a non-Riemannian or a Finslerian metric, these solutions are known as geodesics, since they locally minimise the distance. The second variation can decide if the solution is an extreme one (see [9, Ch.1, Sect.2]). The local expression of the first order Euler-Lagrange equation contains the second derivatives and, in the case of a hyperregular Lagrangian, its solutions are integral curves of a global second order differential equation.

In this paper we consider actions on curves of Pfaff forms instead of Lagrangians. Pfaff forms are differential one forms on tangent spaces of manifolds, or on open subsets of the tangent spaces. In fact, the action of a Pfaff form is the same as the action of a second order Lagrangian, linear in accelerations (see, for example, [4, 6]). In the non-singular case, the local expression of the Euler-Lagrange equation, obtained by a variational method on the action, involves the third derivatives; in the regular case the solutions are integral curves of a global third order differential equation (Proposition 3.1).

The dynamics of Pfaff forms has different behaviors. In the case when the Pfaff form comes from a non-Lagrangian, the dynamics is given by the classical Euler-Lagrange equation of a suitable first order Lagrangian. For a Pfaff form on a one dimensional manifold, the Euler-Lagrange equation (16), admits a standard Lagrangian description (Proposition 3.2). The case of a non-singular Pfaff form is mainly considered in the paper. If the Pfaff form is regular, its action on curves can be described by the action of a second order Lagrangian, linear in the second order velocities (accelerations). The Euler-Lagrange equation of a higher order Lagrangian is originary due to Ostrogradski, then used in a modern form, as in [10] or [4]. For a second order Lagrangian linear in accelerations, the Euler-Lagrange equations are called in the paper as the generalized Euler-Lagrange equations. A formula for the second variational derivative is also given.

As concrete examples, we consider the differential equations of motion of a charge in a field (formulas (17) in [3, Section 17]) and the Euler equations for the rotational dynamics of a rigid body about its center of mass [5]. We prove that there are suitable Pfaff forms in each case, such that the solutions of the considered differential equations are also extremal solutions for the generalized Euler-Lagrange equations of the Pfaff forms, i.e. the second variation has a constant sign along these solutions (Propositions 3.4 and 3.6).

2 Actions of Pfaff forms on curves

Let $L : I\!R \times TM \rightarrow I\!R$ be a (time dependent) Lagrangian. If $\gamma : [a, b] \rightarrow M$ is a curve $\gamma(t) = (t, x^i(t))$, then the action of L on γ is

$$I(\gamma) = \int_a^b L\left(t, x^i(t), \frac{dx^i}{dt}(t)\right) dt. \tag{1}$$

Integrating by parts, one obtain
$I(\gamma) = bL\left(b, x^i(b), \frac{dx^i}{dt}(b)\right) - aL\left(a, x^i(a), \frac{dx^i}{dt}(a)\right) -$
$\int_a^b t\left[\frac{\partial L}{\partial t}\left(t, x^i, \frac{dx^i}{dt}\right) + \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial L}{\partial y^i}\frac{dy^i}{dt}\right] dt$

A Pfaff form is a differentiable form $\omega \in \mathcal{X}^*(I\!R \times TM)$, $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$.

Let us consider a Lagrangian $L : I\!R \times TM \rightarrow I\!R$ and a Pfaff form ω. The action of ω and L on a curve $\gamma : [a, b] \rightarrow M$ can be defined as

$$\begin{aligned} I_1(\gamma) &= bL(b, x^i(b), \frac{dx^i}{dt}(b)) - aL\left(a, x^i(a), \frac{dx^i}{dt}(a)\right) \\ &- \int_a^b t(\omega_0 + \omega_i\frac{dx^i}{dt} + \bar{\omega}_i\frac{d^2x^i}{dt^2})dt. \end{aligned} \tag{2}$$

Let us compute the action (1) in a different way as above. One have
$I(\gamma) = \int_a^b d\tau\left(\int_a^\tau \frac{d}{dt}L\left(t, x^i, \frac{dx^i}{dt}\right)dt + L\left(a, x^i(a), \frac{dx^i}{dt}(a)\right)\right) =$
$\int_a^b (b-t)\left[\frac{\partial L}{\partial t}\left(t, x^i, \frac{dx^i}{dt}\right) + \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial L}{\partial y^i}\frac{d^2x^i}{dt^2}\right]dt + (b-a)L\left(a, x^i(a), \frac{dx^i}{dt}(a)\right)$.

