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The Many Faces of Singletons

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Abstract

Singletons are those unitary irreducible modules of the Poincaré or (anti) de Sitter group that can be lifted to unitary modules of the conformal group. They appear in a wide variety of areas of theoretical physics: AdS/CFT correspondence, higher-spin multiplets, infinite-component Majorana equations, etc. Singletons are reviewed through a list of their many equivalent definitions in order to approach them from various perspectives.

1 Plan of a singleton sightseeing tour

The celebrated singletons are rather "remarkable representations", as coined by Dirac in his seminal paper [1] on the subject. Indeed, these representations of the anti de Sitter spacetime isometry group possess several surprising properties which are so exceptional that they distinguish singletons from all other such representations. Several of these properties are reviewed here, thereby providing an elementary introduction to singletons through a list, presumably inexhaustive, of their distinct but equivalent definitions. Exhibiting the many faces of singletons could give some flavor of their ubiquitous appearances in such seemingly unrelated areas of mathematical physics as the AdS/CFT correspondence, the hydrogen atom spectrum, the infinite-component Majorana equation, the electric-magnetic duality, etc. An exhaustive bibliographical survey of the wide range of results and applications for singletons is by no means attempted here.[1] On the contrary the main focus of this short introduction is on the symmetries of singletons in any dimension and on their manifest realizations. No prior knowledge of singletons is assumed, but some familiarity with the representation theory of Lie algebras is welcome. The plan is as follows:

In order to be as self-contained as possible, the isometry groups of the anti de Sitter spacetime and its conformal boundary are quickly reviewed in Section 2, as well as the corresponding representation theory classifying the elementary particles that may live on these spaces. Then comes the section 3 which presents many faces of singletons: lowest weight modules (subsection 3.1), multiplicity free modules (subsect 3.2), irreducible modules of isometry subalgebras (subsect 3.3), fields on the conformal boundary (subsect 3.4), fields on the ambient space (subsect 3.5) and kernels of the Howe dual algebra (subsect 3.6). The simplest example of singleton is the scalar one and it will serve throughout this review as a useful illustration.

[1] The bibliography has been deliberately focused either on some recent general reviews with indications of the precise location of the relevant information, or on some old seminal papers, in order to give some flavor of the early history though from a modern viewpoint. I do apologize to the experts for the incompleteness of the bibliography.

2 Elementary particles on anti de Sitter spacetime

2.1 Anti de Sitter spacetime

The most transparent realization of AdS_{n+1} ($n \geqslant 1$) is via a global isometric embedding in a flat ambient space:

- The **ambient space** $\mathbb{R}^{n,2}$ is endowed with the
  - Cartesian coordinates X^A (where $A = 0, 0', 1, 2, \dots, n$) and
  - ("mostly plus") metric $\eta_{AB} = \mathrm{diag}(-1, -1, +1, +1, \dots, +1)$.
- The **anti de Sitter spacetime** AdS_{n+1} is the codimension one quadric (more precisely, a one-sheeted hyperboloid)
$$\eta_{AB} X^A X^B = -R^2 \,,$$
where $R > 0$ is the curvature radius, endowed with the induced metric.

So the **isometry algebra** is manifestly the real Lie algebra

$$\mathfrak{o}(n,2) = \mathrm{span}_{\mathbb{R}}\{\mathrm{J}_{AB}\}$$

which can be presented

- by its generators (the ambient "angular momenta")
$$\mathrm{J}_{AB} = -\mathrm{J}_{BA} \quad (\text{where } A, B = 0, 0', 1, 2, \dots, n).$$
- modulo the commutation relations
$$[\mathrm{J}_{AB}, \mathrm{J}_{CD}] = i\,\eta_{BC}\mathrm{J}_{AD} + \text{antisymetrizations}\,.$$

and is **linearly realized** on $\mathbb{R}^{n,2}$ through the generators (the ambient "orbital angular momenta")

$$\mathrm{J}_{AB} = \mathrm{X}_A \mathrm{P}_B - \mathrm{X}_B \mathrm{P}_A$$

where
$$\mathrm{P}_A = -i\,\frac{\partial}{\partial X^A}.$$

Usually, one of the timelike direction of $\mathbb{R}^{n,2}$, say $0'$, is particularized. Equivalently, one of the points of AdS_{n+1}, say of coordinates $X^{0'} = R$ and $X^a = 0$ (where $a = 0, 1, 2, \dots, n$), is particularized. Then the generators decompose in two sets:

- The stabilizer of $X^a = 0$, i.e. the **Lorentz subalgebra**
$$\mathfrak{o}(n,1) = \mathrm{span}_{\mathbb{C}}\{\mathrm{J}_{ab}\}$$
which can be presented
  - by its generators $\mathrm{J}_{ab} = -\mathrm{J}_{ba}$ (where $a, b = 0, 1, 2, \dots, n$)
  - modulo the commutation relations
$$[\mathrm{J}_{ab}, \mathrm{J}_{cd}] = i\,\eta_{bc}\mathrm{J}_{ad} + \text{antisymetrizations}\,.$$
- The **transvections** (the displacements) generated by $\Gamma_a := R\,\mathrm{J}_{0'a}$ and satisfying the commutation relations
$$[\Gamma_a, \Gamma_b] = \frac{i}{R^2}\,\mathrm{J}_{ab}\,.$$

2.2 Conformal boundary

2.2.1 Conformal isometries of Minkowski spacetime

A **conformal metric** is an equivalence class of a metric under the equivalence relation

$$g'_{\mu\nu}(x) \sim \Omega(x) g_{\mu\nu}(x) \qquad (\text{where } \mu,\nu = 0,1,2,\ldots,n-1)$$

with $\Omega(x) > 0$ for all x^μ while a **conformal isometry** is a diffeomorphism such that the metric transforms as

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)\,.$$

In other words, a (conformal) isometry is a diffeomorphism that preserves the (conformal) metric.

From now on, one will restrict to the case $n \geqslant 3$. In such case, the finite conformal isometries of the **Minkowski spacetime** $\mathbb{R}^{n-1,1}$ are generated (see e.g. [2] for a proof) by the:

- **Lorentz transformations** $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\Lambda \in O(n-1,1)$,
- **Translations**: $x'^\mu = x^\mu + a^\mu$, where $a \in \mathbb{R}^{n-1,1}$,
- **Dilatations**: $x'^\mu = \lambda\, x^\mu$, where $\lambda \in \mathbb{R}_0 := \mathbb{R} - \{0\}$,

together with, either the:

- **Special conformal transformations**: $x'^\mu = \frac{x^\mu + x^2 b^\mu}{1+2b_\mu x^\mu + b^2 x^2}$,

  where $b \in \mathbb{R}^{n-1,1}$,

or the:

- **Inversion**: $x'^\mu = \frac{x^\mu}{x^2}$.

The special conformal transformations are the most difficult to visualize but they may be understood indirectly from the following property: *Special conformal transformations are conjugate to translations, via the inversion.*

The algebra of the infinitesimal conformal isometries of $\mathbb{R}^{n-1,1}$ is $\mathfrak{o}(n,2)$ but this is far from obvious in terms of the Cartesian coordinates x^μ. Moreover, the last two transformations (special conformal transformations and inversions) are not well defined everywhere on $\mathbb{R}^{n-1,1}$ because they map some points "at infinity" (when the denominator vanish). In order to make the conformal isometries well defined globally, it is necessary to complete the Minkowski spacetime $\mathbb{R}^{n-1,1}$ by adding "points at infinity". The corresponding conformal compactification of $\mathbb{R}^{n-1,1}$ can be identified with the conformal boundary of AdS_{n+1}.

