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Nonlocal Field Theory and p-Adic Strings

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Abstract

We consider nonlocal field theory aspects of some p-adic strings. In particular, Lagrangians of p-adic open scalar strings, for single p as well as for collective primes p, are reviewed. They contain space-time nonlocality through the d'Alembertian $\Box$ in the argument of exponential and the Riemann zeta function.

1 Introduction

It is well known that standard field theory is a local theory, i.e. Lagrangian depends on fields and their space-time derivatives of the first order, $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$. Standard model of elementary particles is a local quantum field theory. When Lagrangian contains higher than first order derivatives, but still finite, e.g. $\mathcal{L} = \mathcal{L}\Big(\varphi, \partial_\mu\varphi, ..., \partial_\mu^n \varphi\Big)$, then there arise problems with Ostrogradski instabilities. However, when the number of higher derivatives becomes infinite then the situation changes and there are theories which are free of Ostrogradski instabilities. Nonlocal field Lagrangians considered in this paper have the form $\mathcal{L} = \mathcal{L}(\varphi, \mathcal{F}(\Box)\varphi)$, where $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian, and $\mathcal{F}(\Box)$ is related to the operators $\exp(\alpha\Box)$ and $\zeta(\beta\Box)$.

In the last decade or so, nonlocal field theories with infinite number of derivatives have attracted much attention. They have mainly origin in ordinary and p-adic string theory, which emerged in 1987 [1]. Many approaches to p-adic strings have been considered, but the most interesting are strings whose world sheet is p-adic and other properties are real and complex valued. Four-point scattering amplitudes of open scalar ordinary and p-adic strings are connected at the tree level by their product, which is a constant number. In this product formula ordinary and p-adic strings are treated on an equal footing (see, e.g., [2, 3] for a review). Some other systems have been also p-adically modelled and it led to p-adic mathematical physics (for a recent review we refer to [4]).

Unlike ordinary strings, for the open scalar p-adic strings there is an exact effective nonlocal field theory with Lagrangian [5, 6] which describes four-point scattering amplitudes and all higher ones at the tree-level. Note that this Lagrangian does not contain p-adic numbers explicitly, but only a prime number p which can be viewed as a real as a p-adic parameter. Since this Lagrangian is simple and exact at the tree-level, it is significantly used in the last decade and many aspects of p-adic string dynamics have been considered, compared with dynamics of ordinary strings and applied to nonlocal cosmology (see, e.g., [7, 8, 9, 10, 11, 12, 13] and references therein).

The present paper is mainly a brief review of Lagrangians, and some their basic properties, for single and for entire p-adic sector of open scalar strings. Note that a field theory and cosmology based on the Riemann zeta function was proposed in [14].

2 Nonlocality of p-Adic Scalar Strings

Like ordinary string theory, the p-adic one also started with scattering amplitudes by an analogous way. Let $v \in V = \{\infty, 2, 3, ..., p, ...\}$. The crossing symmetric Veneziano amplitude for scattering of two open scalar strings is defined by the Gel'fand-Graev-Tate beta function

$$A_v(a,b) = g_v^2 \int_{\mathbb{Q}_v} |x|_v^{a-1}\, |1-x|_v^{b-1}\, d_v x\,, \tag{1}$$

where $a = -\alpha(s) = -\frac{s}{2} - 1$, $b = -\alpha(t)$ and $c = -\alpha(u)$ are complex-valued kinematic variables with the condition $a + b + c = 1$. $\mathbb{Q}_\infty$ and $\mathbb{Q}_p$ are real and p-adic number fields, respectively. Note that variable x in the integrands is related to the string world-sheet: world-sheet of ordinary and p-adic strings is treated by real and p-adic numbers, respectively (see, e.g. [2, 3] and [15] for basic properties of p-adic numbers and their functions). Thus p-adic strings differ from the ordinary ones by p-adic treatment only of the world-sheet. After calculation of integrals in (1) one obtains

$$A_\infty(a,b) = g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}\,, \tag{2}$$

$$A_p(a,b) = g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}\,, \tag{3}$$

where ζ is the Riemann zeta function. Expression (2) is for the ordinary case and (3) for the p-adic one.

