Physics AUC, vol. 21-Sp.Issue, 171-185 (2011) **PHYSICS AUC**

Gauge Theories on a Noncommutative Poisson Manifold as Spacetime

G. Zet
Physics Department, "Gh. Asachi" Technical University,
Iasi 700050, Romania

Abstract

We construct a model of internal gauge theory defined on a noncommutative Poisson manifold considered as space-time. A covariant star product between Lie algebra valued differential forms is introduced in order to develop the gauge theory. The constraints imposed by the Poisson structure on the connection of the space-time are established and the property of associativity of the covariant star product is verified. As an example, we consider the $U(2)$ noncommutative gauge theory defined on a symplectic space-time manifold endowed only with torsion. It is concluded that the constraints imposed by the Poisson structure of the space-time and the associativity property of the covariant star product completely determine in this case the connection of the space-time. Some comments on the noncommutative gauge theory of gravitation are also made and possible generalizations are emphasized.

1 Introduction

One of the most important problems in the contemporary physics is: how can we describe physics to the Planck scale ($L_P = \sqrt{\frac{G\hbar}{c^3}} = 1,6 \times 10^{-35}m$) ? There are suggestions that this can be done by some generalization of the ordinary spaces which goes under the name of noncommutative geometry [1, 2, 3]. This explains the great attention given to the noncommutative theories and, in particular, to the gauge theory formulated on a noncommutative spaces-time. One important motivation to adopt the idea of noncommutative space-time is the hope that such a framework could offer the possibility to develop a quantum theory of gravity, or at least to give an idea of how this could be achieved [4, 5, 6, 7, 8, 9]. There are two major candidates to quantum gravity: string theory [10] and loop quantum gravity [11].

It is believed that gravity could be quantized if it is formulated in terms of Poisson or symplectic geometry rather than Riemannian geometry, in the context of emergent gravity [27, 28] (for further developments, see [29]). The motivation is that any Poisson manifold can always be quantized at least in the context of deformation quantization [19]. In addition, the emergent gravity is deeply related to the string theory. Many essential aspects of string theory such as AdS/CFT correspondence, open-closed string duality, noncommutative geometry, mirror symmetry, etc. have also been realized in the context of emergent noncommutative geometry. It is even claimed that string theory is simply a "stringy" realization of symplectic or Poisson space-time. This argues again why the quantization of gravity seems to dictate a Poisson (or symplectic) structure to space-time manifold.

Regarding the quantization of gravity we emphasize the possibility that there could be a connection between the gravitational constant G and an intrinsic Poisson structure

$$\theta = \frac{1}{2}\theta^{\mu\nu}(x)\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \tag{1.1}$$

of space-time, since $\theta^{\mu\nu}$ carries the physical dimension of $(length)^2$ in natural units ($\hbar = c = 1$) like G. Mathematically, the quantization of a dynamic system is realized by specifying first an underlying Poisson structure. The dynamic system is described by a Poisson manifold (M, η), where

$$\eta = \hbar\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i} \tag{1.2}$$

defines the Poisson bracket

$$\{f,g\} = \hbar\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i}\right). \tag{1.3}$$

The Poisson structure η is dimensionless like as θ. Then, the physical observables are replaced by self-adjoint operators and the Poisson bracket (1.3) is replaced by a quantum bracket

$$\{f,g\} \to -i\left[\widehat{f},\widehat{g}\right]. \tag{1.4}$$

In the same way one can define a Poisson bracket using the Poisson structure θ of the space-time manifold M. In the case where $\theta^{\mu\nu}$ is a constant co-symplectic matrix of rank $2n$, one can apply the same canonical quantization to the Poisson manifold (M,θ). Now, the space-time M becomes a noncommutative one, i.e.,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}. \tag{1.5}$$

Therefore, the coordinates of space-time are also operators like components of momentum.

Noncommutative geometry and in particular gauge theory of gravity are intimately connected with both these approaches and the overlaps are considerable [5]. String theory is one of the strongest motivations for considering noncommutative space-times geometries and noncommutative gravitation. It has been shown, for example, that in the case when the end points of strings in a theory of open strings are constrained to move on D-branes in a constant B-field background and one considers the low-energy limit, then the full dynamics of the theory is described by a gauge theory on a noncommutative space-time [12].

Recently, it has been argued that the dynamics of the noncommutative gravity arising from string theory [13] is much richer than some versions of the proposed noncommutative gravity. It is suspected that the reason for this is the non-covariance of the Moyal star product under space-time diffeomorphisms. A geometrical approach to noncommutative gravity, leading to a general theory of noncommutative Riemann surfaces in which the problem of the frame dependence of the star product is also recognized, has been proposed in [14] (for further developments, see [15, 16]).

Since the early days of quantum mechanics, the physicists have used star products to build noncommutative generalizations of commuting theories [17]. The first idea has been to consider the quantization as a deformation of the algebra of classical observables of functions on phase space, where the first order term $O(\hbar)$ is taken to be the classical Poisson bracket [18]. Star products have been applied then in many areas of physics, including string theory.

Starting with the works of Kontsevich [19], Cattaneo and Felder [20] and many others, the star product of functions on general Poisson manifolds is well known, in standard coordinates on R^d, to all orders in the deformation parameter. Recently, an explicit form of a covariant star of functions on Poisson manifolds with torsion-free linear connection has been constructed up to the third order $O(\hbar^3)$ [21].

In order to formulate a noncommutative gauge theory it is necessary to generalize the star product to the exterior algebra of differential forms. A covariant star product has been defined in Refs. [22, 23] and the result was extended to case of Lie algebra valued differential forms in Refs. [24, 25, 26]. It has been shown that the graded differential Poisson algebra endows the space-time manifold with a connection having both curvature and torsion (not necessarily torsion-free) [22] and places some constraints upon it. We can try to apply the covariant star product to the case when the space-time is a Poisson or a symplectic manifold which has only curvature, but the torsion vanishes. Then, the restriction imposed by the associativity property of the covariant star product requires also the vanishing curvature. The corresponding connection is flat symplectic and this reduces drastically the applicability area of the covariant star product. Of course, it is possible to have a manifold having both curvature and torsion or only torsion.

In this paper we develop a gauge theory on a Poisson manifold considered as space-time. We introduce a covariant $\star-$*product* between functions or physical fields considering a connection on the space-time M which has only torsion. The noncommutative gauge fields and their strength tensor are expressed in function of the commutative components by using the covariant Seiberg-Witten map [33].

2 Structure of a Poisson manifold

In what follows we will consider a noncommutative space-time M endowed with the coordinates x^{μ}, $\mu = 1, 2, 3, 0$ satisfying the commutation relations

$$[x^{\mu},\ x^{\nu}] = i\ \theta^{\mu\nu}(x)\,, \tag{2.1}$$

where $\theta^{\mu\nu}(x) = -\ \theta^{\nu\mu}(x)$ is a Poisson bivector [22, 23, 30]. We use this bivector to define a Poisson bracket on M

$$\{f, g\} = \theta^{\mu\nu}\partial_{\mu}f\partial_{\nu}g. \tag{2.2}$$

It satisfies the following basic properties

(1) *Skew symmetry* :

$$\{f, g\} = -\{g, f\}\,; \tag{2.3}$$

(2) *Jacobi identity* :

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0; \tag{2.4}$$

(3) Pr*oduct rule* :

$$\{f, gh\} = \{f, g\}\, h + g\,\{f, h\}\,; \tag{2.5}$$

Because the Poisson bracket obeys the Jacobi identity, the bivector $\theta^{\mu\nu}$ must satisfy the following condition

$$\theta^{\mu\rho}\partial_{\rho}\theta^{\nu\sigma} + \theta^{\nu\rho}\partial_{\rho}\theta^{\sigma\mu} + \theta^{\sigma\rho}\partial_{\rho}\theta^{\mu\nu} = 0. \tag{2.6}$$

If a Poisson bracket is defined on M, then M is called a *Poisson manifold* (see [30] for mathematical details).

