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Surprises in Noncommutative Dynamics

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Abstract

We present several unexpected consequences of the noncommutativity of coordinates in classical and quantum mechanics. Classically, a standard Lagrangian variational approach cannot be formulated, dynamics is quite strange, and gauge invariance is broken for a particle minimally coupled to an electromagnetic field. Quantum mechanically, the Schrödinger equation is quite nonstandard, and no configuration-space Feynman formulation exists. Integrating out the momenta in the phase-space path integral one obtains an effective Lagrangian, which however depends also on the accelerations.

1 Introduction

As is well known [1], once the degrees of freedom of a physical system are identified, the dynamics is determined by two elements:
1. The Hamiltonian, which is often quadratic in the momenta and often a finite power series in the coordinates, symbolically

$$H \sim p^2 + V(q). \tag{1}$$

2. The symplectic structure, determined by the Poisson brackets in classical mechanics

$$\{q^i, q^j\} = 0 \qquad \{q^i, p_j\} = \delta^i_j \qquad \{p_i, p_j\} = 0, \tag{2}$$

and by the Heisenberg commutation relations in quantum mechanics

$$[\hat{q}^i, \hat{q}^j] = 0 \qquad [\hat{q}^i, \hat{p}_j] = i\delta^i_j \qquad [\hat{p}_i, \hat{p}_j] = 0. \tag{3}$$

Standard notation is used above, q_i and p_j representing the canonical coordinates and momenta of the system under consideration (and $\hbar \equiv 1$).

A basic but fundamental consequence of the above equations is the following. Due to (3) - more precisely to the fact that coordinates commute - one can use the coordinate representation $\hat{p}_i = -i\frac{\partial}{\partial q_i}$. Given now the relation $\hat{H}\Psi = E\Psi$ it is clear that quadracity of $\hat{H}$ in the momenta $\hat{p}_i$ leads to the usual second order Schrödinger partial differential equation. We stress that we needed simultaneously a Hamiltonian quadratic in the momenta *and* $[\hat{q}^i, \hat{q}^j] = 0$.

In this review, we will mainly give up this second constraint, exploring some of the consequences of the following commutation relations

$$[\hat{q}^i, \hat{q}^j] = i\theta^{ij}(\hat{q}, \hat{p}) \qquad [\hat{q}^i, \hat{p}_j] = i\delta^i_j \qquad [\hat{p}_i, \hat{p}_j] = iF_{ij}(\hat{q}, \hat{p}). \tag{4}$$

In the last years quantum mechanics with noncommuting coordinates attracted much attention [7]-[20], especially for the simpler case of constant θ and F, which we adress here. For partial results concerning nonconstant commutators, one may consult for instance [21]. In its simplest form noncommutative (NC) mechanics follows the structure of ordinary mechanics, but allows in addition for nonzero commutators among the coordinate operators.

In the NC classical version one similarly generalizes the symplectic structure, by allowing further nonvanishing Poisson brackets among coordinates. As will be shown in Section 2, the resulting equations of motion do not admit a standard Lagrangian formulation [20, 4], dynamics becomes occasionally strange and gauge invariance itself gets broken.

The quantum theory is taken up in Section 3. The absence of a classical Lagrangian has as counterpart in the NC quantum mechanical theory the lack of the usual configuration space path integral. Only a phase space path integral can be constructed [19]. Nevertheless, one may search for an effective Lagrangian theory in configuration space, by integrating over the momenta in the phase space path integral. This leads to an acceleration-dependent Lagrangian in configuration space.

2 Classical Dynamics

2.1 Equations of motion

We begin with the classical theory. In addition to the Hamiltonian, one starts from the classical analogue of (4), namely the generalized Poisson brackets

$$\{q^i, q^j\} = \theta^{ij} \qquad \{q^i, p_j\} = \delta^i_j \qquad \{p_i, p_j\} = F_{ij}. \tag{5}$$