2.2.2 Conformal boundary of anti de Sitter spacetime

The most transparent realization of the conformal boundary ∂AdS_{n+1} of the anti de Sitter spacetime AdS_{n+1} is via its global (conformal isometric) embedding in the projectivization of the ambient space $\mathbb{R}^{n,2}$:

- The ambient space is now the projective space $\mathbb{P}(\mathbb{R}^{n,2}) \cong \mathbb{RP}^{n+1}$ endowed with the
  - **Homogeneous coordinates** X^A (where $A = 0, 0', 1, 2, \ldots, n$),
  - Equivalence relation $X^A \sim \lambda\, X^A$ (for any $\lambda \in \mathbb{R}_0$)
  - Conformal metric (i.e. the equivalence class of) η_{AB}

  As usual, points of the projective space $\mathbb{P}(\mathbb{R}^{n,2})$ are rays of $\mathbb{R}^{n,2}$.

- The **Dirac hypercone**[2] is the codimension one quadric (null cone)

$$\eta_{AB}X^A X^B \,=\, 0$$

quotiented by the equivalence relation, endowed with the induced conformal metric. This conformal space is the conformal boundary of the anti de Sitter spacetime AdS_{n+1}. Geometrically, the points of ∂AdS_{n+1} are null rays of the ambient space $\mathbb{R}^{n,2}$. Heuristically, the boundary of the anti de Sitter spacetime is the asymptotic (i.e. located "at infinity") region of intersection between the hyperboloid and the hypercone.

So the **conformal isometry algebra** of the conformal boundary ∂AdS_{n+1} is manifestly the real Lie algebra $\mathfrak{o}(n,2)$ linearly realized on $\mathbb{P}(\mathbb{R}^{n,2})$ through the ("ambient orbital angular momentum") generators and is linearly realized on $\mathbb{R}^{n,2}$ through the generators $\mathrm{J}_{AB}=\mathrm{X}_{[A}\mathrm{P}_{B]}$, where the square bracket denotes the antisymmetrization.

2.2.3 Conformal isometries revisited

The **light-cone coordinates**

$$X^{\pm}:=X^{0'}\pm X^n$$

together with the **inhomogeneous coordinates**

$$x^\mu:=X^\mu/X^- \qquad (\text{where } \mu=0,1,2,\dots,n-1)$$

provide a convenient parametrization of the Dirac hypercone in a neighborhood such that $X^-\neq 0$. The hyperplane $X^-=0$ may be taken as the "hyperplane at infinity" to be added to the affine space $\mathbb{R}^{n+1}$ in order to construct $\mathbb{RP}^{n+1}$. If one identifies the conformal boundary of AdS_{n+1} with the conformal compactification of $\mathbb{R}^{n-1,1}$ then the "hyperplane at infinity" is indeed particularized.

The conformal isometries decompose as follows (see e.g. [4] for a short review):

- The ambient boosts $X'^A=\Lambda^A{}_B X^B$ preserving the hyperplane $X^-=\text{constant}\neq 0$, i.e. the
  - Lorentz transformations:

$$X'^+=X^+,\ X'^\mu=\Lambda^\mu{}_\nu X^\nu \quad\Longleftrightarrow\quad x'^\mu=\Lambda^\mu{}_\nu x^\nu$$

    where $\Lambda\in O(n-1,1)$.
  - Translations:

$$X'^+=a_\mu X^\mu,\ X'^\mu=X^\mu+a^\mu X^- \quad\Longleftrightarrow\quad x'^\mu=x^\mu+a^\mu$$

- The ambient boosts in the plane $0'n\leftrightarrow +-$ which preserve the hyperplane at infinity $X^-=0$, i.e. the dilatations:

$$X'^+=\lambda\,X^+,\ X'^-=\lambda^{-1}\,X^-,\ X'^\mu=X^\mu \quad\Longleftrightarrow\quad x'^\mu=\lambda x^\mu$$

- The remaining transformations, i.e. the special conformal transformations:

$$X'^+=X^+,\, X'^-=b_\mu X^\mu,\, X'^\mu=X^\mu+b^\mu X^+ \Longleftrightarrow x'^\mu=\frac{x^\mu+x^2b^\mu}{1+2b_\mu x^\mu+b^2x^2}$$

- The reflection through the hyperplane $X^n=0$, i.e. the inversion:

$$X'^+=X^-,\ X'^-=X^+,\ X'^\mu=X^\mu \quad\Longleftrightarrow\quad x'^\mu=\frac{x^\mu}{x^2}$$

[2] This construction is the Euclidean analogue of the "Möbius model" in the mathematical literature. It was introduced a long time ago in physics by Dirac in a paper [3] which still remains a splendid introduction to the ambient formulation.

So the infinitesimal generators decompose as follows:

- The stabilizer of any hyperplane $X^- =$constant$\neq 0$, i.e. the **Poincaré subalgebra**
$$\mathfrak{io}(n,1) = \text{span}_{\mathbb{R}}\{\mathrm{P}_\mu, \mathrm{J}_{\mu\nu}\} = \mathbb{R}^d \ni\!\!\!\!\supset \mathfrak{o}(n-1,1)$$
which can be presented
  - by its generators
$$\mathrm{P}_\mu := \mathrm{J}_{+\mu}/2\,, \quad \mathrm{J}_{\mu\nu} = -\mathrm{J}_{\nu\mu} \quad (\mu,\nu = 0,1,2,\dots,n-1)$$
  - modulo the commutation relations
$$\begin{aligned}[\mathrm{P}_\mu, \mathrm{P}_\nu] &= 0\,,\\ [\mathrm{P}_\mu, \mathrm{J}_{\nu\rho}] &= i\,\eta_{\mu\nu}\mathrm{P}_\rho + \text{antisymetrization}\,,\\ [\mathrm{J}_{\mu\nu}, \mathrm{J}_{\rho\sigma}] &= i\,\eta_{\nu\rho}\mathrm{J}_{\mu\sigma} + \text{antisymetrizations}\,.\end{aligned}$$
- The generator of ambient boosts in the plane $0'n \leftrightarrow +-$ which preserve the hyperplane at infinity, i.e. the generator of dilatation
$$\Delta := \mathrm{J}_{+-}$$
- The remaining generators, corresponding to the infinitesimal special conformal transformations
$$\mathrm{K}_\mu := \mathrm{J}_{-\mu}$$

2.2.4 Distinct constant curvature spacetimes as an identical conformal space

Actually, the conformal boundary of AdS_{n+1} may be identified with any of the three constant curvature spacetimes (supplemented by "points at infinity"). These three spacetimes are geometrically realized as quadrics obtained by intersecting the hypercone with an affine hyperplane:

