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Hidden Symmetries in a Gauge Covariant Approach

Mihai Visinescu
Department of Theoretical Physics
National Institute for Physics and Nuclear Engineering
P.O.Box M.G.-6, Magurele, Bucharest, Romania
mvisin@theory.nipne.ro

Abstract

Higher order symmetries in a covariant Hamiltonian framework are investigated. Some nontrivial examples on a three-dimensional space involving Killing tensors of rank 2 are presented. We analyze the possibility for a higher order symmetry to survive when the electromagnetic interactions are taken into account. A concrete realization of this possibility is given by the Killing-Maxwell system.

1 Introduction

The evolution of a dynamical system is described in the phase-space and from this point of view it is natural to go in search of conserved quantities associated with geometrical symmetries of the complete phase-space, not just the configuration one. Such symmetries are related to higher rank symmetric Stackel-Killing (SK) tensors which generalize the Killing vectors. These higher order symmetries are known as *hidden symmetries* and the corresponding conserved quantities are quadratic, or, more general, polynomial in momenta. Also Killing tensors play a fundamental role in the Hamilton-Jacobi theory of separation of variables and the integrability of finite-dimensional Hamiltonian systems [1]. Another natural generalization of the Killing vectors is represented by the antisymmetric Killing-Yano (KY) tensors [2] which in many aspects are more important than the KS tensors.

In the study of motion of charged particles in external gauge fields it has been proved that a gauge covariant Hamiltonian framework [3] is more appropriate. We illustrate the general considerations by some simple but nontrivial examples on a three-dimensional Euclidean space [4].

The covariant approach is also useful to investigate the possibility for a higher order symmetry to survive when the electromagnetic interactions are taken into account. A concrete realization of this possibility is given by the Killing-Maxwell (KM) system [5].

The plan of the paper is as follows. In Section 2 we establish the generalized Killing equations in a covariant framework including external gauge fields and scalar potentials. In Section 3 we exemplify the gauge covariant approach with some nontrivial examples connected with the Kepler/Coulomb potential. In Section 4 we discuss the special role of the KY tensors in generating higher order symmetries and in the next Section we describe the KM system. Finally, the last Section is devoted to conclusions.

2 Symmetries and conserved quantities

Let $(\mathcal{M}, \mathbf{g})$ be a n-dimensional manifold equipped with a (pseudo-)Riemmanian metric $\mathbf{g}$ and denote by

$$H = \frac{1}{2} g^{ij} p_i p_j \,, \tag{1}$$

the Hamilton function describing the geodesic motion in a curved space-time.

Let us consider a conserved quantity of motion expanded as a power series in momenta:

$$K = K_0 + \sum_{k=1}^{p} \frac{1}{k!} K^{i_1 \cdots i_k}(x) p_{i_1} \cdots p_{i_k} \,. \tag{2}$$

It has vanishing Poisson bracket with the Hamiltonian, $\{K, H\} = 0$, which implies

$$K^{(i_1 \cdots i_k ; i)} = 0 \,, \tag{3}$$

where a semicolon denotes the covariant differentiation corresponding to the Levi-Civita connection and round brackets denote full symmetrization over the indices enclosed. A symmetric tensor $K^{i_1 \cdots i_k}$ satisfying (3) is called a SK tensor of rank k. The SK tensors represent a generalization of the Killing vectors and are responsible for the hidden symmetries of the motions, connected with conserved quantities of the form (2) polynomials in momenta. Indeed, using equation (3), for any geodesic γ with tangent vector $\dot{x}^i = p^i$

$$Q_K = K_{i_1 \cdots i_k} \dot{x}^{i_1} \cdots \dot{x}^{i_k} \,, \tag{4}$$

is constant along γ.

The traditional procedure to deal with the coupling to a gauge field F_{ij} expressed (locally) in terms of the potential 1-form A_i

$$F = dA \,, \tag{5}$$

is to replace the Hamiltonian by

$$H = \frac{1}{2} g^{ij} (p_i - A_i)(p_j - A_j) \,, \tag{6}$$

work with the standard Poisson bracket and consider the polynomials (2) in the variables $(p_i - A_i)$ for $i = 1, \cdots, n$.

