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Perturbative QCD and the Determination of $\alpha_{\rm s}$

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Abstract

The precise determination of the coupling $\alpha_{\rm s}$ is one of the most important results in perturbative QCD. After a brief review of some open questions in the extraction of $\alpha_{\rm s}$ from the hadronic decays of the τ lepton, I discuss a new determination of $\alpha_{\rm s}(M_\tau^2)$, based on an improved perturbation expansion of the Adler function.

1 Introduction

At almost 40 years since its birth [1, 2], quantum chromodynamics (QCD), the modern description of the strong interactions, appears to be a consistent theory reaching the level of precise predictions. Since color symmetry is exact, the theory has a small number of fundamental constants. The strong coupling $\alpha_{\rm s}$ is actually the only constant in the particular limit of massless QCD. After renormalization, the dependence of the coupling on the scale is governed by the renormalization group equation (RGE):

$$\mu^2 \frac{d\alpha_{\rm s}}{d\mu^2} = \beta(\alpha_{\rm s}) = -\sum_{j\geq 0} \beta_j \alpha_{\rm s}^{j+2}, \tag{1}$$

which to lowest order predicted the famous "asymptotic freedom" and suggested the "confinement" of the quarks inside the hadrons, the crucial ideas that imposed QCD as a successful theory in the early 70'.

The coefficients β_j, which for $j \geq 2$ depend on the renormalization scheme, have been calculated to four loops in the $\overline{MS}$ scheme [3, 4]:

$$\begin{aligned}
\beta_0 &= \frac{33 - 2N_f}{12\pi}\,, \\
\beta_1 &= \frac{153 - 19N_f}{24\pi^2}\,, \\
\beta_2 &= \frac{77139 - 15099N_f + 325N_f^2}{3456\pi^3}\,, \\
\beta_3 &\approx \frac{29243 - 6946.3N_f + 405.089N_f^2 + 1.49931N_f^3}{256\pi^4},
\end{aligned} \tag{2}$$

where the numerical constants are functions of the $SU(3)_c$ group invariants and N_f is the number of active quark flavours at the scale μ^2.

An approximate solution of the equation (1) to four loops, written as

$$\begin{aligned}
\alpha_{\rm s}(\mu^2) &= \frac{1}{\beta_0 L} - \frac{1}{\beta_0^3 L^2}\beta_1 \ln L \\
&+ \frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2 L - \ln L - 1\right) + \frac{\beta_2}{\beta_0}\right) \\
&+ \frac{1}{\beta_0^4 L^4}\left(\frac{\beta_1^3}{\beta_0^3}\left(-\ln^3 L + \frac{5}{2}\ln^2 L + 2\ln L - \frac{1}{2}\right)\right) \\
&- \frac{1}{\beta_0^4 L^4}\left(3\frac{\beta_1\beta_2}{\beta_0^2}\ln L + \frac{\beta_3}{2\beta_0}\right),
\end{aligned} \tag{3}$$

depends on the arbitrary parameter $\Lambda_{\overline{MS}}$ and is valid at large $L = \ln\left(\mu^2/\Lambda^2_{\overline{MS}}\right)$. An alternative way of writing the solution is based on the expansion in powers of the coupling $a_0 \equiv \alpha_s(\mu_0^2)/\pi$ at a fixed scale:

$$\begin{aligned}\alpha_s(\mu^2) \quad &= a_0 - \beta_0\eta a_0^2\left(\beta_1\eta - \beta_0^2\eta^2\right)a_0^3 \\ &-\left(\beta_2\eta - \frac{5}{2}\beta_0\beta_1\eta^2 + \beta_0^3\eta^3\right)a_0^4 \\ &-\left(\beta_3\eta - \frac{3}{2}\beta_1^2\eta^2 - 3\beta_0\beta_2\eta^2 + \frac{13}{3}\beta_0^2\beta_1\eta^3 - \beta_0^4\eta^4\right)a_0^5.\end{aligned} \tag{4}$$

This representation is useful if the scales μ and μ_0 are close to each other, because then the parameter $\eta = \ln\mu^2/\mu_0^2$ entering the coefficients of the expansion is small.