Given a Pfaff form ω and a Lagrangian L, one can consider a new action I_2 of ω and L on a curve γ by

$$\begin{aligned} I_2(\gamma) = &\int_a^b (b-t)(\omega_0 + \omega_i\frac{dx^i}{dt} + \bar{\omega}_i\frac{d^2x^i}{dt^2})dt + \\ &(b-a)L\left(a, x^i(a), \frac{dx^i}{dt}(a)\right). \end{aligned} \tag{3}$$

The actions (2) and (3) have the form

$$\begin{aligned} I_3(\gamma) = &\int_a^b \varphi(t)(\omega_0 + \omega_i\frac{dx^i}{dt} + \bar{\omega}_i\frac{d^2x^i}{dt^2})dt + \\ &\alpha L\left(b, x^i(b), \frac{dx^i}{dt}(b)\right) - \beta L\left(a, x^i(a), \frac{dx^i}{dt}(a)\right), \end{aligned} \tag{4}$$

where α, $\beta \in I\!R$. The above formula suggests to consider the most general action determined by a Pfaff form ω, a Lagrangian L and α, $\beta \in I\!R$, as follows

$$\begin{aligned} I_0(\gamma) = &\int_a^b (\omega_0 + \omega_i\frac{dx^i}{dt} + \bar{\omega}_i\frac{d^2x^i}{dt^2})dt + \\ &\alpha L\left(b, x^i(b), \frac{dx^i}{dt}(b)\right) - \beta L\left(a, x^i(a), \frac{dx^i}{dt}(a)\right), \end{aligned} \tag{5}$$

If $\omega_i = \bar{\omega}_i \equiv 0$, $\alpha = \beta = 0$, $\omega_0 = L$, then we obtain $I_0 = I$ for the Lagrangian L.

Denoting $L'(t, x^i, y^{(1)i}, y^{(2)i}) = \omega_0(t, x^i, y^{(1)i}) + \omega_i(t, x^i, y^{(1)i})y^{(1)i} + \bar{\omega}_i(t, x^i, y^{(1)i})y^{(2)i}$, we obtain that I_0 is in fact the variation of a second order Lagrangian, affine in the second order velocities (see [4, 6]).

Let us consider two points x, $y \in M$ and $\gamma_0 = (x_0^i(t))$ a curve joining x and y, i.e. $x_0^i(0) = x$ and $x_0^i(1) = y$. Let us consider two classes of variations of γ_0, as curves joining x and y, locally given by $\gamma_\varepsilon = (x_\varepsilon^i(t))$, where $x_\varepsilon^i(t) = x_0^i(t) + \varepsilon h^i(t)$.

We say that a variation is:

– in the *first class* if the following condition holds:

$$h^i(a) = h^i(b) = 0; \tag{6}$$

– in the *second class* if the conditions (6) and

$$\frac{dh^i}{dt}(a) = \frac{dh^i}{dt}(b) = 0 \tag{7}$$

hold.

Obviously, a variation in the second class is also in the first class, i.e. a variation in the second class is more restrictive that a variation in the first class. We say that a variation is *allowed*, if it belongs to one of the above two classes. We consider below these variations according to some properties of the Pfaff form ω.

3 First and second derivatives of a variation in the Pfaff form case

The first variational derivative of the action I_0 has the following form:

$\frac{d}{d\varepsilon} I_0 (\gamma_\varepsilon) |_{\varepsilon=0} =$
$\alpha h^i(b) \frac{\partial L}{\partial x^i}(b, x_0^i(b), \frac{dx^i}{dt}(b)) + \alpha \frac{dh^i}{dt}(b) \frac{\partial L}{\partial y^i}(b, x_0^i(b), \frac{dx^i}{dt}(b)) -$
$\beta h^i(a) \frac{\partial L}{\partial x^i}(a, x_0^i(a), \frac{dx^i}{dt}(a)) - \beta \frac{dh^i}{dt}(a) \frac{\partial L}{\partial y^i}(a, x_0^i(a), \frac{dx^i}{dt}(a)) +$
$\int_a^b (\frac{\partial \omega_0}{\partial x^i} h^i + \frac{\partial \omega_0}{\partial y^i} \frac{dh^i}{dt}) dt + \int_a^b (\frac{\partial \omega_j}{\partial x^i} h^i + \frac{\partial \omega_j}{\partial y^i} \frac{dh^i}{dt}) \frac{dx_0^j}{dt} dt +$
$\int_a^b \omega_i \frac{dh^i}{dt} dt + \int_a^b (\frac{\partial \bar{\omega}_j}{\partial x^i} h^i + \frac{\partial \bar{\omega}_j}{\partial y^i} \frac{dh^i}{dt}) \frac{d^2 x_0^j}{dt^2} dt + \int_a^b \bar{\omega}_i \frac{d^2 h^i}{dt^2} dt.$

3.1 The case of non-Lagrangian systems

A Pfaff form $\omega \in \mathcal{X}^*(I\!R \times TM)$, $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$ is *singular* if its top component $\bar{\omega}_i dy^i$, viewed as a vertical form, is closed, i.e. $\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} = 0$.