Suppose now that the bivector $\theta^{\mu\nu}(x)$ has an inverse $\omega_{\mu\nu}(x)$, i.e.

$$\theta^{\mu\rho}\omega_{\rho\nu} = \delta^{\mu}_{\nu}. \tag{2.7}$$

If the differential 2-form $\omega = \frac{1}{2}\omega_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$ associated to $\omega_{\mu\nu}(x)$ is non-degenerate ($\det\omega_{\mu\nu} \neq 0$) and closed ($d\omega = 0$), then it is called a *symplectic* 2-form and M - a *symplectic manifold*. It can be verified that the condition $d\omega = 0$ is equivalent with the equation (2.6) [22, 23, 30]. In this paper we will consider some applications which correspond to the case when M is symplectic, but many general results will refer to Poisson manifolds.

Because the gauge theories involve Lie-valued differential forms such as gauge potential 1-form $A = A^{a}_{\mu}(x)\, T_{a}dx^{\mu} = A_{\mu}dx^{\mu}$, $A_{\mu} = A^{a}_{\mu}(x)\, T_{a}$, where T_{a} are the infinitesimal generators of a symmetry group G, we need to generalize the definition of the Poisson bracket to differential forms and define then an associative star product for such cases. Many of these problems were solved in Ref. [22, 23, 30]. In Refs. [24, 25] these results have been generalized to the case of Lie algebra valued differential forms. This generalization has the effect that the commutator of differential forms can be a commutator or an anti-commutator, depending on their degrees.

Assuming that $\theta^{\mu\nu}(x)$ is invertible, we can always write the Poisson bracket $\{x, dx\}$in the form [30]

$$\{x^{\mu}, dx^{\nu}\} = -\theta^{\mu\sigma}\Gamma^{\nu}_{\sigma\rho}dx^{\rho}, \tag{2.8}$$

where $\Gamma^{\nu}_{\sigma\rho}$ are some functions of x transforming like a connection under general coordinate transformations. As $\Gamma^{\nu}_{\sigma\rho}$ is generally not symmetric, on can use the 1-forms of connection

$$\widetilde{\Gamma}^{\mu}_{\nu} = \Gamma^{\mu}_{\nu\rho}dx^{\rho},\ \Gamma^{\mu}_{\nu} = dx^{\rho}\Gamma^{\mu}_{\rho\nu}, \tag{2.9}$$

to define two kinds of covariant derivatives $\widetilde{\nabla}$ and ∇, respectively. The curvatures for these two connections are

$$\widetilde{R}^{\nu}_{\lambda\rho\sigma} = \partial_{\rho}\Gamma^{\nu}_{\lambda\sigma} - \partial_{\sigma}\Gamma^{\nu}_{\lambda\rho} + \Gamma^{\nu}_{\tau\rho}\Gamma^{\tau}_{\lambda\sigma} - \Gamma^{\nu}_{\tau\sigma}\Gamma^{\tau}_{\lambda\rho}, \tag{2.10}$$

$$R^{\nu}_{\lambda\rho\sigma} = \partial_{\rho}\Gamma^{\nu}_{\sigma\lambda} - \partial_{\sigma}\Gamma^{\nu}_{\rho\lambda} + \Gamma^{\nu}_{\rho\tau}\Gamma^{\tau}_{\sigma\lambda} - \Gamma^{\nu}_{\sigma\tau}\Gamma^{\tau}_{\rho\lambda}. \tag{2.11}$$

Because the connection coefficients $\Gamma^{\rho}_{\mu\nu}$ are not symmetric ($\Gamma^{\rho}_{\mu\nu} \neq \Gamma^{\rho}_{\nu\mu}$), the symplectic manifold M has also a torsion defined as usually [30] by

$$T = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}. \tag{2.12}$$

The connection satisfies the identity [22]

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] \alpha = -R^{\sigma}_{\rho\mu\nu} \wedge i_{\sigma}\alpha - T^{\rho}_{\mu\nu} \nabla_{\rho}\alpha, \tag{2.13}$$

and an analogous formula applies for $\widetilde{\nabla}$. Here, α is an arbitrary differential k-form

$$\alpha = \frac{1}{k!} \alpha_{\mu_1 \ldots \mu_k} dx^{\mu_1 \ldots \mu_k}, \tag{2.14}$$

and $i_{\sigma}\alpha$ denotes the *interior product* which maps the k-form into a $(k-1)$-form

$$i_{\sigma}\alpha = \frac{1}{(k-1)!} \alpha_{\sigma\mu_2 \ldots \mu_k} dx^{\mu_2 \ldots \mu_k}. \tag{2.15}$$

It has been proven that in order the Poisson bracket satisfies the Leibniz rule

$$d\left\{f, g\right\} = \left\{df, g\right\} + \left\{f, dg\right\}, \tag{2.16}$$

the bivector $\theta^{\mu\nu}(x)$ must obeys the property [22, 23]

$$\widetilde{\nabla}\theta^{\mu\nu} = \partial_{\rho}\theta^{\mu\nu} + \Gamma^{\mu}_{\sigma\rho}\theta^{\sigma\nu} + \Gamma^{\nu}_{\sigma\rho}\theta^{\mu\sigma} \equiv 0. \tag{2.17}$$

Thus $\theta^{\mu\nu}$is covariant constant under $\widetilde{\nabla}$, and $\widetilde{\nabla}$ is named a *symplectic connection*, because it annihilates the symplectic 2-form ω. One can use the Leibniz condition (2.17) together with the Jacobi identity for the Poisson bivector $\theta^{\mu\nu}$ to obtain the cyclic relation for torsion

$$\sum\nolimits_{(\mu,\nu,\sigma)} \theta^{\mu\rho}\theta^{\nu\sigma}T^{\lambda}_{\rho\sigma} = 0. \tag{2.18}$$

Note that while this relation shows that that a torsion-free connection identically satisfies the property (2.18), the Jacobi identity does not require the connection to be torsionless. Also note that (2.17) and the Jacobi identity for the Poisson bivector can be combined to obtain the following cyclicity property

$$\sum\nolimits_{(\mu,\nu,\sigma)} \theta^{\mu\rho}\nabla_{\rho}\theta^{\nu\sigma} = 0. \tag{2.19}$$

If in addition to restriction $\widetilde{\nabla}_{\rho}\theta^{\mu\nu} = 0$, one imposes $\nabla_{\rho}\theta^{\mu\nu} = 0$, the torsion vanishes, $T^{\rho}_{\mu\nu} = 0$, and there is only one covariant derivative $\nabla = \widetilde{\nabla}$. In this paper, we do not require that $\nabla_{\rho}\theta^{\mu\nu} = 0$.

Now we generalize the Poisson bracket to include differential forms. Let us consider some arbitrary differential forms α, β, γ and denote their degrees respectively by $|\alpha|, |\beta|$ and $|\gamma|$. We define then a graded differential Poisson algebra on the manifold M as the set of all differential forms satisfying the following properties

(i) *Bracket degree* : $|\{\alpha, \beta\}| = |\alpha| + |\beta|$;
(ii) *Graded symmetry* : $\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\}$;
(iii) *Graded product rule* : $\{\alpha, \beta\gamma\} = \{\alpha, \beta\}\gamma + (-1)^{|\alpha||\beta|}\beta\{\alpha, \gamma\}$;
(iv) *Leibniz rule* : $d\{\alpha, \beta\} = \{d\alpha, \beta\} + (-1)^{|\alpha|}\{\alpha, d\beta\}$;
(v) *Graded Jacobi identity* :
$\{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha|(|\beta|+|\gamma|)}\{\beta, \{\gamma, \alpha\}\} + (-1)^{|\gamma|(|\alpha|+|\beta|)}\{\gamma, \{\alpha, \beta\}\} = 0.$

These properties naturally combine the defining characteristics of differential forms and the Poisson bracket. The Leibniz rule and the graded Jacobi identity place strong conditions on the Poisson brackets of differential forms. In fact, the properties (i) – (v) uniquely determine the form of the Poisson bracket.