For simplicity in notation we will work in (2+1)-dimensions, although the extension to higher dimensionalities is straightforward. We will denote by x_a, $a = 1, 2, 3, 4$ the phase space coordinates, $x_{1,2,3,4} = q_1, p_1, q_2, p_2$. Since no risk of confusion exists, all indices will be put down from now on. Eqs. (5) can then be rewritten as $\{x_i, x_j\} = \Theta_{ij}$, where

$$\Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix} \quad \text{i.e.} \quad \omega = \frac{1}{1-\theta\sigma} \begin{pmatrix} 0 & -\sigma & 1 & 0 \\ \sigma & 0 & 0 & 1 \\ -1 & 0 & 0 & -\theta \\ 0 & -1 & \theta & 0 \end{pmatrix}. \tag{6}$$

Above, $\Theta_{ij} = (\omega^{-1})_{ij}$, and ω is the symplectic form, which enters the action

$$S = \int dt \left(\frac{1}{2} \omega_{ij} x_i x_j - H(x) \right). \tag{7}$$

Independent variation of S along each x_a provides the equations of motion

$$\dot{x}_i = \{x_i, H\} = \Theta_{ij} \frac{\partial H}{\partial x_j}, \tag{8}$$

more explicitely

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \theta \epsilon_{ij} \frac{\partial H}{\partial q_j} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sigma \epsilon_{ij} \frac{\partial H}{\partial p_j}, \tag{9}$$

with $\epsilon_{12} = -\epsilon_{21} = 1$. If $\theta = \sigma = 0$, (9) are the usual Hamilton equations.

Let us express the equations of motion in terms of q_1 and q_2. Considering Hamiltonians of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{10}$$

one gets the coordinate equations of motion:

$$\ddot{q}_i = -(1-\theta\sigma) \frac{\partial V}{\partial q_i} + \sigma \epsilon_{ij} \dot{q}_j + \theta \epsilon_{ij} \frac{d}{dt} \frac{\partial V}{\partial q_j}, \quad i = 1, 2. \tag{11}$$

For $\theta \neq 0$, it is easy to see that equations (11) are not derivable from a Lagrangian, if the potential V is higher than quadratic in the coordinates. This is shown by finding a Lagrangian formulation (unique up to total derivatives) for one of the two equations in (11), and seeing in which cases the second equation in (11) can be derived from it [18], or by using the so-called Helmholtz conditions which are to be obeyed by a set of equations admitting a variational formulation [20].

We stress that if one would start from the Hamiltonian (10) and perform the usual Legendre transformation, then use (9) to express $p_{1,2}$ in terms of $q_{1,2}$, one would get wrong equations of motion. The usual procedure works correctly only if $\theta = 0$. On the other hand, σ is quite harmless; it plays the role of a constant magnetic field.

The RHS of (11) contains three kinds of terms. The first, $-(1-\theta\sigma)\frac{\partial V}{\partial q_i}$, is the usual Newtonian force, apart from the $(1-\theta\sigma)$ factor. The second term, $\sigma\epsilon_{ij}\dot{q}_j$, mimicks a magnetic field. It is the third term which prevents the Lagrangian formalism from working. However, if it is taken in isolation, it gives

$$\ddot{q}_i = \theta\epsilon_{ij}\frac{d}{dt}\frac{\partial V}{\partial q_j}, \quad \text{i.e.} \quad \dot{q}_i = \theta\epsilon_{ij}\frac{\partial V}{\partial q_j} + c_i, \quad i = 1, 2, \tag{12}$$

c_i being two arbitrary constants. Eqs. (12) allow a first-order Lagrangian formulation. For instance, the equations $\dot{q}_i = \theta\epsilon_{ij}\frac{\partial V}{\partial q_j}$ follow from the Lagrangian $L = \frac{1}{2}(\dot{q}_1 q_2 - q_1\dot{q}_2) - V(q_i)$.

Equations (11) admit a first order Lagrangian description, in the limit

$$\sigma\theta \to 1. \tag{13}$$

In this case, the usual Newtonian force term disappears completely (this is kind of antipodal to usual Hamiltonian dynamics), and (11) becomes

$$\dot{q}_i = \epsilon_{ij}\left(\frac{q_j}{\theta} + \theta\frac{\partial V}{\partial q_j}\right) + C_i. \tag{14}$$

A first-order Lagrangian for (14) is:

$$L = \frac{1}{2}(\dot{q}_1 q_2 - \dot{q}_2 q_1) - \theta V(q_i) - \frac{1}{2\theta}(q_1^2 + q_2^2) + C_2 q_1 - C_1 q_2. \tag{15}$$

This Lagrangian contains a term which is first order in time derivatives, the usual potential V, and an additional two-dimensional harmonic oscillator potential.