The arbitrary constant in the solution of the differential equation (1), chosen either as $\Lambda_{\overline{MS}}$ in (3) or as $\alpha_s(\mu_0^2)$ in (4), can be determined only from experiment. The present determinations of the strong coupling α_s are based on all types of reactions that contain gluons: deep inelastic ep scattering, e^+e^- collisions, $p(\bar{p})$-p collisions, τ hadronic decays, Υ decays. The recent determinations at various scales are in an impressive agreement among each other[1], providing one of the most precise tests of perturbative QCD.

The hadronic decays of the τ lepton allow a particularly interesting determination of α_s, since it is done at a relatively low scale ($M_\tau = 1.78$ GeV). The recent calculation of the Adler function to four loops [7], the same order at which the β function is known [3, 4], stimulated the interest in an updated determination of $\alpha_s(M_\tau^2)$ [8]-[11]. It is interesting to note that the average [6]

$$\alpha_s(M_\tau^2) = 0.330 \pm 0.014, \tag{5}$$

leads to a value $\alpha_s(M_Z^2) = 0.1197 \pm 0.0016$, slightly higher than the world average [5, 6]

$$\alpha_s(M_Z^2) = 0.1184 \pm 0.0007, \tag{6}$$

which includes all the determinations at various scales.

The determination of $\alpha_s(M_\tau^2)$ raises several theoretical problems, among which we mention the validity of the Operator Product Expansion (OPE) near the timelike axis in the complex momentum plane, the magnitude of the nonperturbative contributions (condensates) and the ambiguities of the perturbative expansion, especially the dependence on the renormalization scale. In particular, two choices of the scale, corresponding to the standard fixed-order perturbation theory (FOPT) and to the so-called contour-improved perturbation theory (CIPT) [12, 13], lead to predictions of $\alpha_s(M_\tau^2)$ which differ by about 0.024 [8, 9, 10]. This discrepancy, the largest systematic theoretical uncertainty in the previous determinations of α_s at the M_τ scale, did not go away by adding the recently calculated higher-order term [7]. The problem is complicated by the high order behaviour of the series: from particular classes of Feynman diagrams it is known that the renormalized perturbation series in QCD are divergent and are usually assumed to be asymptotic series. From independent arguments [19], it is known that correlation amplitudes, regarded as functions of the coupling, are singular at $\alpha_s = 0$. For QED these facts are known since a long time [18], but they do not affect the phenomenological predictions since the coupling is very small. By contrast, for a large coupling like $\alpha_s(M_\tau^2)$ in QCD the consequences are nontrivial.

In the present talk I discuss a new perturbation expansion in QCD, which includes theoretical knowledge about the high-order behaviour of the series. I first briefly review the fixed-order and the contour-improved expansions relevant for the determination of α_s from τ decays. Then I define the improved expansions based on the conformal mapping of the Borel plane and apply them for a new determination of α_s.

[1] By convention, the comparison is made at the scale equal to the mass of the Z boson. The values at various scales are evolved to the reference scale by means of the RGE (1).

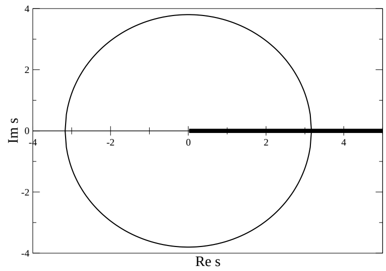

Figure 1: The integration contour of Eq. (9) in the complex cut s-plane.

2 Fixed-order and contour-improved expansions in perturbative QCD

We recall the definition of the Adler function in massless QCD defined as the derivative

$$D(s) = -s\frac{\mathrm{d}\Pi(s)}{\mathrm{d}s}, \tag{7}$$

of the invariant amplitude $\Pi(s)$ of the correlator

$$i\int dx\, e^{iqx}\, \langle\Omega|\, T\{J_\mu(x)\, J_\nu(0)^\dagger\}|\Omega\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(s), \qquad s = q^2, \tag{8}$$

where J_ν is a current operator for u, d or s quarks. From the general principles of quantum field theory (QFT) it follows that $\Pi(s)$ is analytic in the complex s-plane cut along the positive (timelike) axis.