Using the graded product rule (iii), we can prove the following general expression of the Poisson bracket between differential form [22, 23]

$$\{\alpha,\beta\} = \theta^{\mu\nu}\nabla_\mu\alpha\wedge\nabla_\nu\beta + (-1)^{|\alpha|}\,\widetilde{R}^{\mu\nu}\wedge(i_\mu\alpha)\wedge(i_\nu\beta)\,, \tag{2.20}$$

where $|\alpha|$ is the degree of the differential form α, and

$$\widetilde{R}^{\mu\nu} = \frac{1}{2}\widetilde{R}^{\mu\nu}_{\rho\sigma}dx^\rho\wedge dx^\sigma,\quad \widetilde{R}^{\mu\nu}_{\rho\sigma} = \theta^{\mu\lambda}\widetilde{R}^{\nu}_{\lambda\rho\sigma}. \tag{2.21}$$

It can be proven that in order that (2.20) satisfies the properties of the graded differential Poisson bracket, the following restrictions on the connection coefficients must be imposed [23]

(a) $\widetilde{\nabla}$ *is symplectic* : $\widetilde{\nabla}_\rho\theta^{\mu\nu} = 0$;

(b) *satisfies the Jacobi identity* : $\theta^{\mu\rho}\partial_\rho\theta^{\nu\sigma} + \theta^{\nu\rho}\partial_\rho\theta^{\sigma\mu} + \theta^{\sigma\rho}\partial_\rho\theta^{\mu\nu} = 0$;

(c) *The connection has vanishing curvature* : $R^{\nu}_{\lambda\rho\sigma} = 0$;

(d) *The curvature is covariant constant* : $\nabla_\lambda\widetilde{R}^{\mu\nu}_{\rho\sigma} = 0$.

As a consequence of these restrictions, the following condition satisfied by the curvature can be obtained [22, 23]

$$\widetilde{R}^{\mu\nu}\wedge\left(i_\nu\widetilde{R}^{\rho\sigma}\right) + \widetilde{R}^{\rho\nu}\wedge\left(i_\nu\widetilde{R}^{\sigma\mu}\right) + \widetilde{R}^{\sigma\nu}\wedge\left(i_\nu\widetilde{R}^{\mu\rho}\right) = 0. \tag{2.22}$$

Finally, we remark that if a connection exists that satisfies all these properties, then we have completely determined expression of the Poisson bracket between two arbitrary differential forms. This bracket is the only possible bracket between differential forms on a symplectic manifold.

3 Covariant $\star-$ product

What is generally done to construct a noncommutative gauge theory and, in general a noncommutative field theory, is to deform the ordinary pointwise commutative product among functions or differential forms on space-time with the introduction of a star product which is noncommutative and reduces to the usual one in a certain limit. The choice of the star product compatible with the noncommutativity (2.1) is not unique. In this work we use the covariant star product defined in Ref. [22] for differential forms and generalized to the case of Lie algebra valued differential forms in [22] and which has been generalized to Lie algebra valued differential forms in [25, 26].

The covariant star product between arbitrary differential forms that we will consider here has the general form

$$\alpha\star\beta = \alpha\wedge\beta + \sum\nolimits_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n C_n(\alpha,\beta)\,, \tag{3.1}$$

where $C_n(\alpha,\beta)$ are bilinear differential operators satisfying the generalized Moyal symmetry [22, 25]

$$C_n(\alpha,\beta) = (-1)^{|\alpha||\beta|+n}\,C_n(\beta,\alpha)\,. \tag{3.2}$$

The operator $C_1(\alpha,\beta)$ coincides with the Poisson bracket, i.e. $C_1(\alpha,\beta) = \{\alpha,\beta\}$. An expression for $C_2(\alpha,\beta)$ has been obtained also in Ref. [22] (see Appendix) so that the star product (**??**) satisfies the property of associativity

$$(\alpha\star\beta)\star\gamma = \alpha\star(\beta\star\gamma)\,. \tag{3.3}$$

In order to simplify presentation and give some simple illustrative examples, we will consider the case when the symplectic manifold M has only torsion. Since the curvature $R^{\nu}_{\lambda\rho\sigma}$ is vanishing [see Eq. (2.12)], one obtains the following relation between the curvature $\widetilde{R}$ and the torsion T [22]

$$\widetilde{R}^{\sigma}_{\mu\nu\rho} = \nabla_\mu T^{\sigma}_{\nu\rho}. \tag{3.4}$$

This relation shows that the curvature $\widetilde{R}^{\sigma}_{\mu\nu\rho}$ vanishes too if the torsion $T^{\sigma}_{\nu\rho}$ is covariant constant, i.e.

$$\nabla_\mu T^{\sigma}_{\nu\rho} = 0. \tag{3.5}$$

Therefore, if the torsion is covariant constant, the symplectic manifold M has only torsion but not curvature.

For such a symplectic manifold, the bilinear differential operators $C_1(\alpha,\beta)$ and $C_2(\alpha,\beta)$ in the star product (3.1) proposed in Ref. [22] reduce to the simpler forms

$$C_1(\alpha,\beta) = \{\alpha,\beta\} = \theta^{\mu\nu}\nabla_\mu\alpha\wedge\nabla_\nu\beta, \tag{3.6}$$

$$\begin{aligned} C_2(\alpha,\beta) &= \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\rho\alpha\wedge\nabla_\nu\nabla_\sigma\beta + \frac{1}{3}(\theta^{\nu\rho}\nabla_\rho\theta^{\mu\sigma} \\ &+\frac{1}{2}\theta^{\mu\rho}\theta^{\sigma\lambda}T^{\nu}_{\rho\lambda})(\nabla_\mu\nabla_\nu\alpha\wedge\nabla_\sigma\beta - \nabla_\mu\alpha\wedge\nabla_\nu\nabla_\sigma\beta). \end{aligned} \tag{3.7}$$

We can verify that the covariant star product with torsion defined in (**??**)–(**??**) is associative [26].

Now, we extend the above covariant star product to the case of Lie algebra valued differential forms. Suppose that we have an internal gauge group G whose infinitesimal generators T_a satisfy the algebra

$$[T_a, T_b] = if^c_{ab}T_c, \;\; a,b,c = 1,2,....,m \tag{3.8}$$

with the structure constants $f^c_{ab} = -f^c_{ba}$. If $\alpha = \alpha^a T_a$ and $\beta = \beta^b T_b$ are two arbitrary such forms, where α^a and β^b are ordinary differential forms of degrees $|\alpha|$ and $|\beta|$ respectively, then their covariant star product has the expression [24]

$$\begin{aligned} \alpha\star\beta &= \alpha\wedge\beta + \sum\nolimits_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n C_n(\alpha,\beta) \\ &= \alpha^a\wedge\beta^b T_a T_b + \sum\nolimits_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n C_n\left(\alpha^a,\beta^b\right) T_a T_b, \end{aligned} \tag{3.9}$$

where $C_n\left(\alpha^a,\beta^b\right)$ are the bilinear operators given in (3.6)–(3.7) with α and β changed in α^a and β^b respectively. It is important to remark that the operators $C_n\left(\alpha^a,\beta^b\right)$ satisfy the same generalized Moyal symmetry (3.2), i.e.

$$C_n\left(\alpha^a,\beta^b\right) = (-1)^{|\alpha||\beta|+n}\, C_n\left(\beta^b,\alpha^a\right) \tag{3.10}$$

Tacking into account the graded structure of our Poisson algebra, we define the commutator of two Lie algebra valued differential forms $\alpha = \alpha^a T_a$ and $\beta = \beta^b T_b$ by

$$[\alpha,\beta]_\star = \alpha\star\beta - (-1)^{|\alpha||\beta|}\,\beta\star\alpha. \tag{3.11}$$

For example, if α and β are Lie algebra valued differential $one-forms$, we have

$$\begin{aligned} [\alpha,\beta]_\star &= \alpha^a\wedge\beta^b\,[T_a,T_b] + \frac{i\hbar}{2}C_1\left(\alpha^a,\beta^b\right)\{T_a,T_b\} \\ &+\left(\frac{i\hbar}{2}\right)^2 C_2\left(\alpha^a,\beta^b\right)[T_a,T_b] + O\left(\hbar^3\right). \end{aligned} \tag{3.12}$$

This result shows that the star commutator of Lie algebra valued differential forms does not close in general in the Lie algebra but in its universal enveloping algebra. Exceptions are the unitary groups $U(N)$ where this is true. The expressions of the operators $C_1\left(\alpha^a,\beta^b\right)$ and $C_2\left(\alpha^a,\beta^b\right)$ are those given in (3.6) and (3.7) respectively, with α and β exchanged in α^a and β^b.