In fact, by taking at the level of the Hamiltonian equations of motion (9) the limit $\theta\sigma = 1$, which renders Θ (and ω) singular, one observes that

$$\dot{q}_1 = -\theta\dot{p}_2 \quad \text{and} \quad \dot{q}_2 = \theta\dot{p}_1. \tag{16}$$

This means that the limit (13) reduces the number of degrees of freedom of the phase-space by one half, from four to two. Another way to see this is to notice that (14) arises from the one-dimensional Hamiltonian

$$H = \theta V(q,p) + \frac{1}{2\theta}(q^2 + p^2) - C_2 q + C_1 p \tag{17}$$

after relabeling $q_1 = q$, $q_2 = p$. A similar (but not identical) mechanism for dimensional reduction is involved in the Peierls substitution [23], which is based on the noncommutativity of coordinates in an intense magnetic field, (cf. [24], which also refers to earlier work). One also uses the fact that in a two-dimensional first-order system, the coordinates are canonically conjugate to each other.

In conclusion, if $\theta \neq 0$ the equations of motion do not admit the usual variational formulation. One could of course block diagonalize Θ by linear *non-canonical* transformations mixing the q's and p's. A canonical symplectic structure (2) would then result, but also a Hamiltonian not anymore quadratic in the (new) momenta. Hence one would not be able to eliminate the momenta from the equations of Hamilton, and again no explicit Lagrangian formulation would be available.

2.2 Examples

We proceed with examples which do not admit a Lagrangian formulation and display some of their unexpected features [20].

Consider first the anisotropic harmonic oscillator potential, $V = \frac{1}{2}(a_1 q_1^2 + a_2 q_2^2)$, which gives the equations of motion

$$m\ddot{q}_1 = -(1-\theta\sigma)a_1 q_1 + (\sigma + \theta m a_2)\dot{q}_2, \tag{18}$$

$$m\ddot{q}_2 = -(1-\theta\sigma)a_2 q_2 - (\sigma + \theta m a_1)\dot{q}_1. \tag{19}$$

If we chose $\sigma + m\theta a_2 = 0$, then $\sigma + m\theta a_1 \neq 0$, provided $a_1 \neq a_2$. q_1 becomes a harmonic oscillator, whereas q_2 is a harmonic oscillator driven by a periodic force $m\theta(a_1 - a_2)\dot{q}_1$. The solution for q_1 is the usual one, $q_1(t) = q_1(0)\cos\omega_1 t + (q_1'(0)/\omega_1)\sin\omega_1 t$, whereas for q_2 it reads

$$q_2(t) = q_2(0)\cos\omega_2 t + \frac{q_2'(0)}{\omega_2}\sin\omega_2 t + \theta m \frac{q_1'(0)\cos\omega_1 t - \omega_1 q_1(0)\sin\omega_1 t}{1-\theta\sigma}. \tag{20}$$

Above, $m\omega_i^2 = (1-\theta\sigma)a_i$, $i = 1, 2$. If θ is small, the last term in Eq.(20) is a perturbation which produces oscillations around the commutative trajectory. The particle goes on a wiggly path, which averages to the commutative one. If θ is big, or if $|1-\theta\sigma| << 1$, the "perturbation" explodes and dominates the dynamics, which becomes completely different from the commutative one. One sees a qualitative difference between a NC isotropic oscillator (which admits a Lagrangian form) and a NC anisotropic one (no Lagrangian form).

As a second example consider, commutatively speaking, a constant force along q_2, and a harmonic one along q_1, $V = \frac{1}{2}a_1 q_1^2 + b q_2$. The equations of motion are

$$m\ddot{q}_1 = -(1-\theta\sigma)a_1 q_1 + \sigma\dot{q}_2, \tag{21}$$

$$m\ddot{q}_2 = -(1-\theta\sigma)b - (\sigma + \theta m a_1)\dot{q}_1. \tag{22}$$

If $\sigma = 0$, again q_1 is a harmonic oscillator, while q_2 is driven by a constant plus periodic force. The solution is the usual harmonic oscillator for q_1, while for q_2 one has

$$q_2(t) = q_2(0) + [q_2'(0) + q_1(0)\theta a_1]t - \frac{bt^2}{2m} -$$

$$-\theta a_1 \left[\frac{q_1(0)}{\omega_1}\sin\omega_1 t - \frac{q_1'(0)}{\omega_1^2}(1-\cos\omega_1 t)\right]. \tag{23}$$