- **Minkowski spacetime** $\mathbb{R}^{n-1,1}$ endowed with the Cartesian coordinates x^μ as before.
  - **Paraboloid**: intersection with a hyperplane orthogonal to a light-like direction, say $X^- =$ constant$\neq 0$
  - Isometry algebra: Poincaré algebra $\mathfrak{io}(\mathbf{n-1,1})$
- **de Sitter spacetime** dS_n
  - **Hyperboloid**: intersection with a hyperplane orthogonal to a time-like direction, say $X^{0'} = R \neq 0$
  - Isometry algebra: $\mathfrak{o}(\mathbf{n,1})$
- **Anti de Sitter spacetime** AdS_n
  - **Hyperboloid**: intersection with a hyperplane orthogonal to a space-like direction, say $X^n = R \neq 0$
  - Isometry algebra: $\mathfrak{o}(\mathbf{n-1,2})$

The conformal compactifications of the three distinct constant curvature spacetimes $\mathbb{R}^{n-1,1}$, dS_n and AdS_n are identical: they reproduce the flat conformal space ∂AdS_{n+1}. Indeed, all of these spacetimes are conformally flat and they possess the same conformal isometry algebra $\mathfrak{o}(n,2)$. It is important to emphasize this point because, although most of the time the conformal boundary ∂AdS_{n+1} is identified with the conformal compactification of Minkowski spacetime $\mathbb{R}^{n-1,1}$, from the point of view of conformal geometry it can equivalently be taken to be the conformal compactification of (anti) de Sitter spacetime $(A)dS_n$. This remark is useful because most results which will be mentioned here equally apply to all constantly curved spacetimes.

2.3 Unitary irreducible representations

2.3.1 Elementary particles as irreducible modules

Let $\mathcal{M}$ be a maximally symmetric spacetime. A celebrated insight due to Wigner (see e.g. the section 2 of [5] for a review) states that there exists a one-to-one correspondence between the set of equivalence classes of

(i) unitary representation on a unitary module ("representation space") $\mathcal{H}$ of the isometry group of $\mathcal{M}$ and

(ii) linear relativistic wave equations describing the free propagation of a particle on $\mathcal{M}$.

The unitary module $\mathcal{H}$ is the Hilbert space of physical states (i.e. of inequivalent solutions of the wave equation). This identification allows to define an **elementary particle** as an irreducible unitary module of the isometry group of the spacetime $\mathcal{M}$.

So the classification of the free elementary particles in the anti de Sitter spacetime $\mathcal{M} = AdS_{n+1}$ tantamounts to the classification of the irreducible unitary modules of $\mathfrak{o}(n, 2)$. Strictly speaking, the Hilbert space of physical states is usually the direct sum of two irreducible modules: the ones with either positive or negative energy corresponding repsectively to the particle and its antiparticle. Therefore, if the sign of the energy unspecified in the sequel, then the direct sum of the positive energy module and its conjugate should be understood.

2.3.2 Isometry algebra of anti de Sitter spacetime

The **maximal compact subalgebra** $\mathfrak{o}(2) \oplus \mathfrak{o}(n)$ of the real Lie algebra $\mathfrak{o}(n, 2)$ corresponds to the

- **time translations** generated by the **Hamiltonian** $\mathrm{E} = \mathrm{M}_{0'0}$
- **spatial rotations** generated by J_{ij} (where $i, j = 1, 2, \ldots, n$)

The remaining generators can be recast in the form of **ladder operators**

$$\mathrm{J}^{\pm}_j = \mathrm{J}_{0j} \mp i\mathrm{J}_{0'j}\,,$$

raising or lowering the **energy** (= eigenvalue of E) by one unit. Indeed, the real Lie algebra $\mathfrak{o}(n, 2)$ can be presented equivalently

- by the generators E, $\mathrm{J}^{\pm}_i$, J_{jk} (where $i, j, k = 1, 2, \ldots, n$)
- modulo the commutation relations

$$\begin{aligned}
&\big[\mathrm{E}, \mathrm{J}^{\pm}_i\big] = \pm\mathrm{J}^{\pm}_i \quad ; \quad \big[\mathrm{J}_{ij}, \mathrm{J}^{\pm}_k\big] = 2i\delta_{k[j}\mathrm{J}^{\pm}_{i]}\\
&\Big[\mathrm{J}^{-}_i, \mathrm{J}^{+}_j\Big] = 2(i\mathrm{J}_{ij} + \delta_{ij}\mathrm{E})\\
&[\mathrm{J}_{ij}, \mathrm{J}_{kl}] = i\delta_{jk}\mathrm{J}_{il} + \text{antisymetrizations}
\end{aligned}$$

2.3.3 Orthogonal algebra

The classification of the irreducible unitary modules of $\mathfrak{o}(n, 2)$ requires the knowledge of the classification of the irreducible unitary modules of $\mathfrak{o}(n)$, which can be summarized as follows (see e.g. the section 3 of [5] for more details and references):

Irreducible unitary modules of the orthogonal group: *Let $n \geqslant 3$ be a positive integer and $[\frac{n}{2}]$ denote the integer part of $\frac{n}{2}$. Any unitary irreducible module $\mathcal{D}_{\ell_1,\ldots,\ell_{[\frac{n}{2}]}}$ of $O(n)$ is a finite-dimensional highest (and lowest) weight module which is*

- *either tensorial or spinorial,*

- *labeled by a partition of an integer*

$$|\ell| = \ell_1 + \ell_2 + \ldots + \ell_{[\frac{n}{2}]}$$

in $[\frac{n}{2}]$ *parts (where* $\ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_{[\frac{n}{2}]} \geqslant 0$*).*

The converse is also true.

A partition of $|\ell|$ in p parts is usually depicted as a **Young diagram** made of $|\ell|$ boxes arranged in p left-justified rows of non-increasing lengths

$$\ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_p \geqslant 0\,.$$

A renowned example of $\mathfrak{o}(n)$-module is the **spin-s module** $\mathcal{D}_s$ corresponding to a partition of $|\ell| = s \in \mathbb{N}$ in one part corresponds either to a

- tensorial module $\mathcal{D}_s$ of $\mathfrak{o}(n)$ spanned by the components of a symmetric traceless tensor of rank s, or a
- spinorial module $\mathcal{D}_{s+1/2}$ of $\mathfrak{o}(n)$ spanned by the components of a symmetric (gamma)-traceless tensor-spinor of rank s.

2.3.4 Irreducible modules of the isometry algebra

A **Verma module** $\mathcal{V}(E_0;\ell_1,\ldots,\ell_{[\frac{n}{2}]})$ of $\mathfrak{o}(n,2)$ for E_0 positive (or negative)

- is obtained by the action of the **universal enveloping algebra** $\mathcal{U}\big(\mathfrak{o}(n,2)\big)$ on the
- **Lowest (or highest) weight vector** $|E_0;\ell_1,\ldots,\ell_{[\frac{n}{2}]}\rangle$ of $\mathfrak{o}(n,2)$, which is defined as a
  - Lowest (or highest) energy E_0 state

$$\mathrm{J}^-_i\,|E_0;\ell_1,\ldots,\ell_{[\frac{n}{2}]}\rangle \;=\; 0$$

  - Lowest (or highest) weight vector of $\mathfrak{o}(n)$ labeled by

$$\ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_{[\frac{n}{2}]} \geqslant 0.$$

For the general definitions and properties of Verma modules, the reader may look for instance at the concise review in [6]. The ground states of energy E_0 span an irreducible finite-dimensional $\mathfrak{o}(n)$-module labelled by the above partition.