The quantity relevant for the extraction of $\alpha_\mathrm{s}(M_\tau^2)$ is the integral

$$\delta^{(0)} = \frac{1}{2\pi i}\oint\limits_{|s|=M_\tau^2} \frac{ds}{s}\,\omega(s)\,\widehat{D}(s), \tag{9}$$

along the contour shown in Fig. 1, where $\omega(s) = 1-2s/M_\tau^2+2(s/M_\tau^2)^3-(s/M_\tau^2)^4$ and $\widehat{D}(s) = D(s)-1$ is the so-called reduced Adler function. By analyticity, $\delta^{(0)}$ is related to the total τ hadronic width, defined by means of an integral along the real axis up to $s = M_\tau^2$. The advantage of the representation (9) is that along the circle, *i.e.* far from the hadronic thresholds, one case use Operator Product Expansion (OPE) and perturbative QCD for the calculation of the function $\widehat{D}(s)$. It turns out that in the present case the contributions of the condensates is small [9], therefore we concentrate on the pure perturbative part.

We consider the formal expansion

$$\widehat{D}(s) = \sum_{n\geq 1}[K_n + \kappa_n(-s/\mu^2)]\,(a_s(\mu^2))^n, \tag{10}$$

where $a_s(s) \equiv \alpha_\mathrm{s}(s)/\pi$. In the $\overline{MS}$ scheme, for $N_f = 3$, the coefficients K_n calculated up to now to four loops [7] have the values

$$K_1 = 1, \quad K_2 = 1.6398, \quad K_3 = 6.3712, \quad K_4 = 49.076. \tag{11}$$

For the next term the choices $K_5 = 283$ and $K_5 = 275$ were made recently in [9] and [10], respectively. Finally, $\kappa_n(-s/\mu^2)$ entering (10) depend on the renormalization scale μ^2 and the coefficients of the renormalization-group (RG).

By setting in (10) $\mu^2 = M_\tau^2$ one obtains the standard fixed-order perturbation theory (FOPT). Another useful choice is to set $\mu^2 = -s$, which leads to the renormalization-group improved expansion

$$\widehat{D}(s) = \sum_{n\geq 1} K_n\,(a_s(-s))^n. \tag{12}$$

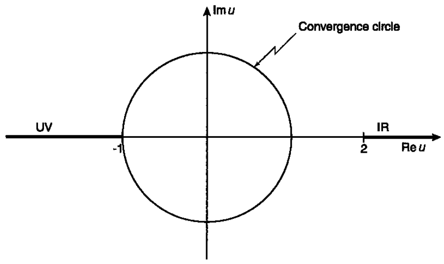

Figure 2: The convergence disc of the expansion (13).

In the present context, this expansion is known also as "contour-improved" perturbation theory (CIPT), because $\alpha_s(s)$ is calculated along the contour of the integral (9) using the solution of the renormalization group equation, applied iteratively starting from $s = -M_\tau^2$. This procedure was proposed in [12, 13] in order to avoid the large imaginary logarithms in the coefficients $\kappa_n(-s/\mu^2)$, responsible for a slow convergence of the expansion (10) near the timelike axis.

3 Improved expansions based on conformal mappings

As already mentioned, from particular classes of Feynman diagrams, and also from independent arguments on the analytic properties in the α_s plane [19], is is known that at large n the coefficients in the expansions (10) and (12) display a factorial increase, $K_n \sim n!$. According to Dyson's proposal [18] from 1952, the divergent series of perturbative QFT are assumed to be asymptotic to the expanded functions.