Again, the NC trajectory wiggles around the commutative one. On the other hand, if $\sigma + \theta m a_1 = 0$, q_2 feels a constant force, while the oscillator q_1 is driven by a linearly time-dependent force $\sigma\dot{q}_2$. One has the solution $q_2(t) = q_2(0) + tq_2'(0) - (1-\theta\sigma)\frac{bt^2}{2m}$, but

$$q_1(t) = q_1(0)\cos\omega_1 t + \frac{q_1'(0)}{\omega_1}\sin\omega_1 t + \frac{\sigma}{a_1}\left[\frac{q_2'(0)}{(1-\theta\sigma)} - \frac{b}{m}t\right] \tag{24}$$

A drastic change occurs: q_1 grows linearly with time (it is not bounded anymore), and oscillates around this path as a commutative oscillator.

As a third example, consider a potential which depends only on one coordinate, say $V = V(q_1)$. If $\sigma = 0$ the equations of motion are

$$m\ddot{q}_1 = -\partial_1 V, \qquad m\ddot{q}_2 = -\theta m\frac{d}{dt}\partial_1 V = -\theta m^2 \frac{d^3 q_1}{dt^3}. \tag{25}$$

If $\theta \neq 0$, q_1 transfers nontrivial dynamics to q_2. More precisely, once $q_1(t)$ is known (its implicit form is $t(q_1) = \int_0^{q_1} \frac{dq'}{\sqrt{V(0)-V(q')}}$), q_2 is fixed by the second equation in (25). To illustrate, consider the quartic potential $V(q_1) = V(0) - \frac{1}{2}m^2 q_1^2 + g q_1^4$. One can not find simple expressions for $q_1(t)$ in a nonlinear problem in general. However, the classical solution satisfying $q_1(t=-\infty) = 0$ and $q_1(t=0) = \frac{m}{\sqrt{g}} = \lambda$ is simple enough

$$q_1(t) = \frac{m}{\sqrt{g}}\frac{2e^{-mt}}{1+e^{-2mt}}. \tag{26}$$

Calculating $q_2(t)$ via (25) one obtains

$$q_2(t) = q_2(0) + q_2'(0)t - \theta m \dot{q}_1(t), \tag{27}$$

radically different from the $\theta = 0$ expression, $q_2(t) = q_2(0) + q_2'(0)t$.

Time-dependent backgrounds appearing "out-of-nowhere" (actually being induced by the dynamics of the other degrees of freedom) are thus possible in NC dynamics.

2.3 Gauge invariance

Another simple type of Hamiltonian worth studying is

$$H = \frac{1}{2} \sum_{i=1,2} (p_i - A_i(q_j))^2, \tag{28}$$

the gauge field A_i being minimally coupled. If the symplectic structure is given by (5, 6) then

$$\dot{q}_i = (p_l - A_l)(\delta_{il} - \theta \epsilon_{ij} \frac{\partial A_l}{\partial q_j}), \tag{29}$$

$$\dot{p}_i = (p_j - A_j)(\frac{\partial A_j}{\partial q_i} + \sigma \epsilon_{ij}). \tag{30}$$

Assuming $\frac{\partial A_j}{\partial t} = 0$ for simplicity, the pair (29) can be rewritten as

$$p_i = A_i + \frac{1}{\Delta} \frac{d}{dt}(q_i + \theta \epsilon_{ij} A_j), \quad i = 1, 2, \tag{31}$$

where $\Delta = 1 + \theta F_{12} + \theta^2 \{A_1, A_2\}_{q_1 q_2}$, with $F_{12} = \partial_1 A_2 - \partial_2 A_1$, $\{A_1, A_2\}_{q_1 q_2} = \frac{\partial A_1}{\partial q_1} \frac{\partial A_2}{\partial q_2} - \frac{\partial A_1}{\partial q_2} \frac{\partial A_2}{\partial q_1}$. Using (31) in (30), and assuming $\frac{\partial A_1}{\partial q_1} = \frac{\partial A_2}{\partial q_2} = 0$, one gets