For physical reasons, an elementary particle on anti de Sitter spacetime is taken to be a positive-energy (lowest weight) unitary module, while its antiparticle is its opposite counterpart, so a negative-energy (highest weight) unitary module. Both cases can be described as **extremal weight** unitary modules.

Irreducible unitary modules of the isometry algebra: *Any extremal weight irreducible module* $\mathcal{D}(E_0;\ell_1,\ldots,\ell_{[\frac{n}{2}]})$ *of* $\mathfrak{o}(n,2)$ *is a quotient of the Verma module* $\mathcal{V}(E_0;\ell_1,\ldots,\ell_{[\frac{n}{2}]})$ *by its maximal submodule* $\mathcal{V}(E'_0;\ell'_1,\ldots,\ell'_{[\frac{n}{2}]})$.

Unitarity imposes some restrictions (see e.g. [7] and refs therein) on the possible values of the extremal energy when the ground state $\mathfrak{o}(n)$-module carries the

- trivial representation ($\ell_1 = \ell_2 = \ldots = \ell_{[\frac{n}{2}]} = 0$): either
  - $E_0 = 0$, corresponding to the trivial $\mathfrak{o}(n,2)$-module, or
  - $|E_0| \geqslant \frac{n}{2} - 1$ corresponding to scalar field $\mathfrak{o}(n,2)$-modules,
- spinor representation ($\ell_1 = \ell_2 = \ldots = \ell_{[\frac{n}{2}]} = 0$):

* $|E_0| \geqslant \frac{n-1}{2}$ corresponding to the spinor field $\mathfrak{o}(n,2)$-modules,

- generalized spin-s representation, labeled by a Young diagram with a upper rectangle made of k rows with length $[s]$ (i.e. a partition such that $[s] = \ell_1 = \ell_2 = \ldots = \ell_k > \ell_{k+1}$ with $1 \leqslant k \leqslant [n/2]$):

  * $|E_0| \geqslant s + n - k - 1$ corresponding to tensorial (or spinorial) modules whether $2s$ is even (or odd).

For more details on the $n = 3$ unitary irreducible modules, an excellent pedagogical introduction is [8]. For more details on the generic construction of $\mathfrak{o}(n,2)$ unitary irreducible modules, one may look at the appendices in [9].

3 Singletons: various definitions

3.1 Singletons as lowest weight modules

Singletons form the very exceptional subclass of the irreducible unitary modules of the algebra $\mathfrak{o}(n,2)$ that saturate the unitary bound and whose ground states are caracterized by a rectangular Young diagram of height $k = \frac{n}{2}$ (when n is even). The group-theoretic definition of singletons dates back to the seminal works of Dirac [1] and Flato & Frønsdal [10]. The higher-dimensional generalization was further developped by a variety of authors (a self-contained and rather complete treatment of the generic case can be found in [11]):

Definition: *A positive-energy* ***singleton of*** $\mathbf{AdS_{n+1}}$ *is a lowest weight unitary irreducible module of* $\mathfrak{o}(n,2)$ *such that:*

- *When n is odd, the spin is*
  - *either $s = \frac{1}{2}$: $\mathcal{D}(\frac{n-1}{2};\frac{1}{2})$ called* ***Di*** *or* ***spinor singleton***
  - *or $s = 0$: $\mathcal{D}(\frac{n}{2}-1;0)$ called* ***Rac*** *or* ***scalar singleton***
- *When n is even, the (generalized) spin s is*
  - *any (half)integer: $\mathcal{D}(s+\frac{n}{2}-1\,;\,[s],\ldots,[s])$ called* ***spin s singleton*** *and labeled by a partition in $\frac{n}{2}$ equal parts, i.e. a rectangular Young diagram made of $\frac{n}{2}$ rows of length $[s]$.*

From now on, whenever singletons of spin $s \geqslant 1$ will be mentioned, the integer n will always be implicitly assumed to be even, as it should.

As a nice illustration, the scalar singleton may deserve a more detailed discussion. A short review of scalar singletons can be found in [12]. The scalar singleton corresponds to the case of a lowest weight vector $|E_0;0\rangle$ of $\mathfrak{o}(n,2)$ annihilated by all generators of the $\mathfrak{o}(n)$ subalgebra, except the energy:

$$(\mathrm{E}-E_0)|E_0;0\rangle = 0 \quad , \quad \mathrm{J}_{ij}|E_0;0\rangle = 0 \quad , \quad \mathrm{J}^-_i|E_0;0\rangle = 0\,.$$

Thus the Verma module is

$$\mathcal{V}(E_0;0) = \mathrm{span}_{\mathbb{R}}\left\{\mathrm{J}^+_{i_1}\ldots\mathrm{J}^+_{i_s}|E_0;0\rangle \;\mid s\in\mathbb{N}\right\}$$

It can be shown (see e.g. [8] for $n = 3$) that unitarity implies $E_0 \geqslant \frac{n}{2}-1$ (or $E_0 = 0$ which corresponds to the trivial representation of $\mathfrak{o}(n,2)$). For the special value $E_0 = \frac{n}{2}-1$ saturating the unitarity bound, the vector $\delta^{ij}\mathrm{J}^+_i\mathrm{J}^+_j|E_0;0\rangle$ is a primitive null vector. The scalar singleton is the unitary module obtained by quotienting the maximal submodule

$$\mathcal{V}\Big(\frac{n}{2}+1;0\Big) \cong \mathrm{span}_{\mathbb{R}}\left\{\delta^{i_1i_2}\mathrm{J}^+_{i_1}\ldots\mathrm{J}^+_{i_s}|E_0;0\rangle \;\mid s\in\mathbb{N}\right\} \subset \mathcal{V}\Big(\frac{n}{2}-1;0\Big)$$

from the Verma module. Concretely, this corresponds to factoring out the trace terms from the Verma module:

$$\mathcal{D}\Big(\frac{n}{2}-1;0\Big)\cong$$
$$\mathrm{span}_{\mathbb{R}}\,\big\{\,\mathrm{J}^+_{i_1}\dots\mathrm{J}^+_{i_s}|n/2-1;0\rangle\;\;|\;\;\delta^{i_1i_2}\mathrm{J}^+_{i_1}\dots\mathrm{J}^+_{i_s}|n/2-1;0\rangle\sim 0\,\big\}$$

3.2 Singletons as multiplicity free modules

A nice corollary of the previous description is that under the restriction of $\mathfrak{o}(n,2)$ to its subalgebra $\mathfrak{o}(n)$, the scalar singleton module is decomposable as follows

$$\mathcal{D}\Big(\frac{n}{2}-1;0\Big)\Big|_{\mathfrak{o}(n)}\cong\bigoplus_{s\in\mathbb{N}}\mathcal{D}_s$$

where each irreducible $\mathfrak{o}(n)$-module $\mathcal{D}_s$ is an eigenspace of distinct energy

$$\Big(\mathrm{E}-\big(s+\frac{n}{2}-1\big)\Big)\mathcal{D}_s=0$$

and appears with multiplicity one. Actually, the terminology "singleton" originates[3] from this absence of degeneracy of $\mathfrak{o}(n)$-modules in the spectrum of the $n=3$ "Di" and "Rac" modules, as observed initially in [13]. This property generalizes to any dimension [14].