The information about the high-order behaviour of the series is encoded in the properties of a new function, $B(u)$, defined by the power series

$$B(u) = \sum_{n=0}^{\infty} b_n u^n \,, \tag{13}$$

with b_n related to the original perturbative coefficients appearing in (12) by

$$b_n = \frac{K_{n+1}}{\beta_0^n \, n!} \,, \quad n \geq 0, \tag{14}$$

where β_0 is the first coefficient in the expansion (1) of the β function.

The $n!$ in the denominator of (14) compensate the increase of K_{n+1}, so that the series in (13) is expected to converge in a disc on nonzero radius with the center at $u = 0$. According to present knowledge, the function $B(u)$, which is called the Borel transform of the Adler function, has branch point singularities in the u-plane, along the negative axis - the ultraviolet (UV) renormalons - and the positive axis - the infrared (IR) renormalons. Specifically, the branch cuts are situated along the rays $u \leq -1$ and $u \geq 2$. The nature of the first branch points was established in [20] and in [21] (see also [9]). Thus, near the first branch points, *i.e.* for $u \sim -1$ and $u \sim 2$, respectively, $B(u)$ behaves as

$$B(u) \sim \frac{r_1}{(1+u)^{\gamma_1}}, \quad B(u) \sim \frac{r_2}{(1-u/2)^{\gamma_2}}, \tag{15}$$

where the residues r_1 and r_2 are not known, but the exponents γ_1 and γ_2 are known positive numbers [20, 21, 9].

The original expansion (12) can be recovered from the definition

$$\widehat{D}(s) \equiv \frac{1}{\beta_0} \, \mathrm{PV} \int_0^{\infty} \mathrm{e}^{-u/(\beta_0 a_s(s))} \, B(u) \, \mathrm{d}u \,, \tag{16}$$

where PV denotes the Principal Value. As shown in [22, 14], the PV prescription is the best choice if one wants to preserve as much as possible the analyticity properties of the correlators in the s-plane, which follow from causality and unitarity.

The expansion (13) converges only in the disc $|u| < 1$, imposed by the first singularity of $B(u)$ at $u = -1$ (see Fig. 2). A series with a larger domain of convergence can be obtained by expanding $B(u)$ in powers of a new variable. As shown in [17] (see also [16]), the variable achieving the conformal mapping of the whole analyticity domain of the expanded function onto a disc in the new complex plane provides an expansion that converges in the whole complex plane, except for the cuts, and has the best asymptotic convergence rate[2].

In the present case, assuming that $B(u)$ has only the above-mentioned singularities on the real axis with a gap, being holomorphic elsewhere, the optimal variable defined in [17] reads:

$$\widetilde{w}(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}. \tag{17}$$

This function maps the u-plane cut for $u \geq 2$ and $u \leq -1$ onto the unit disc $|w| < 1$ in the complex plane $w = \widetilde{w}(u)$, such that $\widetilde{w}(0) = 0$, $\widetilde{w}(2) = 1$ and $\widetilde{w}(-1) = -1$. According to general arguments [17], the expansion

$$B(u) = \sum_{n\geq 0} d_n \, (\widetilde{w}(u))^n \tag{18}$$

converges in the whole disc $|w| < 1$ and has the best asymptotic rate of convergence compared to all the expansions of the function $B(u)$ in powers of other variables.

The series (18) can be used to define an alternative expansion of $\widehat{D}(s)$. This is obtained formally by inserting (18) into (16) and interchanging the order of summation and integration. Thus, we adopt the modified contour-improved expansion defined as [14]

$$\widehat{D}(s) = \sum_{n\geq 0} d_n W_n(s), \tag{19}$$

where

$$W_n(s) = \frac{1}{\beta_0} \mathrm{PV} \int_0^\infty \mathrm{e}^{-u/(\beta_0 a_s(s))} \, (\widetilde{w}(u))^n \, \mathrm{d}u \,. \tag{20}$$

The expansion (18) exploits only the location of the singularities in the Borel plane. However, as shown in (15), some information exists also about the nature of the singularities, and it is convenient to incorporate it explicitly [27]. This is achieved, for instance, by expanding

$$(1+w)^{2\gamma_1}(1-w)^{2\gamma_2} B(\widetilde{u}(w)) = \sum_{n\geq 0} c_n \, w^n, \tag{21}$$

where $\widetilde{u}(w)$ is the inverse of (17).