A *non-Lagrangian system* is given by a Pfaff form ω for which there is a Lagrangian $L : I\!R \times TM \to I\!R$ such that $\omega - dL = \mu_0 dt + \mu_i dx^i$, thus $\omega_0 = \frac{\partial L}{\partial t} + \mu_0$, $\omega_i = \frac{\partial L}{\partial x^i} + \mu_i$ and $\bar{\omega}_i = \frac{\partial L}{\partial y^i}$. One can relax the above condition asking that $\omega - \tilde{\omega} = \mu_0 dt + \mu_i dx^i$, where $\tilde{\omega}$ is closed; but since every closed form is locally exact (local Poincaré Lemma), one can suppose, for brevity, that $\tilde{\omega}$ is exact.

For example, the Pfaff form $\omega = Ldt$, associated with a non-constant Lagrangian L, defines a non-Lagrangian system.

The action I_1, associated with the Pfaff form $\omega = dL + \mu_0 dt + \mu_i dx^i$ and the Lagrangian L, has the form

$$I_1 (\gamma) = \int_a^b (L - t\mu_0 - t\mu_i \frac{dx^i}{dt}) dt. \tag{8}$$

It is easy to see that this action corresponding to the action I, given by (1), is associated with the new Lagrangian $L'(t, x^i, y^i) = L(t, x^i, y^i) - \mu_0(t, x^j, y^j) - y^i \mu_i(t, x^j, y^j)$.

The action I_2, associated with the Pfaff form $\omega = dL + \mu_0 dt + \mu_i dx^i$ and the Lagrangian L, has the form

$$I_2 (\gamma) = \int_a^b (L - (b-t)\mu_0 - (b-t)\mu_i \frac{dx^i}{dt}) dt. \tag{9}$$

This action is the same as the action I given by (1) and associated with the Lagrangian $L'' = L + (b-t)\mu_0 + (b-t)\mu_i y^i$. We extend below this fact.

The action I_3, associated with the real function φ, the Pfaff form $\omega = dL + \mu_0 dt + \mu_i dx^i$, the Lagrangian L and $\alpha = -\varphi(b)$, $\beta = \varphi(a)$, has the form

$$I_3 (\gamma) = \int_a^b (-\varphi' L + \varphi \mu_0 + \varphi \mu_i \frac{dx^i}{dt}) dt. \tag{10}$$

This is the action corresponding to the action I, given by (1), associated with the new Lagrangian

$$\tilde{L}(t, x^i, y^i) = -\varphi'(t)L(t, x^i, y^i) + \varphi(t)\mu_0(t, x^j, y^j) + \varphi(t)\mu_i(t, x^j, y^j)y^i. \tag{11}$$

The Euler-Lagrange equation of $\tilde{L}$ is

$$-\varphi'\frac{\partial L}{\partial x^i} + \varphi\frac{\partial \mu_0}{\partial x^i} + \varphi\frac{\partial \mu_j}{\partial x^i}y^j - \frac{d}{dt}(-\varphi'\frac{\partial L}{\partial y^i} + \varphi\frac{\partial \mu_0}{\partial y^i} + \varphi\frac{\partial \mu_j}{\partial y^i}y^j + \varphi\mu_i) = 0,$$

or

$$\left(\varphi'\frac{\partial L}{\partial x^i} - \frac{d}{dt}(\varphi'\frac{\partial L}{\partial y^i})\right) = \varphi\frac{\partial \mu_0}{\partial x^i} + \varphi\frac{\partial \mu_j}{\partial x^i}y^j - \frac{d}{dt}(\varphi\frac{\partial \mu_0}{\partial y^i} + \varphi\frac{\partial \mu_j}{\partial y^i}y^j + \varphi\mu_i). \tag{12}$$

The Euler-Lagrange equation of the Lagrangian $\tilde{L}$ is obtained using variations in the first class, subject to conditions (2).