$$\ddot{q}_1 = \left(1 + \theta \frac{\partial A_1}{\partial q_2}\right) \left[-\dot{A}_1 + \left(\frac{\partial A_2}{\partial q_1} + \sigma \right) \frac{\dot{q}_2}{1 - \theta \frac{\partial A_2}{\partial q_1}} \right], \tag{32}$$

$$\ddot{q}_2 = \left(1 + \theta \frac{\partial A_2}{\partial q_1}\right) \left[-\dot{A}_2 + \left(\frac{\partial A_1}{\partial q_2} - \sigma \right) \frac{\dot{q}_1}{1 - \theta \frac{\partial A_1}{\partial q_2}} \right]. \tag{33}$$

Let us consider the case of a constant magnetic field, $B = F_{12} = \partial_1 A_2 - \partial_2 A_1$. This can be obtained in different gauges. A striking feature of the equations (32,33) is that they are *not* gauge invariant, unless $\theta = 0$. For instance, in the symmetric gauge, $A_1 = -q_2 B/2, A_2 = q_1 B/2$, one has

$$\ddot{q}_1 = \dot{q}_2(\sigma + B + \theta B^2/4) \qquad \ddot{q}_2 = -\dot{q}_1(\sigma + B + \theta B^2/4), \tag{34}$$

whereas in the gauge $A_1 = 0, A_2 = q_1 B$ one gets

$$\ddot{q}_1 = \dot{q}_2 \frac{(\sigma + B)}{(1 + \theta B)} \qquad \ddot{q}_2 = -\dot{q}_1(\sigma + B)(1 + \theta B), \tag{35}$$

which is not even derivable from a Lagrangian. One sees again that σ is inoffensive - it just adds to B - whereas θ even breaks gauge invariance!

Thus, after the existence of a Lagrangian, a second cherished principle is lost due to $\theta \neq 0$ - gauge invariance. Since a non-zero σ mimicks a constant magnetic field, the remedy we propose is to account for a magnetic field B not through the Hamiltonian - which remains free, but through the symplectic form, by requiring $\{p_1, p_2\} = B$. When $\theta = 0$ this is equivalent to (28), and does not pose problems when $\theta \neq 0$.

A formal remedy for this problem was envisaged in [22]. For further discussion of this topic one can consult [17].

3 Quantum theory

3.1 Quantum mechanics: formulation

We extend the three main formalisms of quantum mechanics (operatorial, Schrödinger, path integral) to the case of noncommuting coordinates.

Operatorial quantization is trivially implemented using Eqs (4,6):

$$\frac{d}{dt}\hat{x}_a = i[\hat{x}_a, H] = i[\hat{x}_a, \hat{x}_b]\frac{\partial H}{\partial \hat{x}_b} = \Theta_{ab}\frac{\partial H}{\partial \hat{x}_b}. \tag{36}$$

The equations of motion (36) are an extension of the usual Heisenberg ones. They are the same as (8), with the coordinates becoming operators.

A Schrödinger (wave function) formulation can easily be constructed, once an appropriate basis is chosen in the Hilbert space on which the operators $\hat{x}_a$ act. For instance, chose a basis in the Hilbert space on which the operators $\hat{x}_a$ act, for instance $|q_1, p_2>$, i.e. the eigenstates of the operators $\hat{q}_1$ and $\hat{p}_2$. Then, for an arbitrary state $|\psi>$, define the wave function (half in coordinate space, half in momentum space)

$$\psi(q_1, p_2, t) \equiv <\psi(t)|q_1, p_2> . \tag{37}$$

The commutation relations (4) imply that the operators $\hat{q}_2$ and $\hat{p}_1$ have the following action on $\psi(q_1, p_2)$:

$$\hat{q}_2\psi = i(\partial_{p_2} - \theta\partial_{q_1})\psi, \qquad \hat{p}_1\psi = i(-\partial_{q_1} + \sigma\partial_{p_2})\psi. \tag{38}$$

If $H = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + V(\hat{q}_1, \hat{q}_2)$, (38) leads to the Schrödinger equation

$$i\frac{d}{dt}\psi = H\psi = \left[\frac{1}{2m}\left(p_2^2 - (\partial_{q_1} - \sigma\partial_{p_2})^2\right) + V(q_1, i\partial_{p_2} - i\theta\partial_{q_1})\right]\psi(q_1, p_2). \tag{39}$$

If $\sigma = 0$, a momentum-space wave function $\psi(p_1, p_2, t)$ also exists; it will be discussed later, along with the examples.