While the expansion (18) is unique, the explicit inclusion of the first singularities of $B(u)$ contains some arbitrariness. The description of the singularities by multiplicative factors is a possibility, but is not a priori necessary. Moreover, the factors are not unique. For a more detailed analysis of this problem see [15, 16].

The expansion (21) suggests the definition of the new CIPT

$$\widehat{D}(s) = \sum_{n\geq 0} c_n \mathcal{W}_n(s), \tag{22}$$

where the expansion functions are defined as

$$\mathcal{W}_n(s) = \frac{1}{\beta_0} \mathrm{PV} \int_0^\infty \mathrm{e}^{-u/(\beta_0 a_s(s))} \, \frac{(\widetilde{w}(u))^n}{(1+\widetilde{w}(u))^{2\gamma_1}(1-\widetilde{w}(u))^{2\gamma_2}} \, \mathrm{d}u \,. \tag{23}$$

[2]For QCD, the use of a conformal mapping in the Borel plane was suggested in [23] and was applied in a more limited context in [24]. Applications of the method were considered also in [25, 26].

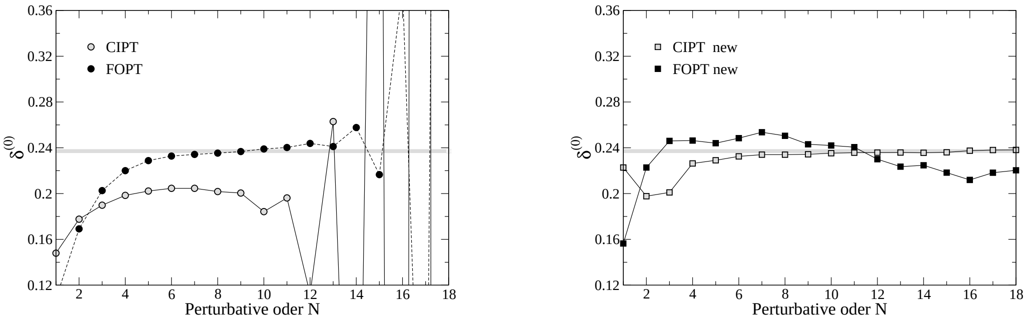

Figure 3: Values of $\delta^{(0)}$ for the model defined in [9], calculated with the standard (left) and the new (right) CIPT and FOPT, for $\alpha_s(M_\tau^2) = 0.34$, as a function of the perturbative order N. The horizontal band is the exact value $\delta^{(0)} = 0.2371$.

The expansions (19) and (22) reproduce the coefficients K_n of the usual expansion (10), when the functions (20) and (23) are expanded in powers of the coupling. In fact, as shown in [14], the new expansion functions are formally represented by asymptotic series in powers of the coupling, much like the expanded correlator itself.

To obtain the FO version of the new expansions, we start from (10) and define the corresponding Borel transform

$$\tilde{B}(u,s) = \sum_{n=0}^{\infty} \tilde{b}_n(s) u^n \,, \tag{24}$$

where

$$\tilde{b}_n(s) = \frac{K_{n+1} + \kappa_{n+1}(-s/\mu^2)}{\beta_0^n \, n!}\,, \quad n \geq 0. \tag{25}$$

By comparing Eqs.(24), (25) with (13), (14), one can see that the singularities of $B(u)$ are present also in the function $\tilde{B}(u,s)$, which we expand as:

$$\tilde{B}(u,s) = \sum_{n\geq 0} \tilde{d}_n(s)\, (\widetilde{w}(u))^n. \tag{26}$$