For $\varphi = 1$, $\alpha = -1$ and $\beta = 1$ the action (5) is the usual action on the Lagrangian $\tilde{L}(t, x^i, y^i) = \mu_0(t, x^j, y^j) + \mu_i(t, x^j, y^j)y^i$. The Euler-Lagrange equation (12) has the form

$$\frac{\partial \mu_0}{\partial x^i} + \frac{\partial \mu_j}{\partial x^i}y^j - \frac{d}{dt}(\frac{\partial \mu_0}{\partial y^i} + \frac{\partial \mu_j}{\partial y^i}y^j + \mu_i) = 0. \tag{13}$$

We can say that a non-Lagrangian system defined by the Pfaff form $\omega = dL + \mu_0 dt + \mu_i dx^i$ is *regular* if the Lagrangian $\tilde{L} = \mu_0 + \mu_i y^i$ is regular. It is easy to see that the the solutions of its Euler-Lagrange equation does not depend on L. Therefore, for L to influence the solutions of the Euler-Lagrange equation of action, one can consider actions when φ is not constant.

Since the actions of non-Lagrangian systems that we considered are reduced only to usual actions of suitable Lagrangians, in the next sections we focus on actions of non-singular Pfaff forms.

3.2 The case of non-singular Pfaff forms

A Pfaff form ω given locally by $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$ is *regular* if the vertical 2-form $(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j})\, dy^i \wedge dy^j$ is regular, i.e. the matrix $(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j})$ is non-singular. If the vertical 2-form $(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j})dy^i \wedge dy^j$ does not vanish, i.e. the matrix $(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j})$ is only non-null, we say that the Pfaff form is *non-singular*.

Let us consider now a variation in the second class. Taking into account of the conditions (6) and (7), we have:

$\frac{d}{d\varepsilon}I_0(\gamma_\varepsilon)|_{\varepsilon=0} = \int_a^b(\frac{\partial \omega_0}{\partial x^i}h^i + \frac{\partial \omega_0}{\partial y^i}\frac{dh^i}{dt})dt + \int_a^b(\frac{\partial \omega_j}{\partial x^i}h^i + \frac{\partial \omega_j}{\partial y^i}\frac{dh^i}{dt})\frac{dx_0^j}{dt}dt +$
$\int_a^b \omega_i\frac{dh^i}{dt}dt + \int_a^b(\frac{\partial \bar{\omega}_j}{\partial x^i}h^i + \frac{\partial \bar{\omega}_j}{\partial y^i}\frac{dh^i}{dt})\frac{d^2x_0^j}{dt^2}dt + \int_a^b \bar{\omega}_i\frac{d^2h^i}{dt^2}dt =$
$\int_a^b(\frac{\partial \omega_0}{\partial x^i} - \frac{d}{dt}\frac{\partial \omega_0}{\partial y^i})h^i dt + \int_a^b(\frac{\partial \omega_j}{\partial x^i}\frac{dx_0^j}{dt} - \frac{d}{dt}(\frac{\partial \omega_j}{\partial y^i}\frac{dx_0^j}{dt}))h^i dt -$
$\int_a^b \frac{d}{dt}\omega_i h^i dt + \int_a^b(\frac{\partial \bar{\omega}_j}{\partial x^i}\frac{d^2x_0^j}{dt^2} - \frac{d}{dt}(\frac{\partial \bar{\omega}_j}{\partial y^i}\frac{d^2x_0^j}{dt^2}))h^i dt + \int_a^b \frac{d^2}{dt^2}\bar{\omega}_i h^i dt.$

Thus we obtain:

$$\begin{aligned}&\frac{\partial \omega_0}{\partial x^i} - \frac{d}{dt}\frac{\partial \omega_0}{\partial y^i} + (\frac{\partial \omega_j}{\partial x^i}\frac{dx_0^j}{dt} - \frac{d}{dt}(\frac{\partial \omega_j}{\partial y^i}\frac{dx_0^j}{dt})) - \frac{d}{dt}\omega_i +\\ &\frac{\partial \bar{\omega}_j}{\partial x^i}\frac{d^2x_0^j}{dt^2} - \frac{d}{dt}(\frac{\partial \bar{\omega}_j}{\partial y^i}\frac{d^2x_0^j}{dt^2}) + \frac{d^2}{dt^2}\bar{\omega}_i = 0.\end{aligned} \tag{14}$$

or

$$\frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i}\frac{dx_0^j}{dt} + \frac{\partial \bar{\omega}_j}{\partial x^i}\frac{d^2x_0^j}{dt^2} - \frac{d}{dt}(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i}\frac{dx_0^j}{dt} + \omega_i + \frac{\partial \bar{\omega}_j}{\partial y^i}\frac{d^2x_0^j}{dt^2}) + \frac{d^2}{dt^2}\bar{\omega}_i = 0. \tag{15}$$