A phase space path integral for systems obeying the commutation relations (4) was constructed in [19]. Since we saw that for generic systems, if $\theta \neq 0$, equations (8) do not admit a Lagrangian formulation, one can at best hope for a phase-space path integral formulation of the quantum theory corresponding to the action (7). This is provided [19] by the path integral

$$Z = \int \prod_{k=1}^{4} Dx_k e^{iS} = \int \prod_{k=1}^{4} Dx_k e^{i\int dt\left(\frac{1}{2}\omega_{ij}x_i\dot{x}_j - H(x)\right)}. \tag{40}$$

To put it briefly the prescription (40) is simple: if $[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}$ then $Z = \int Dx e^{i\int dt(\Theta_{ij}^{-1}\frac{x_i\dot{x}_j}{2} - H)}$, and general: it applies to any Hamiltonian H. The above path integral can be derived by elementary means from the canonical formulation [19]. All one needs to know is that Z represents a transition amplitude between two states of a given Hilbert space, and that time-ordering of operators is enforced, as usual, by the path integral, $\int Dx O_1 O_2 e^{iS} = \langle T\{\hat{O}_1\hat{O}_2\}\rangle$.

Integration of the momenta is particularly transparent in the above path integral, and the result - detailed in Section 3.3 - will be a simple and universal (the correction term is system independent) effective Lagrangian: the *simplest* one not excluded by the no-go argument in [20].

3.2 Quantization: examples

We apply the formalism to the examples considered classically in Section 2.

For an harmonic potential, it can be shown by path integrals [19], or operatorially [5], that the only change induced by NC is an anisotropy of the oscillator. However, *starting* with an anisotropic oscillator, $V = \frac{1}{2}(a_1q_1^2 + a_2q_2^2)$, $a_1 \neq a_2$, makes an important difference. The equations of motion are the same as in (18,19), with $q_{1,2}$ operators. For simplicity, assume $\sigma + m\theta a_2 = 0$; then $\sigma + m\theta a_1 \neq 0$.

$\hat{q}_2$ is driven by a periodic force and, being of the form (20), transitions between the states of the quantum system will appear.

Our second example, $V = \frac{1}{2}a_1 q_1^2 + bq_2$, also exhibits peculiar behaviour. If $\sigma = 0$, the operator solutions of (21,22) again involve transitions which would be absent if $\theta = 0$. If $\sigma + \theta m a_1 = 0$, changes are more dramatic. Eq. (24) shows that the particle is not bounded anymore along q_1, in contrast with the commutative case.

Third, consider the case in which the potential depends only on one coordinate, $V = V(q_1)$. If $\sigma = 0$ an interesting phenomenon takes place. The commutation relations (4) admit a representation in the basis $|p_1, p_2>$, $\psi(p_1, p_2, t) \equiv <\psi(t)|p_1, p_2>$:

$$\hat{q}_1\psi = (i\partial_{p_1} + \theta\alpha p_2)\psi, \qquad \hat{q}_2\psi = (i\partial_{p_2} + \theta(1+\alpha)p_1)\psi(p_1, p_2), \tag{41}$$

with α a parameter, and the Schrödinger equation becomes

$$i\frac{d}{dt}\psi = \left[\frac{1}{2m}\left(p_1^2 + p_2^2\right) + V(i\partial_{p_1} + \theta\Lambda p_2, i\partial_{p_2} + \theta(1+\Lambda)p_1)\right]\psi(p_1, p_2) \tag{42}$$

This equation is (gauge) invariant under shifts of α by Λ,

$$\alpha \rightarrow \alpha - \Lambda \tag{43}$$

combined with multiplications of the momentum-space wave-function by a phase $e^{i\Lambda\theta p_1 p_2}$,

$$\psi(p_1, p_2) \rightarrow e^{i\Lambda\theta p_1 p_2}\psi(p_1, p_2). \tag{44}$$

θ plays the role of a "magnetic field" in momentum space.