Theorem (Angelopoulos & Laoues, 2000): *A positive energy singleton of AdS_{n+1} is a non-trivial lowest weight unitary irreducible module of $\mathfrak{o}(n,2)$ such that its reduction to the compact orthogonal subalgebra $\mathfrak{o}(n)$ is multiplicity free. The corresponding weights of the maximal compact subalgebra $\mathfrak{o}(2)\oplus\mathfrak{o}(n)$ lie along a line in the weight diagram. More precisely,*

$$\mathcal{D}\Big(s+\frac{n}{2}-1\,;\,[s],[s],\dots,[s]\Big)\Big|_{\mathfrak{o}(n)}\cong\bigoplus_{t\in\mathbb{N}}\mathcal{D}_{[s]+t,[s],\dots,[s]}$$

where each irreducible $\mathfrak{o}(n)$-module $\mathcal{D}_{[s]+t,[s],\dots,[s]}$ is an eigenspace of distinct energy

$$\Big(\mathrm{E}-\Big([s]+t+\frac{n}{2}-1\Big)\Big)\,\mathcal{D}_{[s]+t,[s],\dots,[s]}=0$$

and appears with multiplicity one. Moreover, singletons are those non-trivial extremal weight unitary irreducible modules of $\mathfrak{o}(n,2)$ such that their reduction to a compact Cartan subalgebra is multiplicity free: there is at most a single linearly independent eigenvector (i.e. a "singlet") for each weight.

As a side remark, one may observe that the previous properties are the roots of the earliest appearances, but in disguised form, of the $n=3$ singleton module in particle physicist through the so-called infinite-component equations and the spectrum-generating algebras. These ideas were pushed forward in the sixties by the "dynamical group" research programme which revived the seminal work of Majorana [15]. A very concise account of Majorana's publication itself and of the history of infinite-component wave equations can be found in [16]. Such constructions were recently revisited in [17] from a modern and shifted perspective (including a discussion of the higher-dimensional cases $n\geqslant 4$) which will briefly reviewed now.

Because of the commutation relations of the algebra $\mathfrak{o}(n,2)$

$$[\mathrm{J}_{ab},\mathrm{J}_{cd}]\;=\;i\,\eta_{bc}\mathrm{J}_{ad}\;+\;\text{antisymetrizations}$$

$$[\Gamma_a,\Gamma_b]\;=\;i\,\mathrm{J}_{ab}$$

the generators Γ_a can be reinterpreted as infinite-dimensional Dirac matrices from which the "spinning" generators of $\mathfrak{o}(n-1,1)$ are constructed as: $\mathrm{J}_{ab}\;=\;-i\,\Gamma_{[a},\Gamma_{b]}$. If the wave function $\psi(x)$ on

[3]The author thank V.K. Dobrev for pointing out to him that the name "singleton" comes from this fact.

the Minkowski spacetime $\mathbb{R}^{n,1}$ takes values in the scalar singleton module $\mathcal{D}(n/2-1;0)$, then the Dirac-like (i.e. the Majorana infinite-component) equation

$$(\Gamma^a \mathrm{P}_a \, - \, M)\psi(x) = 0$$

possesses a discrete spectrum of massive solutions. As one can see by looking at this equation in the rest frame $(\mathrm{P}_a \, - \, m\,\delta^0_a)\psi(x) = 0$, their masses m are related to their spin s by the relation

$$m \, = \, \frac{M}{s+\frac{n}{2}-1}$$

since the eigenvalue of $\Gamma_0 \, = \, \mathrm{E}$ on the spin-s $\mathfrak{o}(n)$-submodule $\mathcal{D}_s$ is equal to $s+\frac{n}{2}-1$. Unfortunately, the "Regge trajectory" is decreasing so the infinite-component Majorana equation is physically unsatisfactory. Furthermore, tachyonic and continuous spin particles also appear in the spectrum.

As a curiosity, one may mention the following observation [17] based on the above-mentioned decomposition of the scalar singleton module: Consider an infinitely degenerate spectrum of massive particles on the flat spacetime $\mathbb{R}^{n,1}$ with equal mass for all spins (multiplicity one), i.e. a horizontal Regge trajectory. In the rest frame, the massive particle of spin s is described by the irreducible $\mathfrak{o}(n)$-module $\mathcal{D}_s$, therefore the infinite tower of particles fits in an irreducible multiplet of $\mathfrak{o}(n,2)$ corresponding to the scalar singleton module $\mathcal{D}(\frac{n}{2}-1;0)$. One might speculate that such an infinite multipler could come from a highly degenerate (and exotic) spontaneous symmetry breaking of the anti de Sitter isometry algebra [17].

3.3 Singletons as irreducible modules of isometry subalgebras

The following theorem of Angelopoulos & Laoues [11] extends to any $n \geqslant 3$ the previous result for the case $n=3$ of Angelopoulos, Flato, Frønsdal & Sternheimer [18].

Theorem (Angelopoulos & Laoues, 1998): *A positive (or negative) energy singleton of AdS_{n+1} is a non-trivial lowest (or highest) weight unitary irreducible $\mathfrak{o}(n,2)$-module that remains irreducible (or, at most, splits in two) under restriction to any of the following subalgebras: $\mathfrak{io}(n-1,1)$, $\mathfrak{o}(n,1)$ and $\mathfrak{o}(n-1,2)$. Conversely, a singleton on ∂AdS_{n+1} is a unitary irreducible (at most, a sum of two) of any of the previous subalgebras, that can be lifted to a unitary module of $\mathfrak{o}(n,2)$.*

Heuristically, this means that **singletons of $\mathbf{AdS_{n+1}}$** are those fields:

- whose local physical degrees of freedom sit on its conformal boundary (so that one may also call them **singletons on $\partial\mathbf{AdS_{n+1}}$**), and
- which are preserved by the conformal symmetries in spacetime dimension n.

The extremal weight unitary irreducible $\mathfrak{o}(n,2)$-modules that are *not* singletons of AdS_{n+1} provide the **genuine elementary particles on $\mathbf{AdS_{n+1}}$**. The fact that elementary particles may live both on the boundary and in the bulk of anti de Sitter spacetime is the very basis of the AdS_{n+1}/CFT_n correspondence.

Let the conformal boundary ∂AdS_{n+1} be identified with the conformal compactification of $\mathbb{R}^{n-1,1}$. The following theorem was found by Siegel in [19] but the entirely complete and rigorous proof was given later in [11].

Theorem (Siegel, 1989): *A positive-energy singleton on ∂AdS_{n+1} is a positive-energy massless unitary irreducible module of $\mathfrak{io}(n-1,1)$ induced (à la Wigner) by a finite dimensional irreducible representation of the stabilizer $\mathfrak{io}(n-2)$ labeled by a partition in $\frac{n}{2}-1$ equal parts, i.e. by a rectangular Young diagram made of $\frac{n}{2}-1$ rows of length $[s]$.*

The spin s singleton on the conformal compactification of $\mathbb{R}^{n-1,1}$, seen as a representation of the Poincaré subalgebra $\mathfrak{io}(n-1,1)$, is called for

- $n=4$, the **helicity s representation**, and for
- higher even n and $s \geqslant 1$, a **spin s duality-symmetric representation** when $n/2$ is even or **chiral representation** when $n/2$ is odd, because the corresponding fieldstrength span an irreducible $\mathfrak{o}(n-1,1)$-module described by a rectangular Young diagram made of $\frac{n}{2}$ rows for which Hodge self-duality may be defined (more information on this point is provided in the the next subsection).