The above equations become

$\frac{\partial \omega_0}{\partial x^i} - \frac{\partial^2 \omega_0}{\partial t \partial y^i} - \frac{\partial^2 \omega_0}{\partial x^j \partial y^i}\frac{dx_0^j}{dt} - \frac{\partial^2 \omega_0}{\partial y^j \partial y^i}\frac{d^2x_0^j}{dt^2} + (\frac{\partial \omega_j}{\partial x^i} - \frac{\partial^2 \omega_j}{\partial t \partial y^i} - \frac{\partial \omega_j}{\partial x^j \partial y^i} -$
$\frac{\partial \omega_j}{\partial y^j \partial y^i})\frac{dx_0^j}{dt} - \frac{\partial \omega_j}{\partial y^i}\frac{d^2x_0^j}{dt^2} - \frac{\partial \omega_i}{\partial t} - \frac{\partial \omega_i}{\partial x^j}\frac{dx_0^j}{dt} - \frac{\partial \omega_i}{\partial y^j}\frac{d^2x_0^j}{dt^2} + (\frac{\partial \bar{\omega}_j}{\partial x^i} -$

$\frac{\partial^2\bar\omega_j}{\partial t\partial y^i} - \frac{\partial\bar\omega_j}{\partial x^k\partial y^i}\frac{dx_0^k}{dt} - \frac{\partial\bar\omega_j}{\partial y^k\partial y^i}\frac{d^2x_0^k}{dt^2})\frac{d^2x_0^j}{dt^2} - \frac{\partial\bar\omega_j}{\partial y^i}\frac{d^3x_0^j}{dt^3} + \frac{\partial^2\bar\omega_i}{\partial t^2} +$
$\frac{\partial^2\bar\omega_i}{\partial x^j\partial t}\frac{dx_0^j}{dt} + \frac{\partial^2\bar\omega_i}{\partial y^j\partial t}\frac{d^2x_0^j}{dt^2} + (\frac{\partial^2\bar\omega_i}{\partial t\partial x^j} + \frac{\partial^2\bar\omega_i}{\partial x^k\partial x^j}\frac{dx_0^k}{dt} + \frac{\partial^2\bar\omega_i}{\partial y^k\partial x^j}\frac{d^2x_0^k}{dt^2})\frac{dx_0^j}{dt} +$
$\frac{\partial\bar\omega_i}{\partial x^j}\frac{d^2x_0^j}{dt^2} + (\frac{\partial^2\bar\omega_i}{\partial t\partial y^j} + \frac{\partial^2\bar\omega_i}{\partial x^k\partial y^j}\frac{dx_0^k}{dt} + \frac{\partial^2\bar\omega_i}{\partial y^k\partial y^j}\frac{d^2x_0^k}{dt^2})\frac{d^2x_0^j}{dt^2} + \frac{\partial\bar\omega_i}{\partial y^j}\frac{d^3x_0^j}{dt^3} = 0.$

Thus if the Pfaff form ω is non-singular, then the equation is of third order. For a regular Pfaff form one can prove the following result.

Proposition 3.1 *If the Pfaff form ω is regular, then the solutions of the generalized Euler-Lagrange equation (15) are exactly the solutions of a third order equation given by a global second order semi-spray $S : T^2M \to T^3M$.*

Some important class of Pfaff forms are:

– when $\omega_i = \bar\omega_i = 0$, then $\omega = \omega_0 dt$, we recover the classical action of a Lagrangian ω_0.

– when $\omega_0 = 0$; for example, this is the case of time independent Lagrangians $L = L(x^i, y^i)$, since $\omega_0 = \frac{\partial L}{\partial t}$;

– when $\omega_0 = \omega_i = 0$; for example, this is the case of Lagrangians that depend only on direction: $L = L(y^i)$.

If $\omega = \bar\omega_j(y^i)dy^j$, then the equation (14) has the form $\frac{d}{dt}(\frac{\partial\bar\omega_j}{\partial y^i}\frac{d^2x_0^j}{dt^2}) - \frac{d^2}{dt^2}\bar\omega_i = 0$, or $\frac{\partial\bar\omega_j}{\partial y^i}\frac{d^2x_0^j}{dt^2} - \frac{d}{dt}\bar\omega_i = c_i$ $\Leftrightarrow (\frac{\partial\bar\omega_j}{\partial y^i} - \frac{\partial\bar\omega_j}{\partial y^i})\frac{d^2x_0^j}{dt^2} = c_i$.

Example 1. In $I\!R^2$, let us consider coordinates (x, y) on $I\!R^2$ and (x, y, X, Y) on $I\!R^4 = TI\!R^2$. Let $\omega = YdX - XdY$. The equations (14) have the form: $-\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) - \frac{d^2}{dt^2}\left(\frac{dy}{dt}\right) = 0$, or $\frac{d^3y}{d^3t} = 0$, and $\frac{d}{dt}\left(\frac{d^2x}{dt^2}\right) + \frac{d^2}{dt^2}\left(\frac{dx}{dt}\right) = 0$, or $\frac{d^3x}{d^3t} = 0$. Exact solution is: $x(t) = C_1 + C_2t + C_3t^2$, $y(t) = C_4 + C_5t + C_6t^2$.