In particular, when $\Lambda = \alpha$, $\hat{q}_1$ becomes θ-independent. Then, if $V = V(q_1)$ and $\sigma = 0$, the Schrödinger equation is θ-independent. It has consequently the same spectrum with the commutative problem, although classically the NC system does not even admit a Lagrangian formulation! For example, $V(q_1, q_2) = V(q_1) = V(0) - \frac{1}{2}m^2 q_1^2 + g q_1^4$, on a NC space, gives rise to a nonlinear system without classical Lagrangian formulation, but which has the same spectrum as the corresponding commutative (Lagrangian) system.

If $V = V(q_1, q_2)$ the above gauge invariance persists, but does not eliminate θ from the wave equation.

3.3 Effective Lagrangian

We path-integrate over the momenta in (40), to obtain the effective Lagrangian. Starting from the partition function

$$\int Dq_1 Dq_2 Dp_1 Dp_2 e^{iS} \tag{45}$$

with action

$$S = \int_0^T dt[p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{\theta}{2}(p_1\dot{p}_2 - p_2\dot{p}_1) - \frac{p_1^2}{2m} - \frac{p_2^2}{2m} - V(q)], \tag{46}$$

we wish to integrate over the momenta p_1, p_2. The potential part $V(q)$ depends only on q_1 and q_2 and plays no role in what follows (the method is valid for any $V(q)$, or Hamiltonian with separate quadratic dependence upon momenta). We divide the time interval T in n subintervals $\epsilon = \frac{T}{n}$ ($n \rightarrow \infty$ achieves the continuum limit), and choose for simplicity the discrete derivative $\dot{x}_k \equiv \frac{x_{k+1} - x_k}{\epsilon}$ (no issues requiring symmetric operations appear in the following)). The relevant part of the discretized action (excluding $V(q)$ for now) becomes

$$\tilde{S} = \sum_{k=0}^{n}[\epsilon p_1^{(k)} v_1^{(k)} + p_2^{(k)} v_2^{(k)} + \frac{\theta}{2}(p_1^{(k)} p_2^{(k+1)} - p_2^{(k)} p_1^{(k+1)}) - \frac{\epsilon}{2m}(p_1^{(k)})^2 - \frac{\epsilon}{2m}(p_2^{(k)})^2]. \tag{47}$$

The clearest way to proceed with the coupled Gaussian integrals is to introduce matrix notation. Define the column vectors

$$V \equiv \epsilon(v_1^{(0)}, v_1^{(1)}, \ldots, v_1^{(n)} \ldots v_2^{(0)}, v_2^{(1)}, \ldots, v_2^{(n)} \ldots)^T \tag{48}$$

$$P \equiv (p_1^{(0)}, p_1^{(1)}, \dots, p_1^{(n)} \dots p_2^{(0)}, p_2^{(1)}, \dots, p_2^{(n)} \dots)^T \tag{49}$$

and the matrix

$$J = -a \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 & b & 0 & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & 0 & 0 & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \\ 0 & -b & 0 & \cdot & \cdot & 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & -b & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}.$$

where $a = \frac{\epsilon}{2m}$, $b = \frac{m\theta}{\epsilon}$. Its inverse J^{-1} has the same form as above, but with different entries a', b', namely $a' = 1/a$ and $b' = -b$ (the off diagonal part changes sign and the overall factor is reversed). In matrix notation the discrete action becomes

$$\tilde{S} = P^T V + P^T J P. \tag{50}$$

the coordinate transformation

$$\bar{P} \equiv P + \frac{1}{2} J^{-1} V \tag{51}$$

does not change the path integral measure ($D\bar{P} = DP$), and leads to

$$\tilde{S} = \bar{P}^T J \bar{P} - \frac{1}{4} V^T J^{-1} V. \tag{52}$$

The first term is now integrated out - and no more dependency on momenta appears, whereas the second term leads to an exponent of the form (modulo a factor of i)

$$-\frac{1}{4} V^T J^{-1} V = \sum_{k=0}^{n} [\epsilon \frac{m}{2} (v_1^{(k)})^2 + \epsilon \frac{m}{2} (v_2^{(k)})^2 - \frac{\theta m^2}{2} (v_1^{(k)} v_2^{(k+1)} - v_2^{(k)}) v_1^{(k+1)})] \tag{53}$$