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}\,, \quad s = \sigma + i\tau\,, \quad \sigma > 1\,, \tag{4}$$

has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Taking product of p-adic string amplitudes (3) over p and using (4), one obtains

$$\prod_p A_p(a,b) = \frac{\zeta(a)}{\zeta(1-a)} \frac{\zeta(b)}{\zeta(1-b)} \frac{\zeta(c)}{\zeta(1-c)} \prod_p g_p^2\,, \tag{5}$$

that gives a simple formula

$$\prod_v A_v(a,b) = \prod_v g_v^2 = \text{const.} \tag{6}$$

Product of p-adic amplitudes in (5) is divergent [16], but it becomes convergent after an appropriate regularization. Request that product of amplitudes (6) be finite, implies finiteness of the product of coupling constants, i.e. $g_\infty^2 \prod_p g_p^2 = const$. There are three interesting possibilities for g_p^2: (i) $g_p^2 = 1$, (ii) $g_p^2 = \frac{p^2}{p^2-1}$, what gives $\prod_p g_p^2 = \zeta(2)$, (iii) $g_p^2 = |\frac{m}{n}|_p$, where m and n are any two nonzero integers, and it gives $g_\infty^2 \prod_p g_p^2 = |\frac{m}{n}|_\infty \prod_p |\frac{m}{n}|_p = 1$.

From (6) it follows that the ordinary Veneziano amplitude, which is a special function, can be expressed as product of all inverse p-adic counterparts, which are elementary functions. This is a consequence of the Gel'fand-Graev-Tate beta functions and it is not a general property of string scattering amplitudes. In the general case, product of string amplitudes is a function of kinematic variables.

Another interpretation of expression (6) is related to an adelic string. However, an adelic string should have an adelic world-sheet. It has not been obtained so far a scattering amplitude of two open scalar strings with their adelic world-sheets. Thus, concept of an adelic string with adelic world-sheet is not well founded and remains questionable. But p-adic strings with p-adic world-sheet are well defined, and the product of scattering amplitudes for open scalar strings has a useful meaning.

The exact tree-level Lagrangian of the effective scalar field φ, which describes the open p-adic string tachyon, is [5, 6]

$$\mathcal{L}_p = \frac{m^D}{g_p^2} \frac{p^2}{p-1} \Big[-\frac{1}{2}\, \varphi\, p^{-\frac{\Box}{2m^2}}\, \varphi + \frac{1}{p+1}\, \varphi^{p+1} \Big]\,, \tag{7}$$

where p is a prime. The corresponding equation of motion for (7) is

$$p^{-\frac{\Box}{2m^2}}\, \varphi = \varphi^p\,, \tag{8}$$

and it has been investigated by many authors (see, e.g. [9] and references therein). It has trivial solutions $\varphi = 0$, $\varphi = +1$ for all p and another solution $\varphi = -1$ for odd p. There are also inhomogeneous solutions resembling solitons. This equation separates in arguments and for any spatial direction x^i one has

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}} \exp\Big(- \frac{p-1}{2m_p^2\, p \ln p}(x^i)^2\Big).$$

Now we consider construction of Lagrangians which take into account entire p-adic sector of an open scalar string. In particular, an appropriate such Lagrangian should describe scattering amplitude (5), which contains the Riemann zeta function. Consequently, this Lagrangian has to contain the Riemann zeta function with the d'Alembertian in its argument. We have to look for possible Lagrangians which contain the Riemann zeta function and have their origin in p-adic Lagrangian (7). There are additive and multiplicative approach, and in the sequel we shall mainly present the additive one.