Now let the conformal boundary ∂AdS_{n+1} be identified with the conformal compactification of AdS_n. The analogue of the previous theorem was obtained by Metsaev in [20] (see also [11] for a proof).

Theorem (Metsaev, 1995): *A positive-energy singleton on ∂AdS_{n+1} is a finite-component positive-energy unitary irreducible $\mathfrak{o}(n-1,2)$-module $\mathcal{D}(s+\frac{n}{2}-1;[s],\ldots,[s])$ saturating the unitarity bound and whose lowest energy $\mathfrak{o}(n-1)$-module is labeled by a partition in $\frac{n}{2}-1$ equal parts, i.e. by a rectangular Young diagram made of $\frac{n}{2}-1$ rows of length $[s]$.*

Remark: It is important to stress that a singleton *on* ∂AdS_{n+1} is *not* a singleton *of* AdS_n, but of AdS_{n+1}. Indeed, a singleton on ∂AdS_{n+1} has lowest energy $s-1+n/2$ while a singleton of AdS_n has lowest energy $s-1+(n-1)/2$. Moreover, spin $s \geqslant 1$ singletons on ∂AdS_{n+1} exist only for n even while spin $s \geqslant 1$ singletons of AdS_n exist only for n odd.

A similar theorem holds for the conformal compactification of dS_n as well [11].

3.4 Singletons as fields on the conformal boundary

Singletons live on the conformal boundary so they can be described as fields on the corresponding compactified spacetimes. The simplest example is the scalar singleton which can be described as a massless (i.e. harmonic) scalar field $\phi(x)$ on Minkowski spacetime $\mathbb{R}^{n-1,1}$ of conformal weight $1-\frac{n}{2}$ so that the d'Alembert equation

$$\Box_{\mathbb{R}^{n-1,1}}\,\phi(x)=0 \quad \text{is preserved by the conformal algebra } \mathfrak{o}(n,2)\,.$$

Equivalently, the scalar singleton may be described on $(A)dS_n$ through a linear wave equation involving the conformal (or Yamabe) Laplacian

$$\Big(\Box_{(A)dS_n}\pm\frac{n(n-2)}{4\,R^2}\Big)\phi(x)=0$$

where $\Box$ denotes the Laplace-Beltrami operator.

The spin-s singleton $\mathfrak{io}(n-1,1)$-module can be realized as a space of harmonic irreducible multiforms [21] (see e.g. the section 5 of [5] for a review of the general construction of Poincaré modules). For $n=4$, this construction reproduces the famous Bargmann-Wigner equations which are known to be conformally symmetric since a long time ago [22]. For definiteness, one will focus on tensorial singletons, i.e. integer spin $s\in\mathbb{N}$. Let θ^μ_i be a set (where $\mu,\nu=0,1,2,\ldots,n-1$ and $i=1,2,\ldots,s-1,s$) of fermionic coordinates

$$\theta^\mu_i\theta^\nu_j+\theta^\nu_j\theta^\nu_i=0\,,$$

on $\Pi(\mathbb{R}^{n,2}\otimes\mathbb{R}^s)$ (where Π reverses the Grassmann parity).

Definitions: *A (differential)* ***multiform*** *on Minkowski spacetime $\mathbb{R}^{n-1,1}$ is*

- *a function $\psi(x^\mu,\theta^\nu_i)$ on the superspace $\mathbb{R}^{n-1,1}\oplus\Pi(\mathbb{R}^{n-1,1}\otimes\mathbb{R}^s)$, i.e. tensor fields on $\mathbb{R}^{n-1,1}$ with components described by a product of s columns. A multiform is:*
- ***closed*** *if it is annihilated by all operators $d_i=\vartheta^\mu_i\frac{\partial}{\partial x^\mu}$*
- ***coclosed*** *if it is annihilated by all operators $d_i^\dagger=\frac{\partial}{\partial\vartheta^i_\mu}\frac{\partial}{\partial x^\mu}$.*

- ***harmonic*** *if it is closed and coclosed.*

The components of a multiform span an irreducible $GL(n)$-module described by a rectangular Young diagram made of s columns and $\frac{n}{2}+1$ rows iff it is annihilated by the operators $\theta^\mu_i\frac{\partial}{\partial\theta^\mu_j}-\delta^j_i\,\frac{n}{2}$ that span the algebra $\mathfrak{gl}(s)$. Moreover, it is further irreducible under $O(n-1,1)$ iff it is also annihilated by the operators $\theta^\mu_i\theta^j_\mu$ and $\frac{\partial}{\partial\theta^\mu_i}\frac{\partial}{\partial\theta^j_\mu}$ which, together with the previous ones, span the algebra $\mathfrak{o}(2s)$.

Poincaré covariant equations for singletons [21]: *The spin-s singleton* $\mathfrak{io}(n-1,1)$*-module can be realized as a space of multiforms* $\psi(x,\theta)$ *on the Minkoswki spacetime* $\mathbb{R}^{n-1,1}$ *which are harmonic and whose components span an irreducible* $\mathfrak{o}(n-1,1)$*-module* $\mathcal{D}_{[s],\dots,[s]}$ *labeled by a rectangular Young diagram made of* $[s]$ *columns and* $\frac{n}{2}$ *rows.*

Remark: The multiform associated with a singleton of spin $s\geqslant 1$ is physically interpreted as its **fieldstrength** (or **curvature tensor**). An irreducible $O(n-1,1)$-module labeled by a rectangular Young diagram made of $[s]$ columns and $\frac{n}{2}$ rows decomposes a sum of two irreducible $\mathfrak{o}(n-1,1)$-modules when $n/2$ is odd. This subtlety is related to the involutive property of the Hodge operator. In order to treat both cases uniformly, one should consider the complexification of the $O(n-1,1)$-module when $n/2$ is even. Then both modules are eigenspaces (of eigenvalue ± 1) of the involutive Hodge duality, with a factor i included when $n/2$ is even (see e.g. the article [23] containing a concise introduction to the $s=1$ case).

So the singleton modules of spin $s\geqslant 1$ are either said to be duality-symmetric when $n/2$ is even or chiral when $n/2$ is odd. This duality properties extend to constantly curved spacetimes, therefore the singletons of spin $s\geqslant 1$ are those finite-component unitary irreducible representations of one of the isometry algebras $\mathfrak{io}(n-1,1)$, $\mathfrak{o}(n,1)$ or $\mathfrak{o}(n-1,2)$ which are either duality-symmetric or chiral. This deep connection between conformal symmetry and electric-magnetic duality somehow explains the appearance of singletons in many celebrated models of high-energy theoretical physics, such as maximally supersymmetric theories.

3.5 Singletons as fields on the ambient space

The main drawback of the description of singletons as fields on the conformal boundary (presented in the previous subsection) is that the conformal symmetry is not manifest (dilatation symmetry is obvious but not the special conformal and inversion symmetries). To circumvent this defect, one may describe singletons as fields on the ambient space.