Upon taking the continuum limit $\epsilon \to 0$ we obtain

$$\int Dq_1 Dq_2 Dp_1 Dp_2 e^{iS} = N \int Dq_1 Dq_2 e^{i \int_0^T dt L_{eff}(q_i, v_i, a_i)} \tag{54}$$

with

$$L_{eff} = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{\theta m^2}{2}(\dot{q}_1 \ddot{q}_2 - \dot{q}_2 \ddot{q}_1) - V(q_1, q_2) \tag{55}$$

and N a constant not depending on the q's. We have reintroduced the potential term. The second term in (55)

$$\Delta L = -\frac{1}{2} \theta m^2 (\dot{q}_1 \ddot{q}_2 - \dot{q}_2 \ddot{q}_1). \tag{56}$$

is the correction due to noncommutativity and it has an universal character as it is independent of the potential V.

The term (56) was previously studied in detail in [7] starting from different considerations, and in fact its appearance can be traced back to earlier developments (cf. [4, 8]). Lukierski et al. [7] added (56) to a free Lagrangian $\frac{m}{2}\vec{v}^2$, to provide a dynamical realization of a centrally extended (2+1)-dimensional Galilean algebra. Upon constrained quantization of the resulting higher order action (which circumvents the no-go theorem of [20]) noncommutative dynamics was shown to emerge for appropriate choices of canonical variables. The negative energy resulting from two "internal modes" posed no problem, since they were easily shown to decouple from the four relevant degrees of freedom. Interactions were subsequently introduced in a constrained way, in order to keep the ghosts harmless. (the potential V was contrained in their analysis to depend only on the "would-be" NC coordinates).

We went in the opposite direction; starting from an *arbitrary* system with Heisenberg noncommutativity of the coordinates, we deduced the correction (56) term by path integral methods (this lecturer is not aware of any canonical approach doing the same thing). Incidentally, the path integral derivation gives a technical understanding of why even higher order terms are forbidden in the effective

Lagrangian. It starts *ab initio* with arbitrary potentials $V(q_1, q_2)$, in contrast to the inverse route of Ref. [7], where one has to carefully pin down the (in the end NC) variables on which interactions must depend.

The price to be paid for the initial noncommutativity of the coordinates is the appearance of second order time derivatives in the effective action, and the ensuing lack of appropriate boundary/initial conditions for the two (fortunately ghost-like) additional degrees of freedom. This is not an issue, however, since the "internal modes" have to be eliminated anyway [7].

More precisely, the equations of motion engendered by (55) are of third order in time derivatives,

$$\epsilon_{ij}\theta m^2 \frac{d^3 q_j}{dt^3} + m\ddot{q}_i + \partial_{q_i} V = 0. \tag{57}$$

No fourth-order time derivatives arise for q_1, q_2, and this leads to two constraints in the Hamiltonian formulation. Six constants (BC/IC) are still required, two more in comparison with the commutative case. We are not able to provide them, since we can at the very beginning start with only four constants (for instance the initial and final values of q_1 and p_2). This apparent indeterminacy is a consequence of the initial nonocommutativity of q_1 and q_2, but poses no serious problem, since one can show [7] that exactly those constants are needed for the two "internal" modes. Now these modes must be eliminated for consistency (an elementary analysis appears in [8]).

The equations of motion (57) engendered by the effective Lagrangian (obtained via path integration over momenta) need to be identical to the equations of motion (11), obtained via elimination of *classical* momenta. In the first place, they are not of the same order. Nevertheless, a comparison reveals some intriguing relationship. For Hamiltonians of the form (10) the completely classical coordinate equations of motion (11) read, for $\sigma = 0$,

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i} + \theta\epsilon_{ij}\frac{d}{dt}\frac{\partial V}{\partial q_j}, \quad i = 1, 2. \tag{58}$$

By taking one more time derivative, we obtain

$$\epsilon_{ij}\theta m^2 \frac{d^3 q_j}{dt^3} + m\ddot{q}_i + \partial_{q_i} V + m\theta^2 \frac{d^2}{dt^2}\frac{\partial V}{\partial q_i} = 0. \tag{59}$$

In the limit of small θ, in which θ^2 terms can be neglected, (57) and (59) are identical! The effective classical solutions appear to be given by a small θ limit of the purely classical solutions!

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