Example 2. In $I\!R^2$, let us consider coordinates (x, y) on $I\!R^2$ and (x, y, X, Y) on $I\!R^4 = TI\!R^2$. Let $\omega = -ydx + xdy + YdX - XdY$. The equations (14) have the form $\frac{\partial\omega_j}{\partial x^i}\frac{dx_0^j}{dt} - \frac{d}{dt}(\omega_i + \frac{\partial\bar\omega_j}{\partial y^i}\frac{d^2x_0^j}{dt^2}) + \frac{d^2}{dt^2}\bar\omega_i = 0$.

For $j = 1$, $\frac{dy}{dt} - \frac{d}{dt}(-y - \frac{d^2y}{dt^2}) + \frac{d^2}{dt^2}\left(\frac{dy}{dt}\right) = 0$, or $\frac{dy}{dt} + \frac{d^3y}{d^3t} = 0$ and

For $j = 2$, $-\frac{dx}{dt} - \frac{d}{dt}(x + \frac{d^2x}{dt^2}) - \frac{d^2}{dt^2}\left(\frac{dx}{dt}\right) = 0$, or $\frac{dx}{dt} + \frac{d^3x}{d^3t} = 0$.

The general solution is $x(t) = c_1\cos t + c_3\sin t + c_5$, $x(t) = c_2\cos t + c_4\sin t + c_6$. The integral curves are ellipses and straight lines. If $t_1 < t_2 < t_3$ are given, then for every 3 distinct points $A_\alpha(x_\alpha, y_\alpha) \in I\!R^2$, $\alpha = \overline{1,3}$, there is only one integral curve in the family passing through these points, i.e. $t \to (x(t), y(t))$, $x(t_\alpha) = x_\alpha$, $y(t_\alpha) = y_\alpha$, $\alpha = \overline{1,3}$.

This feature characterizes the dynamics generated by a third order differential equation, when an integral curve is determinated in general by three distinct points. Analogously, an integral curve of a second order differential equation is generally determinated by two distinct points.

Let us consider now the case $\dim M = 1$. In this case, since the only sqew-symmetric matrix of first order is the null matrix, the equation (14) is always of second order, for every Pfaff form $\tilde\omega = \omega_0 dt + \omega dx + \bar\omega dy$, having the form

$(\frac{\partial^2\bar\omega}{\partial t\partial y} - 2\frac{\partial\omega}{\partial y} + 2\frac{\partial\bar\omega}{\partial x})\frac{d^2x_0}{dt^2} + \frac{\partial^2\bar\omega}{\partial x^2}(\frac{dx_0}{dt})^2 + (-\frac{\partial^2\omega}{\partial t\partial y} - \frac{\partial^2\omega}{\partial x\partial y} - \frac{\partial\omega}{\partial y^2} + 2\frac{\partial^2\bar\omega}{\partial x\partial t})\frac{dx_0}{dt} + \frac{\partial\omega_0}{\partial x} - \frac{\partial^2\omega_0}{\partial t\partial y} - \frac{\partial\omega}{\partial t} + \frac{\partial^2\bar\omega}{\partial t^2} = 0.$

In the case when the local functions ω_0, ω and $\bar\omega$ does not depend on y, the above equation becomes

$$2\frac{\partial\bar\omega}{\partial x}\frac{d^2x_0}{dt^2} + \frac{\partial^2\bar\omega}{\partial x^2}\left(\frac{dx_0}{dt}\right)^2 + 2\frac{\partial^2\bar\omega}{\partial x\partial t}\frac{dx_0}{dt} + \frac{\partial\omega_0}{\partial x} - \frac{\partial\omega}{\partial t} + \frac{\partial^2\bar\omega}{\partial t^2} = 0. \tag{16}$$

According to [1, Section 2.], a *standard* Lagrangian has the form

$$L(t, x, y) = \frac{1}{2}P(t, x)y^2 + Q(t, x)y + R(t, x). \tag{17}$$

The following result can be proved by a straightforward computation using [1].

Proposition 3.2 *The equation (16) admits a standard Lagrangian description.*

3.3 The second derivative of the variation

We continue with the second derivative of $I(\gamma_\varepsilon)$.