In the next Section we apply this covariant star product in order to develop a noncommutative internal gauge theory.

4 Noncommutative gauge theory

We suppose that G is a gauge group with the equations of structure given in (3.8) and denote the Lie algebra valued infinitesimal parameter by

$$\widehat{\lambda} = \widehat{\lambda}^a T_a. \tag{4.1}$$

We use the hat symbol "^" to denote the non-commutative quantities of our gauge theory. The parameter $\widehat{\lambda}$ is a 0-form, i.e. $\widehat{\lambda}^a$ are functions of the coordinates x^μ on the symplectic manifold .

Now, we define the gauge transformation of parameter $\widehat{\lambda}$ of the non-commutative Lie valued gauge potential

$$\widehat{A} = \widehat{A}^a_\mu (x) T_a dx^\mu = \widehat{A}_\mu dx^\mu,\ \widehat{A}_\mu = \widehat{A}^a_\mu (x) T_a, \tag{4.2}$$

by

$$\widehat{\delta}\widehat{A} = d\widehat{\lambda} - i \left[\widehat{A}, \widehat{\lambda}\right]_\star . \tag{4.3}$$

Here we consider the definition of the commutator $[\alpha, \beta]_\star$ of two arbitrary differential forms α and β given in (3.11). Then, using the definition (3.9) of the covariant star product and the equations of structure (3.8) of the gauge group, we can write (4.3) as

$$\widehat{\delta}\widehat{A}^a = d\widehat{\lambda}^a + f^a_{bc}\widehat{A}^b\widehat{\lambda}^c + \frac{\hbar}{2} d^a_{bc} C_1\left(\widehat{A}^b, \widehat{\lambda}^c\right) - \frac{\hbar^2}{4} f^a_{bc} C_2\left(\widehat{A}^b, \widehat{\lambda}^c\right) + O\left(\hbar^3\right), \tag{4.4}$$

where we noted $\{T_a, T_b\} = d^a_{bc} T_c$. In fact, this notation is valid if the Lie algebra closes also for anticommutator, as it happens for example in the case of unitary groups $U(N)$. In general, the commutators like $\left[\widehat{A}, \widehat{\lambda}\right]_\star$ take values in the enveloping algebra [10]. Therefore, the gauge field $\widehat{A}$ and the parameter $\widehat{\lambda}$ take values in this algebra. Let us write for instance $\widehat{A} = \widehat{A}^I T_I$ and $\widehat{\lambda} = \widehat{\lambda}^J T_J$. Then we have

$$\left[\widehat{A}, \widehat{\lambda}\right]_\star = \frac{1}{2}\left\{\widehat{A}^I, \widehat{\lambda}^J\right\}_\star [T_I, T_J] + \frac{1}{2}\left[\widehat{A}^I, \widehat{\lambda}^J\right]_\star \{T_I, T_J\}. \tag{4.5}$$

Thus, all products of the generators T_I will be necessary in order to close the enveloping algebra. Its structure can be obtained by successively computing the commutators and anti-commutators starting from the generators of Lie algebra, until it closes [31, 32],

$$[T_I, T_J] = i f^K_{IJ} T_K,\ \ \{T_I, T_J\} = d^K_{IJ} T_K.$$

Therefore, in our above notations and in what follows we understand this structure in general.

The operators $C_1\left(\widehat{A}^b, \widehat{\lambda}^c\right)$ and $C_2\left(\widehat{A}^b, \widehat{\lambda}^c\right)$ have the expressions [see Eqs. (3.6)–(3.7)]

$$C_1\left(\widehat{A}^b, \widehat{\lambda}^c\right) = \left\{\widehat{A}^b, \widehat{\lambda}^c\right\} = \theta^{\mu\nu}\nabla_\mu\widehat{A}^b \wedge \nabla_\nu\widehat{\lambda}^c, \tag{4.6}$$

$$\begin{aligned} C_2\left(\widehat{A}^b, \widehat{\lambda}^c\right) \ &= \ \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\rho\widehat{A}^b \wedge \nabla_\nu\nabla_\sigma\widehat{\lambda}^c + \frac{1}{3}(\theta^{\nu\rho}\nabla_\rho\theta^{\mu\sigma} \\ &\quad + \frac{1}{2}\theta^{\mu\rho}\theta^{\sigma\lambda}T^\nu_{\rho\lambda})(\nabla_\mu\nabla_\nu\widehat{A}^b \wedge \nabla_\sigma\widehat{\lambda}^c - \nabla_\mu\widehat{A}^b \wedge \nabla_\nu\nabla_\sigma\widehat{\lambda}^c). \end{aligned} \tag{4.7}$$

Here we use the definition of the covariant derivative

$$\nabla_\mu\widehat{A}^a = \left(\partial_\mu\widehat{A}^a_\nu - \Gamma^\rho_{\mu\nu}\widehat{A}^a_\rho\right) dx^\nu, \tag{4.8}$$

and $\nabla_\nu\widehat{\lambda}^c \equiv \partial_\nu\widehat{\lambda}^c$is understood.

We define also the curvature 2-form $\widehat{F}$ of the gauge potentials by

$$\widehat{F} = \frac{1}{2} dx^\mu \wedge dx^\nu \widehat{F}_{\mu\nu} = d\widehat{A} - \frac{i}{2}\left[\widehat{A}, \widehat{A}\right]_\star . \tag{4.9}$$

Then, using the definition (3.9) of the star product and the property (3.10) of the bilinear operators $C_n\left(\alpha^a,\beta^b\right)$, we obtain from (4.9)

$$\begin{aligned}\widehat{F}^a &= d\widehat{A}^a+\frac{1}{2}f^a_{bc}\widehat{A}^b\wedge\widehat{A}^c+\frac{1}{2}\frac{\hbar}{2}d^a_{bc}C_1\left(\widehat{A}^b,\widehat{A}^c\right)\\ &\quad -\frac{1}{2}\frac{\hbar^2}{4}f^a_{bc}C_2\left(\widehat{A}^b,\widehat{A}^c\right)+O\left(\hbar^3\right).\end{aligned}\tag{4.10}$$

More explicitly, in terms of components we have

$$\begin{aligned}\widehat{F}^a_{\mu\nu} &= \nabla_\mu\widehat{A}^a_\nu-\nabla_\nu\widehat{A}^a_\mu+f^a_{bc}\widehat{A}^b_\mu\widehat{A}^c_\nu+\widehat{A}^a_\rho T^\rho_{\mu\nu}+\frac{\hbar}{2}d^a_{bc}C_1\left(\widehat{A}^b_\mu,\widehat{A}^c_\nu\right)\\ &\quad -\frac{\hbar^2}{4}f^a_{bc}C_2\left(\widehat{A}^b_\mu,\widehat{A}^c_\nu\right)+O\left(\hbar^3\right).\end{aligned}\tag{4.11}$$

where we used the definition , with

$$C_1\left(\widehat{A}^b_\mu,\widehat{A}^c_\nu\right)=\theta^{\rho\sigma}\nabla_\rho\widehat{A}^b_\mu\nabla_\sigma\widehat{A}^c_\nu,\tag{4.12}$$

$$\begin{aligned}C_2\left(\widehat{A}^b_\mu,\widehat{A}^c_\nu\right) &= \frac{1}{2}\theta^{\rho\sigma}\theta^{\lambda\tau}\nabla_\rho\nabla_\lambda\widehat{A}^b_\mu\nabla_\sigma\nabla_\tau\widehat{A}^c_\nu+\frac{1}{3}(\theta^{\rho\tau}\nabla_\tau\theta^{\sigma\lambda}\\ &\quad +\frac{1}{2}\theta^{\sigma\tau}\theta^{\lambda\phi}T^\rho_{\tau\phi})(\nabla_\rho\nabla_\sigma\widehat{A}^b_\mu\nabla_\lambda\widehat{A}^c_\nu-\nabla_\sigma\widehat{A}^b_\mu\nabla_\rho\nabla_\lambda\widehat{A}^c_\nu).\end{aligned}\tag{4.13}$$