The disadvantage of this approach is that the canonical momenta p_i and implicitly the Hamilton equations of motion are not manifestly gauge covariant. This drawback can be removed using van Holten's receipt [3] by introducing the gauge invariant momenta:

$$\Pi_i = p_i - A_i \,. \tag{7}$$

The Hamiltonian (6) becomes

$$H = \frac{1}{2} g^{ij} \Pi_i \Pi_j + V(x) \,, \tag{8}$$

where for completeness we included a scalar potential $V(x)$. The equations of motion are derived using the Poisson bracket

$$\{P, Q\} = \frac{\partial P}{\partial x^i} \frac{\partial Q}{\partial \Pi_i} - \frac{\partial P}{\partial \Pi_i} \frac{\partial Q}{\partial x^i} + q F_{ij} \frac{\partial P}{\partial \Pi_i} \frac{\partial Q}{\partial \Pi_j} \,. \tag{9}$$

In consequence the fundamental Poisson brackets are

$$\{x^i, x^j\} = 0 \,,\ \{x^i, \Pi_j\} = \delta^i_j \,,\ \{\Pi_i, \Pi_j\} = F_{ij} \,, \tag{10}$$

showing that the momenta Π_i are not canonical.

Searching for conserved quantities (2) expanded rather into powers of the gauge invariant momenta Π_i we get the following series of constraints

$$K^i V_{,i} = 0 \,, \tag{11a}$$

$$K_0{}^{,i} + F_j{}^i K^j = K^{ij} V_{,j} \,. \tag{11b}$$

$$K^{(i_1 \cdots i_l ; i_{l+1})} + F_j{}^{(i_{l+1}} K^{i_1 \cdots i_l) j} = \frac{1}{(l+1)} K^{i_1 \cdots i_{l+1} j} V_{,j} \,, \quad for \quad l = 1, \cdots (p-2) \,, \tag{11c}$$

$$K^{(i_1 \cdots i_{p-1} ; i_p)} + F_j{}^{(i_p} K^{i_1 \cdots i_{p-1}) j} = 0 \,, \tag{11d}$$

$$K^{(i_1 \cdots i_p ; i_{p+1})} = 0 \,. \tag{11e}$$

Examining the above hierarchy of constraints some remarks are in order. First of all, the last equations is satisfied by a SK tensor (3), while the rest of the equations mixes up the terms of K with the gauge field strength F_{ij}. Also it is worth mentioning that equations (11) separate into two groups: one involves the terms of K of odd degree in the momenta Π_i and the other involves only terms of K of even degree in the momenta [6].

Several applications using van Holten's covariant framework [3] are given in [4, 7, 8, 9].

3 Explicit examples

Let us illustrate these general considerations by some nontrivial examples. In what follows we consider $\mathcal{M}$ to be a 3-dimensional Euclidean space $\mathbb{E}^3$ and in these circumstances in this Section we get rid of a difference between covariant and contravariant indices. The Coulomb potential will be the basis of our examples superposing different types of electric and magnetic fields. The hidden symmetries which will be found involve SK tensors of rank 2 looking for constants of motion of the form

$$K = K_0 + K_i\Pi_i + \frac{1}{2}K_{ij}\Pi_i\Pi_j\,. \tag{12}$$

3.1 Coulomb potential

To put in a concrete form, we consider the Hamiltonian for the motion of a point charge q of mass M in the Coulomb potential produced by a charge Q

$$H = \frac{M}{2}\dot{\mathbf{r}}^2 + q\frac{Q}{r}\,. \tag{13}$$

We start with (11e) for $p = 2$ which is satisfied by a SK tensor of rank 2. For the Coulomb problem it proved that the following form of the SK tensor is adequate [10]:

$$K_{ij} = 2\delta_{ij}\mathbf{n}\cdot\mathbf{r} - (n_i r_j + n_j n_i)\,, \tag{14}$$

written in spherical coordinates with $\mathbf{n}$ an arbitrary constant vector.

Corresponding to this SK tensor the non relativistic Coulomb problem admits the Runge-Lenz vector constant of motion

$$\mathbf{K} = \mathbf{\Pi}\times\mathbf{L} + MqQ\frac{\mathbf{r}}{r}\,, \tag{15}$$

where

$$\mathbf{L} = \mathbf{r}\times\mathbf{\Pi}\,, \tag{16}$$

is the angular momentum.