Such a description was initiated by Dirac in [3] and can be summarized for the scalar singleton as follows (see e.g. the section 3 of [24] for a review of this elegant construction): On the one hand, any space of functions of the inhomogeneous coordinates on $\mathbb{RP}^{n+1}$ can be realized in terms of the homogeneous coordinates as a space of homogeneous functions on $\mathbb{R}^{n+2}$ of some fixed degree. On the other hand, any space of functions on the null cone can be realized as a space of equivalence classes of functions on the ambient space modulo the functions which vanish on the null cone. The homogeneity degree is fixed by the requirement that the Laplace-Beltrami operator on the ambient space $\mathbb{R}^{n,2}$ preserves the latter equivalence relation, so that this operator induces the conformal Laplacian on the conformal boundary ∂AdS_{n+1}.

Ambient construction of the scalar singleton (Dirac, 1936): *The scalar singleton* $\mathfrak{o}(n,2)$*-module can be realized as a space of functions* $\Phi(X)$ *on the ambient space* $\mathbb{R}^{n,2}$ *which are*

- *harmonic:* $\Box_{\mathbb{R}^{n,2}}\,\Phi(X)=0$
- *of homogeneity degree* $1-\frac{n}{2}$*:* $(X^A\partial_A+\frac{n}{2}-1)\Phi(X)=0$
- *quotiented by the equivalence relation*

$$\Phi(X)\sim\Phi(X)\;+\;(X^AX_A)\,\Xi(X)$$

where $\Xi(X)$ *is of homogeneity degree* $-1-\frac{n}{2}$.

Remark: The operators $\Box$, $X \cdot \partial_X + \frac{n+2}{2}$, X^2 are called (first class) **constraints** and they span the symplectic algebra $\mathfrak{sp}(2)$. This property will be made manifest via Howe duality. These first-class constraints find a natural interpretation in the "two-time physics" research programme of Bars (see e.g. [25]). Actually, all constraints can equivalently be imposed on the physical states.

Ambient construction of the scalar singleton: *The scalar singleton module can be realized as a space of distributions*

$$\Psi(X) := \delta(X^2)\Phi(X)$$

on ambient space $\mathbb{R}^{n,2}$ *which are*

- *harmonic:* $\Box\,\Psi(X) = 0$
- *of homogeneity degree* $-1 - \frac{n}{2}$*:* $(X \cdot \partial + \frac{n+2}{2})\Psi(X) = 0$
- *annihilated by the quadratic form:* $X^2\,\Psi(X) = 0$

The generalization of this construction to any spin [28] can be performed in the language of multiform. Let ϑ_i^A be a set (where $i = 1, 2, \ldots, s-1, s$) of fermionic coordinates

$$\vartheta_i^A \vartheta_j^B + \vartheta_j^B \vartheta_i^A = 0\,,$$

on $\Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$.

Definitions:*An* ***ambient multiform*** *is*

- *a multiform on the ambient space* $\mathbb{R}^{n,2}$*, i.e. a function* $\Psi(X^A, \vartheta_i^B)$ *on the superspace* $\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$.
- ***tangent to the anti de Sitter spacetime*** AdS_{n+1} *if it is annihilated by all operators* $X^A \frac{\partial}{\partial \vartheta_i^A}$.
- ***tangent to the conformal boundary*** $(\partial AdS)_n$ *if it is annihilated by all operators* $X^A \frac{\partial}{\partial \vartheta_i^A}$ *and* $\frac{\partial}{\partial X^A} \frac{\partial}{\partial \vartheta^i_A}$.

The definitions of (co)closure and harmonicity for ambient multiforms are the analogues of the ones for spacetime multiforms.

Ambient construction (Arvidsson & Marnelius, 2006): The tensorial singleton $\mathfrak{o}(n, 2)$-module can be realized as a space of multiforms $\Psi(X, \vartheta)$ on the ambient space $\mathbb{R}^{n,2}$

- which are
  - harmonic
  - of homogeneity degree $-1 - \frac{n}{2}$
  - annihilated by X^2
  - tangent to the conformal boundary
- whose components span an irreducible $\mathfrak{o}(n, 2)$-module described by a rectangular Young diagram made of s columns and $\frac{n}{2} + 1$ rows.

This formulation is appealing because conformal symmetry is manifest, unfortunately the price to pay is that locality is not manifest any more. However, there exists a formulation [26] where both conformal invariance and locality are manifest. This is made possible by an ambient space construction in the fiber rather than in the spacetime, along the lines of the parent approach [27].

Remark: The various operators $\Box$, $X \cdot \partial_X + \frac{n+2}{2}$, X^2, $\frac{\partial}{\partial X} \cdot \vartheta_i$, $\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i}$, $X \cdot \frac{\partial}{\partial \vartheta_i}$, $\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i}$. $\vartheta_i \cdot \frac{\partial}{\partial \vartheta_j} - \delta_i^j\, \frac{d+2}{2}$, $\vartheta_i \cdot \vartheta_j$, and $\frac{\partial}{\partial \vartheta_i} \cdot \frac{\partial}{\partial \vartheta_j}$ which annihilate the module span the orthosymplectic superalgebra $\mathfrak{osp}(2s|2)$ of constraints. This superalgebra finds a natural interpretation, on the mathematical side, in terms of Howe duality, and, on the physical side, in terms of the $\mathfrak{o}(2s)$ extended supersymmetric spinning particle (see e.g. [28, 29] and refs therein).

3.6 Singletons as kernels of the Howe dual algebra

An important message is that the orthosymplectic $\mathfrak{osp}(2s|2)$ (super)algebra of constraints annihilating the spin-s singleton module is the Howe dual of the conformal algebra $\mathfrak{o}(n,2)$ acting on the singleton irreducible module. For a concrete description of Howe duality, one may look e.g. at the the section 3 of the review [30]. For the sake of simplicity, let us turn back to the scalar singleton.

Let $T^*\mathbb{R}^{n,2}$ be the (trivial) **cotangent bundle** of the ambient space with canonical

- **Darboux coordinates** $Y^A_\alpha = (X^A, P_A)$ (where $\alpha = 1,2$)
- **Poisson bracket**
$$\{Y^A_\alpha, Y^B_\beta\} = \varepsilon_{\alpha\beta}\,\eta^{AB} \quad \Longleftrightarrow \quad \{X^A, P_B\} = \delta^A_B$$

where $\varepsilon_{\alpha\beta}$ is the symplectic form of $\mathfrak{sp}(2)$. The **Weyl algebra** A_{n+2} is the algebra of (polynomial) differential operators $\mathrm{O}(\mathrm{X},\mathrm{P})$ on the ambient space $\mathbb{R}^{n,2}$, where $\mathrm{P}_A = -\,i\,\partial/\partial X^A$. The Weyl algebra is isomorphic to the space of **Weyl symbols** $O(X,P)$, i.e. (polynomial) functions on the cotangent bundle $T^*\mathbb{R}^{n,2}$, endowed with the **Moyal star product** $* = \exp i\{\ ,\ \}$.