Under the gauge transformation (4.3) the curvature 2-form $\widehat{F}$ transforms as

$$\widehat{\delta}\widehat{F}=i\left[\widehat{\lambda},\widehat{F}\right]_\star,\tag{4.14}$$

where we used the Leibniz rule

$$d\left(\widehat{\alpha}\star\widehat{\beta}\right)=d\widehat{\alpha}\star\widehat{\beta}+(-1)^{|\alpha|}\,\widehat{\alpha}\star d\widehat{\beta},\tag{4.15}$$

which we admit to be valid to all orders in $\hbar$. In terms of the components (4.14) becomes

$$\widehat{\delta}\widehat{F}^a=f^a_{bc}\widehat{F}^b\widehat{\lambda}^c+\frac{\hbar}{2}d^a_{bc}C_1\left(\widehat{F}^b,\widehat{\lambda}^c\right)-\frac{\hbar^2}{4}f^a_{bc}C_2\left(\widehat{F}^b,\widehat{\lambda}^c\right)+O\left(\hbar^3\right).\tag{4.16}$$

In the zeroth order, the formula (4.16) reproduces therefore the result of the commutative gauge theory

$$\delta F^a_{\mu\nu}=f^a_{bc}F^b_{\mu\nu}\lambda^c\quad\Rightarrow\quad\delta F=i\left[\lambda,F\right].\tag{4.17}$$

Using again the Leibniz rule, we obtain the deformed Bianchi identity

$$d\widehat{F}+i\left[\widehat{F},\widehat{A}\right]_\star=0.\tag{4.18}$$

If we apply the definition (3.11) of the star commutator, we obtain

$$d\widehat{F}+i\left[\widehat{F},\widehat{A}\right]=\left[\frac{\hbar}{2}d^a_{bc}C_1\left(\widehat{F}^b,\widehat{A}^c\right)-\frac{\hbar^2}{4}f^a_{bc}C_2\left(\widehat{F}^b,\widehat{A}^c\right)\right]T_a+O\left(\hbar^3\right),\tag{4.19}$$

or in terms of components

$$d\widehat{F}^a-f^a_{bc}\widehat{F}^b\wedge\widehat{A}^c=\frac{\hbar}{2}d^a_{bc}C_1\left(\widehat{F}^b,\widehat{A}^c\right)-\frac{\hbar^2}{4}f^a_{bc}C_2\left(\widehat{F}^b,\widehat{A}^c\right)+O\left(\hbar^3\right).\tag{4.20}$$

We remark that in zeroth order we obtain from (4.19) the usual Bianchi identity

$$dF+i\left[F,A\right]=0\tag{4.21}$$

In addition, if the gauge group is $U(1)$, the Bianchi identity (4.18) becomes

$$d\widehat{F} = \hbar C_1\left(\widehat{F}, \widehat{A}\right) + O\left(\hbar^3\right). \tag{4.22}$$

This result is also in accord with that of Ref. [30].

Having established the previous results, we can construct a noncommutative Yang-Mills (NCMY) action. We will consider therefore the case when the gauge group is $U(N)$. Let $G^{\mu\nu}$ a metric on the noncommutative space-time M [24]. We suppose that the metric $G^{\mu\nu}$ belongs to the *adjoint representation* of $U(1) \subset U(N)$, in sense that $G^{\mu\nu} = G^{\mu\nu}I$, where I is the unity matrix of $U(N)$ in this representation. Therefore, we consider the components of $G^{\mu\nu}$ as Lie algebra-valued 0-forms. The covariant derivative of the metric $G^{\mu\nu}$ is

$$\nabla_\mu G^{\nu\rho} = \partial_\mu G^{\nu\rho} + G^{\nu\sigma}\Gamma^\rho_{\mu\sigma} + G^{\sigma\rho}\Gamma^\nu_{\mu\sigma} \tag{4.23}$$

If $G^{\mu\nu}$ is not constant we have to modify it to be a gauge covariant metric for the (NCYM) action [25, 30] in sense that it transforms like $\widehat{F}$ (see (4.18))

$$\widehat{\delta}\widehat{G}^{\mu\nu} = i\left[\widehat{\lambda}, \widehat{G}^{\mu\nu}\right]'_\star \tag{4.24}$$

Then, using the definition (3.11) for the $\star$ - commutator, we obtain from (4.24)

$$\widehat{\delta}\widehat{G}^{\mu\nu} = \theta^{\rho\sigma}\nabla_\rho \widehat{G}^{\mu\nu}\partial_\sigma\widehat{\lambda} + O\left(\hbar^3\right). \tag{4.25}$$

We can use the $Seiberg - Witten$ *map with covariant* $\star-$ *product* for a field which is in the adjoint representation (as we consider to be $G^{\mu\nu}$) to obtain [33]

$$\widehat{G}^{\mu\nu} = G^{\mu\nu} - A^0_\rho\theta^{\rho\sigma}\nabla_\sigma G^{\mu\nu} + O\left(\hbar^3\right), \tag{4.26}$$

where A^0_ρ is the gauge field in the sector $U(1)$ of $U(N)$.

In order to construct the NCYM action for the gauge fields $A^a_\mu(x)$, $\mu = 1,2,3$, $a = 0,1,2,...,N^2-1$, we use the definition for the integration of a function f (or of another quantity) over the noncommutative space M as (for details see [35])

$$\langle\cdot\rangle \equiv Tr = \int d^4x \left|Pf(B)\right|(\cdot) \tag{4.27}$$

where $B = \theta^{-1}$ and $Pf(B)$ denotes the $Pfaffian$ of B, i.e. $Pf(B) = \sqrt{\det(B)}$.

The notation $B = \theta^{-1}$ is in connection with the very important result that for a $D - brane$ in a B field background (with B constant or not constant), its low energy effective theory lives on a noncommutative space-time with the Poisson structure $\theta = B^{-1}$[34, 35, 36]. More exactly, it is shown that the metric G introduced on the Poisson manifold M is connected with the metric g appearing in the fundamental string (open or closed) action by relation $G = -B^{-1}gB^{-1}$ [12, 34, 35].

Now, we define the NCYM action by (see [24, 34])

$$\begin{aligned}\widehat{S}_{NCYM} &= -\frac{1}{2{g_c}^2}\left\langle tr\left(\widehat{G}\star\widehat{F}\star\widehat{G}\star\widehat{F}\right)\right\rangle = \\ &= -\frac{1}{4{g_c}^2}\left\langle\left(\widehat{G}^{\mu\rho}\star\widehat{F}^a_{\rho\nu}\star\widehat{G}^{\nu\sigma}\star\widehat{F}_{a\sigma\mu}\right)\right\rangle,\end{aligned} \tag{4.28}$$

where g_c is the Yang-Mills gauge coupling constant, and we have used the normalization property

$$tr\left(T_aT_b\right) = \frac{1}{2}\delta_{ab}I. \tag{4.29}$$

Using the properties of gauge covariance (4.14) and (4.24) for $\widehat{F}$ and $\widehat{G}$ respectively, we obtain

$$\widehat{\delta}\widehat{S}_{NCYM} = -\frac{\hbar}{2{g_c}^2}\left\langle C_1\left(tr\left(\widehat{G}\widehat{F}\widehat{G}\widehat{F}\right), \widehat{\lambda}\right)\right\rangle + O\left(\hbar^3\right). \tag{4.30}$$

Now, since the integral is cyclic in the Poisson limit [34], i.e.

$$C_1\left(tr\left(\widehat{G}\widehat{F}\widehat{G}\widehat{F}\right),\widehat{\lambda}\right)=0, \tag{4.31}$$

then the Eq. (4.30) becomes

$$\widehat{\delta}\widehat{S}_{NCYM}=0+O\left(\hbar^3\right). \tag{4.32}$$

Therefore, the action $\widehat{S}_{NCYM}$ is invariant up to the second order in $\hbar$. The expression (4.28) of the action can be further simplified as [24, 34]

$$\begin{aligned}\widehat{S}_{NCYM} &= -\frac{1}{2{g_c}^2}\left\langle tr\left(\widehat{G}\widehat{F}\widehat{G}\widehat{F}\right)\right\rangle+O\left(\hbar^3\right)= \\ &= -\frac{1}{4{g_c}^2}\left\langle\left(\widehat{G}^{\mu\rho}\widehat{F}^a_{\rho\nu}\widehat{G}^{\nu\sigma}\widehat{F}_{a\sigma\mu}\right)\right\rangle+O\left(\hbar^3\right).\end{aligned} \tag{4.33}$$

Using the previous results we can obtain solutions for the noncommutative gauge field equations. An example is given in Section 5 using the symplectic manifold M endowed with a covariant constant torsion.