3.2 Constant electric field

The next more involved example consists of an electric charge q moving in the Coulomb potential in the presence of a constant electric field $\mathbf{E}$. The corresponding Hamiltonian is:

$$H = \frac{1}{2M}\mathbf{\Pi}^2 + q\frac{Q}{r} - q\mathbf{E}\cdot\mathbf{r}\,, \tag{17}$$

with $\mathbf{\Pi} = M\dot{\mathbf{r}}$ in spherical coordinates of $\mathbb{E}^3$.

Again it is adequate to take for the SK tensor of rank 2 the simple form (14) choosing $\mathbf{n} = \mathbf{E}$. Using this form for K_{ij} after a straightforward calculation

$$K_0 = \frac{MqQ}{r}\mathbf{E}\cdot\mathbf{r} - \frac{Mq}{2}\mathbf{E}\cdot[\mathbf{r}\times(\mathbf{r}\times\mathbf{E})]\,. \tag{18}$$

Concerning equation (11a), it is automatically satisfied by a vector $\mathbf{K}$ of the form

$$\mathbf{K} = \mathbf{r}\times\mathbf{E}\,, \tag{19}$$

modulo an arbitrary constant factor. This vector $\mathbf{K}$ contribute to a conserved quantity with a term proportional to the angular momentum $\mathbf{L}$ along the direction of the electric field $\mathbf{E}$.

In conclusion, when a uniform constant electric field is present, the Coulomb system admits two constants of motion $\mathbf{L}\cdot\mathbf{E}$ and $\mathbf{C}\cdot\mathbf{E}$ where $\mathbf{C}$ is a generalization of the Runge-Lenz vector (15):

$$\mathbf{C} = \mathbf{K} - \frac{Mq}{2}\mathbf{r}\times(\mathbf{r}\times\mathbf{E})\,. \tag{20}$$

3.3 Spherically symmetric magnetic field

Another configuration which admits a hidden symmetry is the superposition of an external spherically symmetric magnetic field

$$\mathbf{B} = f(r)\mathbf{r}\,, \tag{21}$$

over the Coulomb potential acting on a electric charge q. This configuration is quite similar to the Dirac charge-monopole system.

For K_{ij} we use again the form (14) and F_{ij} in this case is

$$F_{ij} = \epsilon_{ijk}B_k = \epsilon_{ijk}r_k f(r)\,. \tag{22}$$

The system of constraint (11) can be solely solved only for a definite form of the function $f(r)$

$$f(r) = \frac{g}{r^{5/2}}\,, \tag{23}$$

with g a constant connected with the strength of the magnetic field.

With this special form of the function $f(r)$ we get

$$K_0 = \left[\frac{MqQ}{r} - \frac{2g^2q^2}{r}\right](\mathbf{n}\cdot\mathbf{r})\,, \tag{24}$$

and

$$K_i = -\frac{2gq}{r^{1/2}}(\mathbf{r}\times\mathbf{n})_i\,. \tag{25}$$

Collecting the terms K_0, K_i, K_{ij} the constant of motion (12) becomes

$$K = \mathbf{n}\cdot\left(\mathbf{K} + \frac{2gq}{r^{1/2}}\mathbf{L} - 2g^2q^2\frac{\mathbf{r}}{r}\right)\,, \tag{26}$$

with $\mathbf{n}$ an arbitrary constant unit vector and $\mathbf{K}, \mathbf{L}$ given by (15), (16) respectively. The angular momentum $\mathbf{L}$ [3] is not separately conserved, entering the constant of motion (26).

3.4 Magnetic field along a fixed direction

The last example consists in a magnetic field directed along a fixed unit vector $\mathbf{n}$

$$\mathbf{B} = B(\mathbf{r}\cdot\mathbf{n})\mathbf{n}\,, \tag{27}$$

where, for the beginning, $B(\mathbf{r}\cdot\mathbf{n})$ is an arbitrary function.

Again we are looking for a constant of motion of the form (12) with the SK tensor of rank 2 (14).

Equations (11) prove to be solvable only for a particular form of the magnetic field

$$\mathbf{B} = \frac{\alpha}{\sqrt{\alpha\mathbf{r}\cdot\mathbf{n} + \beta}}\,\mathbf{n}\,, \tag{28}$$

with α and β two arbitrary constants.