Taking into account of the conditions (6) and (7), one obtain:

$\frac{d^2}{d\varepsilon^2} I_0(\gamma_\varepsilon)|_{\varepsilon=0} =$
$\frac{1}{2}\int_a^b (\frac{\partial^2\omega_0}{\partial x^i \partial x^j} h^i h^j - \frac{d}{dt}(\frac{\partial^2\omega_0}{\partial x^i \partial y^j} + \frac{\partial^2\omega_0}{\partial y^i \partial x^j}) h^i h^j + \frac{\partial^2\omega_0}{\partial y^i \partial y^j}\frac{dh^i}{dt}\frac{dh^j}{dt})dt +$
$\frac{1}{2}\int_a^b (\frac{\partial^2\omega_k}{\partial x^i \partial x^j}\frac{dx_0^k}{dt} h^i h^j - \frac{d}{dt}((\frac{\partial^2\omega_k}{\partial x^i \partial y^j} + \frac{\partial^2\omega_k}{\partial y^i \partial x^j})\frac{dx_0^k}{dt}) h^i h^j + \frac{\partial^2\omega_k}{\partial y^i \partial y^j}\frac{dx_0^k}{dt}\frac{dh^i}{dt}\frac{dh^j}{dt})dt +$
$\frac{1}{2}\int_a^b (-\frac{d}{dt}(\frac{\partial\omega_j}{\partial x^i} + \frac{\partial\omega_i}{\partial x^j}) h^i h^j + (\frac{\partial\omega_j}{\partial y^i} + \frac{\partial\omega_i}{\partial y^j})\frac{dh^i}{dt}\frac{dh^j}{dt})dt +$
$\frac{1}{2}\int_a^b (\frac{\partial^2\bar\omega_k}{\partial x^i \partial x^j}\frac{d^2x_0^k}{dt^2} h^i h^j - \frac{d}{dt}((\frac{\partial^2\bar\omega_k}{\partial x^i \partial y^j} + \frac{\partial^2\bar\omega_k}{\partial y^i \partial x^j})\frac{d^2x_0^k}{dt^2}) h^i h^j + \frac{\partial^2\bar\omega_k}{\partial y^i \partial y^j}\frac{d^2x_0^k}{dt^2}\frac{dh^i}{dt}\frac{dh^j}{dt})dt +$
$\frac{1}{2}\int_a^b (-\frac{d^2}{dt^2}(\frac{\partial\bar\omega_j}{\partial x^i} + \frac{\partial\bar\omega_i}{\partial x^j}) h^i h^j - 2(\frac{\partial\bar\omega_j}{\partial x^i} + \frac{\partial\bar\omega_i}{\partial x^j})\frac{dh^i}{dt}\frac{dh^j}{dt} - \frac{d}{dt}(\frac{\partial\bar\omega_j}{\partial y^i} + \frac{\partial\bar\omega_i}{\partial y^j})\frac{dh^i}{dt}\frac{dh^j}{dt})dt$

3.4 Some examples

If $\omega = \bar\omega_j(y^i)dy^j$, then the equation (14) becomes $\frac{d}{dt}(\frac{\partial\bar\omega_j}{\partial y^i}\frac{d^2x_0^j}{dt^2}) - \frac{d^2}{dt^2}(\bar\omega_i)=0$.

We have: $\frac{\partial\bar\omega_j}{\partial y^i}\frac{d^2x_0^j}{dt^2} - \frac{d}{dt}\bar\omega_i = c_i$, or

$$\left(\frac{\partial\bar\omega_j}{\partial y^i} - \frac{\partial\bar\omega_i}{\partial y^j}\right)\frac{d^2x_0^j}{dt^2} = c_i. \tag{18}$$

In this case, the second derivative of the variation is

$$\begin{aligned}\frac{d^2}{d\varepsilon^2} I_0(\gamma_\varepsilon)|_{\varepsilon=0} &= \left(\frac{\partial^2\bar\omega_k}{\partial y^i \partial y^j}\frac{d^2x_0^k}{dt^2} - \frac{d}{dt}(\frac{\partial\bar\omega_i}{\partial y^j} + \frac{\partial\bar\omega_j}{\partial y^i})\right)\frac{dh^i}{dt}\frac{dh^j}{dt} \\ &= (\frac{\partial^2\bar\omega_k}{\partial y^i \partial y^j} - \frac{\partial^2\bar\omega_i}{\partial y^k \partial y^j} - \frac{\partial^2\bar\omega_j}{\partial y^k \partial y^i})\frac{d^2x_0^k}{dt^2}\frac{dh^i}{dt}\frac{dh^j}{dt}.\end{aligned}$$

The matrix of the quadratic form is

$$\left(\left(\frac{\partial^2\bar\omega_k}{\partial y^i \partial y^j} - \frac{\partial^2\bar\omega_i}{\partial y^k \partial y^j} - \frac{\partial^2\bar\omega_j}{\partial y^k \partial y^i}\right)\frac{d^2x_0^k}{dt^2}\right)_{i,j=\overline{1,m}}, \tag{19}$$

where $m = \dim M$.