We can add, as usually, fields in our noncommutative gauge model. As an example, we mention the case when the noncommutative $U(N)$ gauge theory is coupled to a Higgs multiplet $\widehat{\Phi}(x)=\widehat{\Phi}^a(x)\,T_a$ in the adjoint representation. The integral of action for $\widehat{\Phi}(x)$ is [37]

$$\widehat{S}_{HIGGS}=-\frac{1}{2{g_c}^2}\left\langle tr\left(\widehat{D}_\mu\widehat{\Phi}\star\widehat{G}^{\mu\nu}\star\widehat{D}_\nu\widehat{\Phi}\right)\right\rangle, \tag{4.34}$$

where

$$\widehat{D}_\mu\widehat{\Phi}=\partial_\mu\widehat{\Phi}-ig_c\left[\widehat{\Phi},\widehat{A}_\mu\right]_\star \tag{4.35}$$

is the noncommutative gauge covariant derivative $\widehat{\Phi}(x)$. Because this derivative is gauge covariant, in the sense

$$\widehat{\delta}\left(\widehat{D}_\mu\widehat{\Phi}\right)=i\left[\widehat{\lambda},\widehat{D}_\mu\widehat{\Phi}\right]_\star, \tag{4.36}$$

the action $\widehat{S}_{HIGGS}$ is invariant as well as $\widehat{S}_{NCYM}$ up to the second order $O\left(\hbar^2\right)$. The action of the noncommutative $U(N)$ gauge fields coupled to Higgs multiplet $\widehat{\Phi}(x)$ reads

$$\widehat{S}_{NC}=-\frac{1}{2{g_c}^2}\left\langle\left(\widehat{G}^{\mu\rho}\star\widehat{F}_{\rho\nu}\star\widehat{G}^{\nu\sigma}\star\widehat{F}_{\sigma\mu}+\widehat{D}_\mu\widehat{\Phi}\star\widehat{G}^{\mu\nu}\star\widehat{D}_\nu\widehat{\Phi}\right)\right\rangle. \tag{4.37}$$

This action can be used to obtain solutions for the noncommutative version of the Yang-Mills-Higgs model using the commutative $\star$-product defined on the manifold M by extending the results of [37] where one uses the usual Moyal $\star$-product.

5 Example: noncommutative $U(2)$ gauge theory

As a very simple example we consider the Poincaré gauge theory to construct the manifold M . Then, suppose that we have the gauge fields e^a_μ and fix the gauge $\omega^{ab}_\mu=0$[38]. We define the connection coefficients

$$\Gamma^\rho_{\mu\nu}=\bar{e}^\rho_a\partial_\nu e^a_\mu, \tag{5.1}$$

where $\bar{e}^\rho_a$ denotes the inverse of e^a_μ. Obviously, the connection Γ defined by these coefficients is not symmetric, i.e. $\Gamma^\rho_{\mu\nu}\neq\Gamma^\rho_{\nu\mu}$. Define then the torsion by formula

$$T^\rho_{\mu\nu}=\Gamma^\rho_{\mu\nu}-\Gamma^\rho_{\nu\mu}. \tag{5.2}$$

In order to simplify the calculation, we consider the case of spherically symmetry and choose the gauge fields e^a_μ as

$$e^a_\mu=\left(A,1,1,\frac{1}{A}\right),\quad \bar{e}^\rho_a=\left(\frac{1}{A},1,1,A\right), \tag{5.3}$$

where $A = A(r)$ is a function depending only on the radial coordinate r. Then, denoting the spherical coordinates on M by $(x^\mu) = (r, \theta, \phi, t)$, $\mu = 1, 2, 3, 0$, the non-null components of the connection coefficients are

$$\Gamma^0_{10} = -\frac{A'}{A}, \quad \Gamma^1_{11} = \frac{A'}{A} \tag{5.4}$$

It is easy to see that the only non-null components of the torsion are

$$T^0_{01} = -T^0_{10} = \frac{A'}{A}. \tag{5.5}$$

Also, using the definitions (2.10) and (2.11) of the curvatures, we obtain

$$\widetilde{R}^0_{101} = -\widetilde{R}^0_{110} = \frac{AA'' - 2A'^2}{A^2}, \quad R^\lambda_{\mu\nu\rho} = 0, \tag{5.6}$$

and all other components of $\widetilde{R}^\lambda_{\mu\nu\rho}$ are vanishing. In these expressions, we denoted the first and second derivatives of by A' and A'' respectively. The vanishing of the curvature $R^\lambda_{\mu\nu\rho}$ agrees with the constraint imposed on the connection ∇.

Introduce then the noncommutative parameters $\theta^{\mu\nu}$ and suppose that we choose them so that

$$(\theta^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{A(r)} \\ 0 & 0 & -b & 0 \\ 0 & b & 0 & 0 \\ \frac{1}{A(r)} & 0 & 0 & 0 \end{pmatrix}, \quad \mu, \nu = 1, 2, 3, 0, \tag{5.7}$$

where b is a non-vanishing constant. Then, we have

$$\widetilde{\nabla}_1 \theta^{01} = -\widetilde{\nabla}_1 \theta^{10}, \quad \nabla_1 \theta^{01} = -\nabla_1 \theta^{10} = \frac{A'}{A^2}. \tag{5.8}$$

This agrees with the constraint (2.17) that $\theta^{\mu\nu}$ is covariant constant under $\widetilde{\nabla}$.

Finally, if we impose also the condition of vanishing of the curvature $\widetilde{R}^\lambda_{\mu\nu\rho}$, then from (5.6) we obtain the following differential equation of the second order for the unknown function $A(r)$:

$$AA'' - 2A'^2 = 0. \tag{5.9}$$

The solutions of this equation is

$$A(r) = -\frac{1}{C_1 r + C_2}, \tag{5.10}$$

where C_1 and C_2 are two arbitrary constants of integration. Therefore, in our simple example, the conditions necessary to define a covariant star product on a symplectic manifold M completely determine its connection. In addition, it is very interesting to see that the covariant derivative of the torsion, defined as

$$\nabla_\mu T^\nu_{\rho\sigma} = \partial_\mu T^\nu_{\rho\sigma} + \Gamma^\nu_{\lambda\mu} T^\lambda_{\rho\sigma} - \Gamma^\lambda_{\rho\mu} T^\nu_{\lambda\sigma} - \Gamma^\lambda_{\sigma\mu} T^\nu_{\rho\lambda}, \tag{5.11}$$

has the following non-null components

$$\nabla_1 T^0_{01} = -\nabla_1 T^0_{10} = \frac{AA'' - 2A'^2}{A^2}. \tag{5.12}$$

Then, tacking into account the equation (5.9), we conclude that the torsion is covariant constant, $\nabla_\mu T^\nu_{\rho\sigma} = 0$, a result which is in concordance with the condition (3.5).

We develop now a noncommutative $U(2)$ gauge theory on the space-time manifold M constructed in the previous example. Denote the generators of group $U(2)$ by T_a, $a = k, 0$, with $k = 1, 2, 3$; here $T_k = \sigma_k$ (σ_k- Pauli matrices) generates the $SU(2)$-sector, and $T_0 = I$ ($I-$ the unit matrix) - the U(1)-sector of the gauge group $U(2)$. These generators satisfy the algebra (3.8), where only the structure constant $f^i_{jk} = 2\varepsilon_{ijk}$ (ε_{ijk} - total antisymmetric Levi-Civita symbols) of the $SU(2)$-sector are non-vanishing, the other components of f^a_{bc} being equal to zero. The anti-commutator $\{T_a, T_b\} = d^c_{ab}T_c$ also belongs to the algebra of $U(2)$, where $d^0_{bc} = 2\delta_{ab}$, $d^a_{b0} = 2$ are the only non-vanishing components.