Consequently we get for K_0 and K_i

$$K_0 = \frac{MqQ}{r}(\mathbf{r}\cdot\mathbf{n}) + \alpha q^2(\mathbf{r}\times\mathbf{n})^2\,, \tag{29}$$

$$K_i = -2q\sqrt{\alpha\mathbf{r}\cdot\mathbf{n} + \beta}\,(\mathbf{r}\times\mathbf{n})_i\,. \tag{30}$$

The final form of the conserved quantity in this case is:

$$K = \mathbf{n} \cdot \left[\mathbf{K} + 2q\sqrt{\alpha \mathbf{r} \cdot \mathbf{n} + \beta}\ \mathbf{L} \right] + \alpha q^2 (\mathbf{r} \times \mathbf{n})^2 \,. \tag{31}$$

As in the previous example the angular momentum $\mathbf{L}$ is forming part of the constant of motion K (31).

4 Killing-Yano tensors

KY tensors are a different generalization of Killing vectors which can be studied on a manifold. They were introduced by Yano [2] from a purely mathematical perspective and later on it turned out they have many interesting properties relevant to physics. The existence of higher rank KY tensors indicated the presence of dynamical symmetries which are not isometries. Here we shall point out the role of KY tensors in construction of conserved quantities paying a special attention to the KM system introduced by Carter [5].

A KY tensor is a p-form $Y (p \leq n)$ which satisfies

$$\nabla_X Y = \frac{1}{p+1} X \lrcorner\, dY \,, \tag{32}$$

for any vector field X, where 'hook' operator $\lrcorner$ is dual to the wedge product. This definition is equivalent with the property that $\nabla_j Y_{i_1 \cdots i_p}$ is totally antisymmetric or, in components,

$$Y_{i_1 \cdots i_{p-1}(i_p;j)} = 0 \,. \tag{33}$$

The first connection with the symmetry properties of the geodesic motions is the observation that along every geodesic γ in $\mathcal{M}$, $Y_{i_1 \cdots i_{p-1} j}\, \dot{x}^j$ is parallel.

These two generalizations SK and KY of the Killing vectors could be related. Let $Y_{i_1 \cdots i_p}$ be a KY tensor, then the symmetric tensor field

$$K_{ij} = Y_{i i_2 \cdots i_p} Y_j^{\ i_2 \cdots ip} \,, \tag{34}$$

is a SK tensor and it sometimes refers to this SK tensor as the associated tensor with $Y_{i_1 \cdots i_p}$. However, the converse statement is not true in general: not all SK tensors of rank 2 are associated with a KY tensor.

Having in mind the special role of null geodesic for the motion of massless particles, it is convenient to look for conformal generalization of KY tensor. Let us mention also that recently a lot of interest focuses on higher dimensional black holes. It was demonstrated the remarkable role of the conformal Killing-Yano (CKY) tensors in the study of the properties of such black holes (see e. g. [11, 12, 13] and the cites contained therein).

A CKY tensor of rank p is a p-form which satisfies

$$\nabla_X Y = \frac{1}{p+1} X \lrcorner\, dY - \frac{1}{n-p+1} X^\flat \wedge d^* Y \,, \tag{35}$$

where $X^\flat$ denotes the 1-form dual with respect to the metric to the vector field X and d^* is the exterior co-derivative. Let us recall that the Hodge dual maps the space of p-forms into the space of $(n-p)$-forms. The square of $*$ on a p-form Y is either $+1$ or -1 depending on n, p and the signature of the metric [14, 15]

$$* * Y = \epsilon_p Y \quad , \quad *^{-1} Y = \epsilon_p * Y \,, \tag{36}$$

with the number ϵ_p

$$\epsilon_p = (-1)^p *^{-1} \frac{detg}{|detg|} \,. \tag{37}$$

With this convention, the exterior co-derivative can be written in terms of d and the Hodge star:

$$d^* Y = (-1)^p *^{-1} d * Y \,. \tag{38}$$

Comparing definitions (32) and (35) we remark that all KY tensors are co-closed but not necessarily closed. From this point of view CKY tensors represent a generalization more symmetric in the pair of notions. CKY equation (35) is invariant under Hodge duality that if a p-form Y satisfies it, then so does the $(n-p)$-form $*Y$. Moreover the dual of a CKY tensor is a KY tensor if and only if it is closed.