On the one hand, the algebra $\mathfrak{o}(n,2) = \mathrm{span}\{\mathrm{L}^{AB}\}$ is linearly realized on $\mathbb{R}^{n,2}$ via the generators $\mathrm{L}^{AB} = \mathrm{X}^A\mathrm{P}^B - \mathrm{X}^B\mathrm{P}^A$ whose Weyl symbols are the bilinears

$$L^{AB} = \varepsilon^{\alpha\beta} Y^A_\alpha Y^B_\beta = X^A P^B - X^B P^A$$

On the other hand, the Lie algebra
$$\mathfrak{sp}(2) = \mathrm{span}\{\mathrm{U}_{\alpha\beta}\}$$
can be presented

- by its generators $\mathrm{U}_{\alpha\beta} = \mathrm{U}_{\beta\alpha}$ (where $\alpha,\beta = 1,2$)
- modulo the commutation relations
$$[\mathrm{U}_{\alpha\beta}, \mathrm{U}_{\gamma\delta}] \;=\; i\,\varepsilon_{\beta\gamma}\mathrm{U}_{\alpha\delta} \;+\; \text{symmetrizations}\,.$$

The Weyl symbols of the operators $\Box\,,\ X\cdot\partial_X + \frac{n+2}{2}\,,\ X^2$ are the bilinears

$$U_{\alpha\beta} = \eta_{AB} Y^A_\alpha Y^B_\beta$$

In both cases, the Weyl commutators (or Poisson brackets) of generators reproduce the corresponding commutation relations. These respective realizations of $\mathfrak{o}(n,2)$ and $\mathfrak{sp}(2)$ are maximal commutants in the algebra of quadratic Weyl symbols: they form a **Howe dual pair**.

Ambient construction of the scalar singleton: *The scalar singleton $\mathfrak{o}(n,2)$-module is a space of distributions $\Psi(X)$ on the ambient space $\mathbb{R}^{n,2}$ which are annihilated by the $\mathfrak{sp}(2)$ algebra, Howe dual to $\mathfrak{o}(n,2)$ in the algebra of linear operators on $\mathbb{R}^{n,2}$:*

$$\mathrm{U}_{\alpha\beta}\Psi(X) = 0$$

The generalization to any integer spin $s\in\mathbb{N}$ is analogous [26]: The Grassmann even indices A,B will still correspond to the $(n+2)$-dimensional ambient space $\mathbb{R}^{n,2}$ with metric η^{AB} but the letters α,β will now be superindices corresponding to a $(2|2s)$-dimensional symplectic superspace

$$T^*\mathbb{R}^{1|s} \cong \mathbb{R}^{2|2s}$$

with symplectic form $\mathcal{J}_{\alpha\beta}$. The symplectic form on the superspace $\mathbb{R}^{2|2s}$ can be seen as a metric form on the superspace $\mathbb{R}^{2s|2} \cong \Pi(\mathbb{R}^{2|2s})$ with opposite Grassmann parity. Therefore, the symplectic form $\mathcal{J}_{\alpha\beta}$ is manifestly preserved by the orthosymplectic algebra $\mathfrak{osp}\,(2s\,|\,2)$. The multiforms are functions on the superspace

$$\mathbb{R}^{n,2}\,\oplus\,\Pi(\mathbb{R}^{n,2}\otimes\mathbb{R}^s) \cong \mathbb{R}^{n+2\,|\,s(n+2)}$$

with

- $n+2$ even coordinates X^A on $\mathbb{R}^{n,2}$
- $s(n+2)$ odd coordinates ϑ^A_i on $\Pi(\mathbb{R}^{n,2}\otimes\mathbb{R}^s)$.

Let $(P_A|\pi^i_B)$ be the conjugates of the supercoordinates $(X^A|\theta^B_i)$. The phase (super)space coordinates on the cotangent bundle $T^*\mathbb{R}^{n+2|s(n+2)}$ are collectively denoted by

$$Z^A_\alpha := (X^A, P_B|\theta^A_i, \pi^j_B)$$

where the superindex α takes $2+2s$ values. The graded Poisson bracket originating from the symplectic structure on the phase superspace is

$$\{Z^A_\alpha, Z^B_\beta\} = \eta^{AB}\mathcal{J}_{\alpha\beta}$$

$$\iff \quad \{X^A, P_B\} = -\{P_B, X^A\} = \delta^A_B\,, \quad \{\theta^A_i, \pi^j_B\} = \{\pi^j_B, \theta^A_i\} = \delta^A_B\delta^i_j\,.$$

The phase space coordinates Z^A_α are natural coordinates on the tensor product $\mathbb{R}^{n,2}\otimes\mathbb{R}^{2|2s}$. The algebra $\mathfrak{o}(n,2)$ is linearly realized on $\mathbb{R}^{n,2}\oplus\Pi(\mathbb{R}^{n,2}\otimes\mathbb{R}^s)$ as

$$\mathrm{J}^{AB} = \mathrm{X}^A\mathrm{P}^B - \mathrm{X}^B\mathrm{P}^A - i\,\vartheta^A_i\frac{\partial}{\partial\vartheta^i_B} + i\,\vartheta^B_i\frac{\partial}{\partial\vartheta^i_A}$$

The Weyl symbols of these generators of the algebra $\mathfrak{o}(n,2)$ are the bilinears

$$J^{AB} = \mathcal{J}^{\alpha\beta}Z^A_\alpha Z^B_\beta$$

The Lie superalgebra

$$\mathfrak{osp}(2s|2) = \mathrm{span}\{\mathrm{T}_{\alpha\beta}\}$$

can be presented

- by its generators $\mathrm{T}_{\alpha\beta} = \mathrm{T}_{\beta\alpha}$ (where $\alpha,\beta = 1,2$)
- modulo the graded commutation relations

$$[\mathrm{T}_{\alpha\beta}, \mathrm{T}_{\gamma\delta}] = i\,\mathcal{J}_{\beta\gamma}\mathrm{T}_{\alpha\delta} + \text{(anti)symmetrizations}\,.$$

The Weyl symbols of the operators $\Box$, $X\cdot\partial_X + \frac{n+2}{2}$, X^2, $\frac{\partial}{\partial X}\cdot\vartheta_i$, $\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial\vartheta_i}$, $X\cdot\frac{\partial}{\partial\vartheta_i}$, $\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial\vartheta_i}$. $\vartheta_i\cdot\frac{\partial}{\partial\vartheta_j} - \delta^j_i\,\frac{d+2}{2}$, $\vartheta_i\cdot\vartheta_j$, and $\frac{\partial}{\partial\vartheta_i}\cdot\frac{\partial}{\partial\vartheta_j}$ are the bilinears

$$T_{\alpha\beta} = \eta_{AB}Z^A_\alpha Z^B_\beta$$

The Weyl graded commutators (or graded Poisson brackets) of generators reproduce the corresponding graded commutation relations. The respective realizations of $\mathfrak{o}(n,2)$ and $\mathfrak{osp}(2s|2)$ are maximal commutants in the algebra of quadratic Weyl symbols: they form a Howe dual pair.

Ambient construction of tensorial singletons [26]: *The spin $s\in\mathbb{N}$ singleton module $\mathcal{D}(s+\frac{n}{2}-1\,;\,s,\dots,s)$ can be realized as a space of distributions on the superspace $\mathbb{R}^{n,2}\oplus\Pi(\mathbb{R}^{n,2}\otimes\mathbb{R}^s)$ which are annihilated by the $\mathfrak{osp}(2s|2)$ superalgebra, which is Howe dual to $\mathfrak{o}(n,2)$ in the superalgebra of linear operators on $\mathbb{R}^{n,2}\oplus\Pi(\mathbb{R}^{n,2}\otimes\mathbb{R}^s)$.*

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