We chose the 1-form gauge potential of $U(2)$ of the form [39, 40]

$$A = uT_3dt + w(T_2d\theta - \sin\theta T_1 d\phi) + vT_0dt, \tag{5.13}$$

where u, w, v are functions depending only on the radial coordinate r. We consider the metric $G_{\mu\nu}$ and its inverse $G^{\mu\nu}$ of the form

$$G_{\mu\nu} = diag\left(\frac{1}{N}, r^2, r^2\sin^2\theta, -N\right), \quad G^{\mu\nu} = diag\left(N, r^2, r^2\sin^2\theta, -\frac{1}{N}\right), \tag{5.14}$$

respectively, where N is also a function depending only on r. For example, the following set of functions

$$u = u_0 + \frac{\sqrt{q^2+g^2-1}}{r}, \ w = 0, \ v = 0, \ N = 1 - \frac{2M}{r} + \frac{q^2+g^2}{r^2}, \tag{5.15}$$

(u_0 being an arbitrary constant) describes a colored black hole in -sector [39, 40]. The metric $G_{\mu\nu}$ is of Reissner-Nordström type with electric charge q and magnetic charge g [40]. It is the simplest solution of the Einstein-Yang-Mills field equations with a nontrivial gauge field.

Imposing then the variational principle $\delta\widehat{S}_{NCYM} = 0$, we can obtain the noncommutative Yang-Mills field equations and their solutions. However, it is much simpler and equivalent to use the Seiberg-Witten map and determine order by order the noncommutative gauge fields $\widehat{A}_\mu$, the field strength $\widehat{F}_{\mu\nu}$ and the metric $\widehat{G}^{\mu\nu}$.

To end this, we denote the noncommutative quantities of our model by $\widehat{\lambda} = \widehat{\lambda}^a T_a$ (the gauge parameter), $\widehat{A} = \widehat{A}_\mu dx^\mu = \widehat{A}^a_\mu T_a dx^\mu$ (the 1-form gauge potential) and $\widehat{G}^{\mu\nu} = \widehat{G}^{\mu\nu} I$ (the metric), and expand them as formal power series in $\hbar$ (or equivalently in $\theta^{\mu\nu}$)

$$\widehat{\lambda} = \lambda + \hbar\lambda^{(1)} + \hbar^2\lambda^{(2)} + \cdots, \tag{5.16}$$

$$\widehat{A}_\mu = A_\mu + \hbar A^{(1)}_\mu + \hbar^2 A^{(2)}_\mu + \cdots, \tag{5.17}$$

$$\widehat{G}^{\mu\nu} = G^{\mu\nu} + \hbar G^{\mu\nu(1)} + \hbar^2 G^{\mu\nu(2)} + \cdots, \tag{5.18}$$

where the zeroth order terms λ, A_μ and $G^{\mu\nu}$ are the ordinary (commutative) counterparts of $\widehat{\lambda}$, $\widehat{A}_\mu$ and $\widehat{G}^{\mu\nu}$ respectively. Using the Seiberg-Witten map for the noncommutative gauge theory with covariant star product [33], we obtain the following expressions for the first order deformations

$$\lambda^{(1)} = \frac{1}{4}\theta^{\rho\sigma}\{\partial_\rho\lambda, A_\sigma\}, \tag{5.19}$$

$$A^{(1)}_\mu = -\frac{1}{4}\theta^{\rho\sigma}\{A_\rho, \nabla_\sigma A_\mu + F_{\sigma\mu}\}, \tag{5.20}$$

$$G^{\mu\nu(1)} = -\theta^{\rho\sigma}A^0_\rho\nabla_\sigma G^{\mu\nu}. \tag{5.21}$$

Here we mention that the solution (5.10) and the particular form of the parameters $\theta^{\mu\nu}$ introduce in fact three noncommutativity parameters in our model: C_1, C_2, and b. From now on we denote them by: $C_1 = \theta_1$ (of dimension T), $C_2 = \theta_2$ (of dimension LT) and $b = \theta_3$(dimensionless).

The first order deformations of the field strength $\widehat{F}_{\mu\nu}$ can be obtained from the definition (4.9) by using (5.20):

$$F^{(1)}_{\mu\nu} = -\frac{1}{4}\theta^{\rho\sigma}\left(\{A_\rho, \nabla_\sigma F_{\mu\nu} + D_\sigma F_{\mu\nu}\} - 2\{F_{\mu\rho}, F_{\nu\sigma}\}\right), \tag{5.22}$$

where

$$\nabla_\sigma F_{\mu\nu} = \partial_\sigma F_{\mu\nu} - \Gamma^\rho_{\sigma\mu}F_{\rho\nu} - \Gamma^\rho_{\sigma\nu}F_{\mu\rho} \tag{5.23}$$

is the covariant derivative (it concerns the space-time manifold M) and

$$D_\sigma F_{\mu\nu} = \nabla_\sigma F_{\mu\nu} - i[A_\sigma, F_{\mu\nu}] \tag{5.24}$$

is the gauge covariant derivative (it concerns the gauge group $U(2)$).

In particular, for the colored black hole solution (5.15) we obtain

$$A_0^{a(1)} = \left(0, 0, \frac{\theta_1}{2}\sin(2\theta), -\frac{\left(Au^2\right)'}{2A^2}\right) \tag{5.25}$$

$$G^{\mu\nu(1)} = 0, \tag{5.26}$$

$$F_{\mu\nu}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{(u\,u')'T_0}{A} \\ 0 & 0 & \theta_1 \sin(2\theta)\,T_0 & 0 \\ 0 & -\,\theta_1\sin(2\theta)\,T_0 & 0 & 0 \\ \frac{(u\,u')'T_0}{A} & 0 & 0 & 0 \end{pmatrix} \tag{5.27}$$

Here we denoted the derivative of $u(r)$ with respect to r by u'. In the case when $v(r) \neq 0$, i.e. the $U(1)$ - sector is not empty, we obtain that the first order deformation $G^{\mu\nu(1)} \neq 0$. In particular, we can use the solution (5.15) for the colored black hole, or other solutions with quantum numbers $\overrightarrow{n} = (n_1, n_2)$, to obtain the corresponding first order deformations in (5.25)–(5.27). All these results can be also extended to the higher order deformations by using the "covariant" Seiberg-Witten map [33].

Taking into account the solution (5.10) for $A(r)$, we can verify that all these first order deformations vanish if the noncommutativity parameters θ_1, θ_2, $\theta_3 \to 0$. However, this limit cannot be achieved because the symplectic structure of the space-time M imposes the condition $\det\left(\theta^{\mu\nu}\right) \neq 0$. But we can discover the commutative limit considering $\hbar \to 0$.

Finally, we mention that colored black holes and their generalizations with rotation and cosmological term, as well as solutions with cylindrical and plane symmetries have been also obtained [40]. It would be of interest to extend these results to a noncommutative gauge theory by using the above formalism with covariant $\star$ - product.

6 Conclusions and discussions

We constructed a model of noncommutative internal gauge theory by using a $\star-$ product between Lie algebra valued differential forms defined on a Poisson manifold. We followed the same way as in our recent paper [24]. To simplify the calculations, we considered a space-time endowed only with torsion. We have showed that, in order to satisfy the restrictions imposed by the associativity property of the covariant star product, the torsion of the space-time has to be covariant constant, $\nabla_\mu T^\nu_{\rho\sigma} = 0$. On the other hand, we argued that a covariant star product defined in the case when the space-time is a symplectic manifold endowed only with curvature is not possible. This is due to the restrictions imposed by the associativity property of the covariant star product which requires also the vanishing curvature. The corresponding connection is therefore flat symplectic and this reduces the applicability area of the covariant star product.

An illustrative example has been presented starting from the commutative Poincaré gauge theory. Using the gauge fields e^a_μ and fixing the gauge $\omega^{ab}_\mu = 0$ [38], we defined the non-symmetric connection $\Gamma^\rho_{\mu\nu} = \bar{e}^\rho_a \partial_\nu e^a_\mu$. We deduced that, in this case, the conditions necessary to define a covariant star product on a symplectic manifold M completely determine its connection.