There is also a conformal generalization of the SK tensors, namely a symmetric tensor $K_{i_1\cdots i_p} = K_{(i_1\cdots i_p)}$ is called a conformal Killing (CSK) tensor if it obeys the equation

$$K_{(i_1\cdots i_p;j)} = g_{j(i_1}\tilde{K}_{i_2\cdots i_p)}\,, \tag{39}$$

where the tensor $\tilde{K}$ is determined by tracing the both sides of equation (39). Let us note that in the case of CSK tensors, the quantity (4) is constant only for null geodesics γ. There is also a similar relation between CKY and CSK tensors as in equation (34).

For what follows it is necessary to mention an interesting construction involving CKY tensors of rank 2 in 4 dimensions. Let us consider equation (35) for this particular case:

$$Y_{i(j;k)} = -\frac{1}{3}\left(g_{jk}Y^l{}_{i;l} + g_{i(k}Y_{j)}{}^l{}_{;l}\right)\,, \tag{40}$$

and let us denote

$$Y_k := Y^l{}_{k;l}\,. \tag{41}$$

The trace in ij in equation (40) leads to the following result [16]:

$$Y_{(i;j)} = \frac{3}{2}R_{l(i}Y_{j)}{}^l\,. \tag{42}$$

It is obvious that in a Ricci flat space ($R_{ij} = 0$) or in an Einstein space ($R_{ij} \sim g_{ij}$), Y_k is a Killing vector and we shall refer to it as the *primary Killing vector.* In Carter's construction [5] of a primary Killing vector it is used a CKY tensor which in turn is the dual of an ordinary KY tensor.

5 KM system

Returning to the system of equations (11) we should like to find the conditions of the electromagnetic tensor field F_{ij} to maintain the hidden symmetry of the system. More precisely, we are looking for favorable conditions under which the terms $F_j{}^{(i_l}K^{i_1\cdots i_{l-1})j}$ do not contribute to equations (11) regulating the conserved quantities. To make things more specific, let us assume that the system admits a hidden symmetry encapsulated in a SK tensor of rank 2, K_{ij} associated with a KY tensor Y_{ij} according to (34). The sufficient condition of the electromagnetic field to preserve the hidden symmetry is [4]

$$F_{k[i}Y_{j]}{}^k = 0\,. \tag{43}$$

where the indices in square bracket are to be antisymmetrized.

We mention that this condition appeared in many different contextes as conformal Killing spinors [17], pseudo-classical spinning point particles models [18], Dirac-type operators that commute with the standard Dirac operator [19].

A concrete realization of (43) is presented by the KM system [5]. In Carter's construction a primary Killing vector (41) is identified, modulo a rationalization factor, with the source current j^i of the electromagnetic field

$$F^{ij}{}_{;j} = 4\pi j^i\,. \tag{44}$$

Therefore the KM system is defined assuming that the electromagnetic field F_{ij} is a CKY tensor which, in addition, is a closed 2-form (5). Its Hodge dual

$$Y_{ij} = *F_{ij}\,, \tag{45}$$

is a KY tensor (see Section 4).

Finally, the KM system possesses a hidden symmetry associated with the KY tensor (45). It is quite simple to observe that $F_{ij}Y_k{}^j \sim F_{ij} * F_k{}^j$ is a symmetric matrix (in fact proportional with the unit matrix) and therefore condition (43) is fulfilled.

6 Concluding comments

To conclude let us discuss shortly some problems that deserve a further attention. An obvious extension of the gauge covariant approach to hidden symmetries is represented by the non-abelian dynamics using the appropriate Poisson brackets [3, 7]. We worked out some examples in an Euclidean 3-dimensional space and restricted to SK tensors of rank 2. More elaborate examples working in a N-dimensional curved space and involving higher ranks of SK tensors [20] will be presented elsewhere [21].

It is interesting to note that the conserved quantities associated with Killing tensors do not generally transfer to the quantized systems producing so-called *quantum anomalies* [22, 23, 24]. Quantum anomalies in the presence of non-abelian gauge fields, higher order ($k > 2$) symmetries, skew-symmetric torsions are a few possible extensions which deserve further study. Let us mention that the concept of generalized (C)KY symmetry of spaces with a skew-symmetric torsion is more widely applicable and may become very powerful [25].

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