Let us consider now two examples. Even the equation of motions in the examples has the second order, their integral curves are obtained from some suitable equations of Pfaff forms.

First, we consider a system that has the form

$$\begin{cases} x'' = c_1 + cy' - bz' \\ y'' = c_2 + az' - cx' \\ z'' = c_3 + bx' - ay' \end{cases} \tag{20}$$

where the coefficients are constants. The equations of motion of a charge in a field (formulas (17) in [3, Section 17]) have this form.

Proposition 3.3 *There is a Pfaff form $\omega = \bar\omega_j(y^i)dy^j$ on $I\!R^3$ such that the solutions of the system (20) are solutions of the generalized Euler-Lagrange equation (18).*

Some Pfaff forms with the property asked in Proposition 3.3 are

$$\begin{aligned}\omega_1(y^j) &= c_3y^2 + by^1y^2 + cy^1y^3, \\ \omega_2(y^j) &= c_1y^3 + ay^1y^2 + cy^2y^3, \\ \omega_3(y^j) &= -c_2y^1 + ay^1y^3 + by^2y^3.\end{aligned}$$

Concerning the second derivative of the variation, we have the following result.

Proposition 3.4 *Let us consider the system (20).*

Then there is a Pfaff form $\omega = \bar{\omega}_j(y^i)dy^j$ *defined for* $(y^i) \in I\!R^3$, *such that the solutions of the system (20) are extremal solutions for the generalized Euler-Lagrange equation (18), i.e. the second variation has a constant sign along these solutions.*

An other example is constructed using the Euler equations for the rotational dynamics of a rigid body about its center of mass, as follows.

Let us consider a system of the form

$$\begin{cases} x'' = \beta_1 y' z' \\ y'' = \beta_2 z' x' \\ z'' = \beta_3 x' y' \end{cases} . \tag{21}$$

According to [5] the equations of a rigid body have the above form (21), where $\beta_1 = \frac{I_2 - I_3}{I_1}$, $\beta_2 = \frac{I_3 - I_1}{I_2}$, $\beta_3 = \frac{I_1 - I_2}{I_3}$.

Proposition 3.5 *There is a Pfaff form* $\omega = \bar{\omega}_j(y^i)dy^j$ *on* $I\!R^3$ *such that the solutions of the system (21) are solutions of the generalized Euler-Lagrange equation (18).*

In the case of the Euler equations for the rotational dynamics of a rigid body about its center of mass, when $\beta_1 = \frac{I_2 - I_3}{I_1}$, $\beta_2 = \frac{I_3 - I_1}{I_2}$, $\beta_3 = \frac{I_1 - I_2}{I_3}$, one can take

$\omega_1(y^1, y^2, y^3) = \frac{I_1}{2I_3} y^1 \left(y^2\right)^2 + \frac{I_1}{2I_2} y^1 \left(y^3\right)^2 + \frac{\delta_1 (I_3 - I_2)}{6} \left(y^1\right)^3$,

$\omega_2(y^1, y^2, y^3) = \frac{I_2}{2I_1} y^2 \left(y^3\right)^2 + \frac{I_2}{2I_3} y^2 \left(y^1\right)^2 + \frac{\delta_2 (I_1 - I_3)}{6} \left(y^2\right)^3$,

$\omega_3(y^1, y^2, y^3) = \frac{I_3}{2I_2} y^3 \left(y^1\right)^2 + \frac{I_3}{2I_1} y^3 \left(y^2\right)^2 + \frac{\delta_3 (I_2 - I_1)}{6} \left(y^3\right)^3$.

Considering the second derivative of the variation. we take $\gamma_i = 0$.

Proposition 3.6 *Let us consider the system (21) coming from the Euler equations for the rotational dynamics of a rigid body about its center of mass, in an bounded domain* U *where* $y^1 y^2 y^3 \neq 0$.

Then there is a Pfaff form $\omega = \bar{\omega}_j(y^i)dy^j$ *defined for* $(y^i) \in U$, *such that the solutions of the system (21) are extremal solutions for the generalized Euler-Lagrange equation (18), i.e. the second variation has a constant sign along these solutions.*

Using the Lagrangian description given in our paper, one can follow a dual Hamiltonian approach. Other intersting applications can be given in applied mechanics, as for example in biomechanics, in the study of sinovial joints of a human body, following the Lagrangian description given in [2].

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