Some other possibilities of applying this covariant star product have been also analyzed. *First*, it will be very important to generalize the Seiberg-Witten map to the case when the ordinary derivatives are replaced with covariant derivatives and the Moyal star product is the covariant one. *Second*, we can try to develop a noncommutative gauge theory of gravity considering the symplectic manifold M as the background space-time. For such a purpose, we have to verify if the non-commutative field equations do not impose too many restrictive conditions on the connection $\Gamma^\rho_{\mu\nu}$, in addition to those required by the existence of the covariant star product. However, it remain unsolved the problem of the gauge group which we can choose. The Poincaré group can not be used because it does not close with respect to star product. A possibility will be to choose the group $GL(2, C)$, but in this case we obtain a complex theory of gravitation [41, 42]. Another possibility is to consider the universal enveloping of Poincaré group, but this is infinite dimensional and we must find criteria to reduce the number of the freedom degrees to a finite one. Some possible ideas are given for the case of SU(N) or *GUT* theories in Ref. [43], where it is argued that the infinite number of parameters can in fact all expressed in terms of right number of classical parameters and fields via the Seiberg-Witten maps.

A similar aim of applying covariant $\star-$ products has been pursued in a recent paper [44], though with a different approach. Specifically, there are studied covariant $\star-$ products on spaces of tensor fields defined over a Fedosov manifold with a given symplectic structure and a given flat torsionless symplectic connection.

Acknowledgments. The author acknowledges the support by the CNCSIS-UEFISCSU grant 620 of the Minister of Education, Research and Youth of Romania.

References

[1] A. Connes, Noncommutative Geometry, Academic Press, San Diego CA, 1994

[2] G. Landi, An introduction to noncommutative spaces and their geometry, Springer-Verlag, Berlin Heidelberg, 1997.

[3] J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa, Elements of noncommutative geometry, Birkhäuser Boston, 2001.

[4] R. J. Szabo, Quantum gravity, field theory and signatures of noncommutative spacetime, Gen. Relativ. Gravit. 42, 1-29 (2010), hep-th/0906.2913.

[5] F. Lizzi, The structure of space-time and noncommutative geometry, hep-th/0811.0268.

[6] V. O. Rivelles, Noncommutative field theories and gravity, Phys. Lett. B558 (2003) 191-196, hep-th/0212262.

[7] D. V. Vassilevich, Diffeomorphism covariant star products and noncommutative gravity, Class. Quant. Grav. 26 (2009) 145010, hep-th/0904.3079.

[8] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, A gravity theory on noncommutative spaces, Class. Quant. Grav. 22 (2005) 3511-3532, hep-th/0504183.

[9] P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Noncommutative geometry and gravity, Class. Quant. Grav. 23 (2006) 1883-1912, hep-th/0510059.

[10] J. Polchinschi, String theory, Cambridge University Press, 1998.

[11] C. Rovelli, Quantum gravity, Cambridge University Press, 2004.

[12] N. Seiberg, E. Witten, String theory and noncommutative geometry, JHEP 09 (1999) 032, hep-th/9908142.

[13] L. Álvarez-Gaumé, F. Meyer, M. A. Vázquez-Mozo, Comments on noncommutative gravity, Nucl. Phys. B753 (2006) 92.

[14] M. Chaichian, A. Tureanu, R. B. Zhang, X. Zhang, Riemannian geometry of noncommutative surfaces, J. Math. Phys. 49 (2008) 07351, hep-th/0612128.

[15] D. Wang, R. B. Zhang, X. Zhang, Quantum deformations of the Schwarzschild and Schwarzschild-de Sitter spacetimes, Class. Quant. Grav. 26 (2009) 085014, hep-th/0809.0614.

[16] R. B. Zhang, X. Zhang, Projective module description of embedded noncommutative spaces, hep-th/0810.2357.

[17] G. Dito, D. Sternheimer, Deformation quantization: Genesis, Developments and metamorphoses, Proceedings of the meeting between mathematicians and theoretical physicists, Strasbourg, 2001, IRMA Lectures in Math. Theoret. Phys., vol. 1, Walter De Gruyter, Berlin 2002, math.QA/0201168.

[18] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization. Deformations of symplectic structures, Annals Phys. 111 (1978) 61

[19] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Mat. Phys. 66 (2003) 157, q-alg/9709040.

[20] A. S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000) 591, math.QA/9902090.

[21] M. Ammar, V. Chloup, S. Gutt, Universal star products, Lett. Math. Phys. 84 (2008) 199, math.SG/0804.1300.

[22] S. McCurdy, B. Zumino, Covariant star product for exterior differential forms on symplectic manifolds, hep-th/0910.0459.

[23] A. Tagliaferro, A star product for differential forms on symplectic manifolds, hep-th/0809.4717.

[24] M. Chaichian, A. Tureanu, G. Zet, Gauge field theories with covariant star product, JHEP 07 (2009) 084, hep-th/0905.0608.

[25] M. Chaichian, M. Oksanen, A. Tureanu, G. Zet, Noncommutative gauge theory using covariant star product defined between Lie valued differential forms, Phys. Rev. D81 (2010) 085026, hep-th/1001.0508

[26] M. Chaichian, M. Oksanen, A. Tureanu, G. Zet, Covariant star product on symplectic and Poisson spacetime manifolds, math-ph/0503

[27] J. Lee, H. S. Yang, Quantum gravity from noncommutative space-time, hep-th/1004.0745.

[28] H. Grosse, H. Steinacker, M. Wohlgenannt, Emergent gravity, matrix model UV/IR mixing, JHEP 04 (2008) 083, hep-th/0802.0973.

[29] C. Barcelo, S. Liberati, M. Visser, Analogue Gravity, Living Reviews in General Relativity 8 (2005) 12, gr-qc/0505065.

[30] C.-S Chu, P.-M. Ho, Poisson algebra of differential forms, Int. J. Mod. Phys. 12 (1997) 5573, q-alg/9612031.

[31] H. García-Compeán, O. Obregón, C. Ramírez, M. Sabido, Noncommutative self-dual gravity, Phys. Rev. D68 (2003) 044015, hep-th/0302180.

[32] B. Jurco, L. Möller, S. Schraml, P. Schupp, J. Wess, Construction of non-Abelian gauge theories on noncommutative spaces, Eur.Phys.J. C21 (2001) 383, hep-th/0104153.

[33] M. Chaichian, A. Tureanu, G. Zet, The Seiberg-Witten map with covariant star product (in preparation).

[34] P.-M Ho, S.-P Miao, Noncommutative differential calculus for D-brane in non-constant B field background, Phys. Rev. D64 (2001) 126002, hep-th/0105191.

[35] P.-M. Ho, Y.-T. Yeh, Noncommutative D-brane in non-constant NS-NS B field background, Phys. Rev. Lett. 85 (200) 5523, hep-th/0005159.

[36] A. S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000) 591, math.qa/9902090.

[37] L. Cieri, F. A. Schaposnik, The dyon charge in noncommutative gauge theories, Res. Lett. Phys. (2008) 890916, hep-th/0706.0449.

[38] G. Zet, Schwarzschild solution on a space-time with torsion, gr-qc/0308078.

[39] G. Zet, Self-duality equations for spherically symmetric gauge fields, Eur. Phys. J. A15 (2002) 405, hep-th/0201212.

[40] M. S. Volkov, D. V. Gal'tsov, Gravitating non-Abelian solitons and black holes with Yang-Mills fields, Phys. Rept. 319 (1999) 1, hep-th/9810070.

[41] A. Stern, Particle classification and dynamics in GL(2,C) gravity, Phys. Rev. D79 (2009) 105017, hep-th/0903.0882.

[42] A. Chamseddine, SL(2,C) gravity with complex vierbein and its noncommutative extension, Phys. Rev. D69 (2004) 024015, hep-th/0309166.

[43] X. Calmet, B. Jurco, P. Schupp, J. Wess, M. Wohlgenannt, The standard model on noncommutative spaetime, Eur. Phys. J. C23 (2002) 363-376, hep-th/0111115.

[44] D. V. Vassilevich, Tensor calculus on noncommutative spaces, hep-th/1001.0766