2.1 Additive approach

Prime number p in (7) can be replaced by any natural number $n \geq 2$ and consequences also make sense. Let us introduce a Lagrangian which incorporates all the above Lagrangians (7), with p replaced by $n \in \mathbb{N}$. The corresponding sum of all Lagrangians $\mathcal{L}_n$ is

$$L = \sum_{n=1}^{+\infty} C_n \, \mathcal{L}_n = m^D \sum_{n=1}^{+\infty} \frac{C_n}{g_n^2} \frac{n^2}{n-1} \Big[-\frac{1}{2}\, \phi \, n^{-\frac{\Box}{2m^2}} \, \phi + \frac{1}{n+1} \, \phi^{n+1} \Big] , \tag{9}$$

whose concrete form depends on the choice of coefficients C_n and coupling constants g_n. Denote $\frac{C_n}{g_n^2} \frac{n^2}{n-1} = D_n, \; n = 1, 2, \ldots$. The following simple cases lead to the Riemann zeta function: $D_n = 1$, $D_n = n+1$, $D_n = \mu(n)$, $D_n = -\mu(n)(n+1)$, $D_n = (-1)^{n-1}$, $D_n = (-1)^{n-1}(n+1)$, where $\mu(n)$ is the Möbius function.

The case $D_n = 1$ was considered in [17, 18]. Obtained Lagrangian is

$$L = m^D \Big[-\frac{1}{2}\, \phi \, \zeta\Big(\frac{\Box}{2\, m^2}\Big)\, \phi + \mathcal{AC} \sum_{n=1}^{+\infty} \frac{\phi^{n+1}}{n+1} \Big] , \tag{10}$$

where $\mathcal{AC}$ denotes analytic continuation.

The case $D_n = n+1$ was investigated in [19] and the corresponding Lagrangian is

$$L = m^D \Big[-\frac{1}{2}\, \phi \, \Big\{ \zeta\Big(\frac{\Box}{2\, m^2} - 1\Big) + \zeta\Big(\frac{\Box}{2\, m^2}\Big) \Big\} \, \phi + \frac{\phi^2}{1-\phi} \Big] . \tag{11}$$

The cases with the Möbius function $\mu(n)$ are presented in [20] and [21]. Recall that its explicit definition is

$$\mu(n) = \begin{cases} 0, & n = p^2 m \, , \\ (-1)^k, & n = p_1 p_2 \cdots p_k, \; p_i \neq p_j \, , \\ 1, & n = 1, \; (k=0) \, , \end{cases} \tag{12}$$

and it is related to the inverse Riemann zeta function by the following way:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1 \, . \tag{13}$$

The corresponding Lagrangian for $D_n = \mu(n)$ is

$$L = m^D \Big[-\frac{1}{2}\, \phi \, \frac{1}{\zeta\Big(\frac{\Box}{2m^2}\Big)} \, \phi + \int_0^\phi \mathcal{M}(\phi)\, d\phi \Big], \tag{14}$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n)\, \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots$.

When $D_n = -\mu(n)\,(n+1)$ then the Lagrangian is

$$L = m^D \Big\{ \frac{1}{2}\, \phi \Big[\frac{1}{\zeta\Big(\frac{\Box}{2m^2} - 1\Big)} + \frac{1}{\zeta\Big(\frac{\Box}{2m^2}\Big)} \Big] \, \phi - \phi^2 \, F(\phi) \Big\} , \tag{15}$$

where $F(\phi) = \sum_{n=1}^{+\infty} \mu(n)\phi^{n-1} = 1 - \phi - \phi^2 - \phi^4 + ...$

The case with $D_n = (-1)^{n-1}\,(n+1)$ was introduced in [22]. Recall that

$$\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n^s} = (1-2^{1-s})\,\zeta(s), \quad s = \sigma + i\tau\,, \quad \sigma > 0\,, \tag{16}$$

which has analytic continuation to the entire complex s plane without singularities, and can be presented by series [24]

$$(1-2^{1-s})\,\zeta(s) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c} n \\ k \end{array}\right)(k+1)^{-s} \tag{17}$$

convergent for all real values of the variable s. At point $s = 1$, one has $\lim_{s\to 1}(1-2^{1-s})\,\zeta(s) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n} = \log 2$. Applying (16) to (9) and using analytic continuation one obtains

$$\begin{aligned} L = -\,m^D\Big[\frac{1}{2}\,\phi\Big\{\Big(1-2^{2-\frac{\Box}{2m^2}}\Big)\,\zeta\Big(\frac{\Box}{2\,m^2}-1\Big) \\ +\Big(1-2^{1-\frac{\Box}{2m^2}}\Big)\,\zeta\Big(\frac{\Box}{2\,m^2}\Big)\Big\}\,\phi\;-\;\frac{\phi^2}{1+\phi}\Big]\,. \end{aligned} \tag{18}$$