This leads us to the definition of a modified FOPT, analogous to the CIPT expansion (19):

$$\widehat{D}(s) = \sum_{n\geq 0} \tilde{d}_n(s)\, \tilde{W}_n, \tag{27}$$

in terms of the functions

$$\tilde{W}_n = \frac{1}{\beta_0} \mathrm{PV} \int_0^{\infty} \mathrm{e}^{-u/(\beta_0 a_s(\mu^2))}\, (\widetilde{w}(u))^n \, \mathrm{d}u \,. \tag{28}$$

If we impose explicitly the behaviour (15), we obtain the new FO expansion:

$$\widehat{D}(s) = \sum_{n\geq 0} \tilde{c}_n(s)\, \tilde{\mathcal{W}}_n, \tag{29}$$

in terms of the functions

$$\tilde{\mathcal{W}}_n = \frac{1}{\beta_0} \mathrm{PV} \int_0^{\infty} \mathrm{e}^{-u/(\beta_0 a_s(\mu^2))}\, \frac{(\widetilde{w}(u))^n}{(1+\widetilde{w}(u))^{2\gamma_1}(1-\widetilde{w}(u))^{2\gamma_2}} \, \mathrm{d}u \,. \tag{30}$$

The properties of the new expansions are illustrated in Figs. 3-5 using a realistic model of the Adler function proposed in [9]. The comparison of the standard CIPT and FOPT with the new CIPT and FOPT is seen from Fig. 3 where we show $\delta^{(0)}$ calculated as a function of the order up to

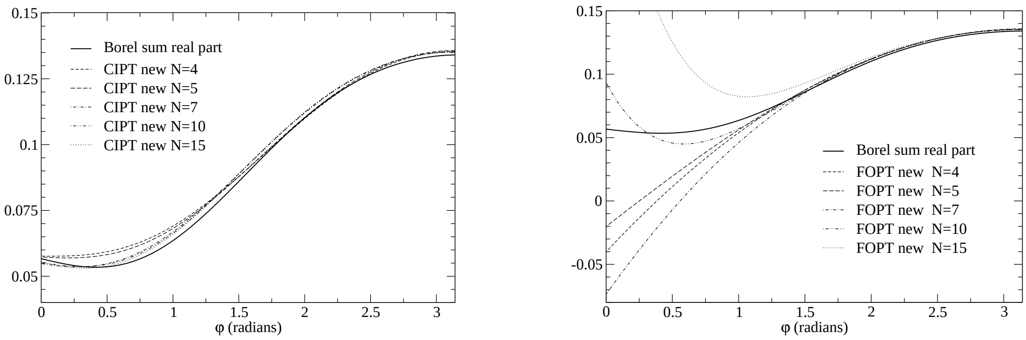

Figure 4: Real part of $\widehat{D}(s)$ calculated with the new CIPT (left) and the new FOPT (right) along the circle $s = M_\tau^2 \exp(i\varphi)$.

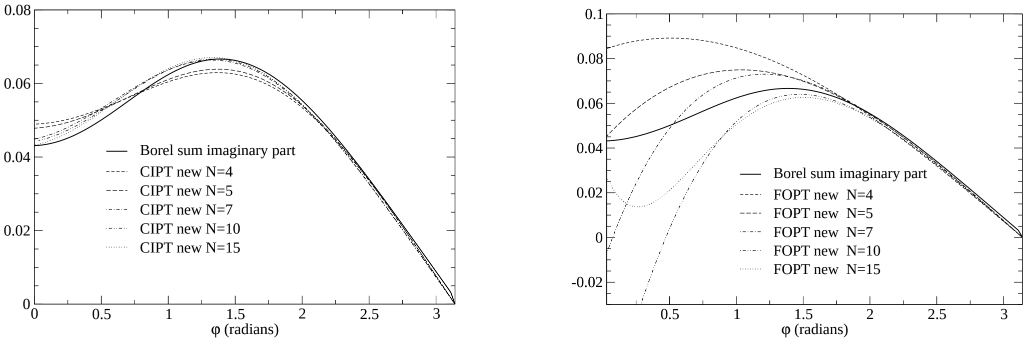

Figure 5: Imaginary part of $\widehat{D}(s)$ calculated with the new CIPT (left) and the new FOPT (right) along the circle $s = M_\tau^2 \exp(i\varphi)$.