Recently it was considered the case $D_n = (-1)^{n-1}$ [23]. The corresponding Lagrangian is

$$L = m^D\,\Big[-\frac{1}{2}\phi\Big(1-2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi\,+\phi - \frac{1}{2}\,\log(1+\phi)^2\Big]. \tag{19}$$

The potential is

$$V(\phi) = -L(\Box = 0) = m^D\Big[\frac{1}{4}\,\phi^2 - \phi + \frac{1}{2}\log(1+\phi)^2\Big], \tag{20}$$

which has one local maximum $V(0) = 0$ and one local minimum at $\phi = 1$. It is singular at $\phi = -1$, i.e. $V(-1) = -\infty$, and $V(\pm\infty) = +\infty$. The equation of motion is

$$\Big(1-2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi = \frac{\phi}{1+\phi}, \tag{21}$$

which has two trivial solutions: $\phi = 0$ and $\phi = 1$.

2.2 Multiplicative approach

In the multiplicative approach the Riemann zeta function emerges through its product form in (4). We start by p-adic Lagrangian (7) with $g_p^2 = \frac{p^2}{p^2-1}$ and rewrite (7) in the form

$$\begin{aligned} \mathcal{L}_p = & m^D\,\Big\{\frac{1}{2}\,\varphi\,\Big[\Big(1-p^{-\frac{\Box}{2m^2}+1}\Big)+\Big(1-p^{-\frac{\Box}{2m^2}}\Big)\Big]\,\varphi \\ & -\varphi^2\Big(1-\varphi^{p-1}\Big)\Big\}\,. \end{aligned} \tag{22}$$

Taking products

$$\prod_p(1-p^{-\frac{\Box}{2m^2}+1})\,,\quad \prod_p(1-p^{-\frac{\Box}{2m^2}})\,,\quad \prod_p(1-\varphi^{p-1}) \tag{23}$$

in (22) at the relevant places one obtains Lagrangian

$$\mathcal{L} = \frac{m^D}{g^2}\,\Big\{\frac{1}{2}\,\phi\Big[\frac{1}{\zeta\Big(\frac{\Box}{2m^2}-1\Big)}+\frac{1}{\zeta\Big(\frac{\Box}{2m^2}\Big)}\Big]\,\phi - \phi^2\,\Phi(\phi)\Big\}\,, \tag{24}$$

where $\Phi(\phi) = \prod_p\Big(1-\phi^{p-1}\Big) = 1 - \phi - \phi^2 + \phi^3 - \phi^4 + ...$ Lagrangian obtained by this way [21] is similar to the above one (15). These two Lagrangians describe the same field theory in the week field approximation.

3 Concluding remarks

In the previous section we presented some Lagrangians which can be further used for investigation of the p-adic sector of open scalar strings. They contain the Riemann zeta function and they are also interesting examples of what we call zeta field theory. The corresponding potentials, which are $V(\phi) = -L(\Box = 0)$, and equations of motions are considered in cited references. All these nonlocal field theory models contain tachyons.

Two above Lagrangians presented in (18) and (19) seems to be very intersting. Unlike other Lagrangians, these have no singularity with respect to the d'Alembertian $\Box$ and it is easier to apply pseudodifferential treatment. This Lagrangian should be useful in its application to nonlocal cosmology, in particular using linearization procedure (see, e.g. [25] and references therein).

We would like also to point out Lagrangians (15) and (24), since they are mutually very similar. These Lagrangians describe the same model at the weak field approximation, although they are constructed using rather different approaches.

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