which the series have been summed. The figure in the left panel shows that the standard CIPT does not approach the true value, staying below it up to the orders at which the results start to exhibit large oscillations. By contrast, the new CIPT gives very good results which approach the true value with great accuracy when N increases. As concerns FOPT, the new approach gives results somewhat poorer than the standard one at low orders. At large orders, when the standard FOPT shows large oscillations, the new FOPT leads to values closer to the true result, but not as good as those obtained with the new CIPT.

In order to understand these results, we calculated the Adler function $\widehat{D}(s)$ for complex s along the integration contour in the integral (9). In Figs. 4 and 5 we present the real and the imaginary part part of $\widehat{D}(s)$ calculated with the new contour-improved and fixed-order expansions, for s along the upper semicircle in the definition (9) of $\delta^{(0)}$ ($s = M_\tau^2 \mathrm{e}^{\mathrm{i}\varphi}$, for $0 \leq \varphi \leq \pi$).

From Figs. 4 and 5, one can see that the new CIPT give approximations of the real and imaginary parts of $\widehat{D}(s)$ that improve continously with increasing N along the whole contour. As concerns FOPT, it gives a very good approximation of $\widehat{D}(s)$, which improves continously with increasing N, for φ close to π, *i.e.* near the spacelike axis. However, the description deteriorates as φ approaches 0, *i.e.* near the timelike axis. This can be understood by the large imaginary logarithms of the coefficients κ in the expansion (10) when s is close to the timelike axis.

4 Determination of $\alpha_{\mathrm{s}}(M_\tau^2)$

The determination of $\alpha_{\mathrm{s}}(M_\tau^2)$ amounts to the calculation of $\delta^{(0)}$ defined in (9) using a specific expansion of $\widehat{D}(s)$ and solving the equation $\delta^{(0)} = \delta^{(0)}_{\mathrm{phen}}$ with respect to the coupling. We shall consider here the new CIPT and FOPT expansions defined in (22) and (29), respectively.

Using as input the phenomenological value quoted in [9]

$$\delta^{(0)}_{\rm phen} = 0.2042 \pm 0.0050, \tag{31}$$

the first four K_n in the $\overline{\rm MS}$ scheme given in (11), and the estimate $K_5 \approx 283$ from [9], we obtain:

$$\begin{aligned} \alpha_s(M_\tau^2) &= 0.3198\,^{+0.0113}_{-0.0094}\,, &\quad& \text{new CIPT,} \\ \alpha_s(M_\tau^2) &= 0.3113\,^{+0.0114}_{-0.0050}\,, && \text{new FOPT.} \end{aligned} \tag{32}$$

As discussed in [11], CIPT is more sensitive to the uncertainty of the last perturbative term, while FOPT is sensitive to the change of the renormalization scale μ^2 around the value $\mu^2 = M_\tau^2$.

It is remarkable that the difference between the central values in (32) is only 0.009, while for the standard CIPT and FOPT the difference is 0.024. The new expansions remove thus the most intriguing theoretical discrepancy in the determination of α_s from τ decays. We note that both values in (32) are closer to the standard FOPT than to the standard CIPT.

We take as best result the value given in (32) by the new CIPT:

$$\alpha_s(M_\tau^2) = 0.320\ \pm 0.011\,, \tag{33}$$

which, evolved to the M_Z scale, leads to the prediction $\alpha_s(M_Z^2) = 0.1180\ \pm 0.0015$, very close to the world average (6).

The result (33) is based on a systematic perturbation theory, and its uncertainty is related mainly to the error of the last perturbative term. So, the accuracy of the prediction is expected to increase when more perturbative terms for the Adler function in QCD will be available.

In conclusion, the determinations of α_s from various processes at different scales provide a solid test of QCD. The perturbative expansion of QCD can be improved by including information about the high-order behaviour of the series, increasing the precision of its predictions.

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