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Alternative integrable discretisation of Korteweg de Vries equation

Corina Băbălîc[1], A. S. Cârstea[2]
[1] *University of Craiova, Romania*
[2] *Dept. Theor. Physics, IFIN-HH, 77125, Magurele, Bucharest, Romania*

Abstract

We present an alternative integrable discretization of differential-difference KdV equation based on Hirota bilinear formalism. It is shown that using two tau functions the direct discretisation of the bilinear equations gives immediately the well known discrete KdV equation. We comment also on integrability and relation with the classical bilinear form involving only one tau function.

1 Introduction

One of the main difficulties in the topic of integrable systems is to obtain an integrable discretisation to a given partial differential or differential-difference equation. Applying integrability criteria like complexity growth [1], or singularity confinement [2] is not always quite easy because the discrete lattice equations have in general complicated forms. However is much convenient to start with some general lattice equation and impose some strong integrability requirements (like for instance cube consistency [3]).

Among the methods of finding integrable discretizations one of the powerful ones is the Hirota bilinear method [4]. The idea is quite simple. Once the differential/differential-difference bilinear form is given then, in a first step, replace differential Hirota operators with discrete ones preserving gauge invariance. Of course the resulting bilinear discrete equation is not necessarily integrable so in the second step the multisoliton solution is to be found. If this exists then the bilinear equation is integrable and in a third step which is rather complicated the nonlinear form is recovered. This method has been applied successfully to many equations [5]-[9].

In the first part of this paper we will give a brief presentation of the KdV equation and its soliton solutions. Then we will discuss on semidiscrete KdV equation and its bilinear form. Afterwords, the discussion will be on bilinear discretisation, showing the main steps. Finally the construction of a new bilinear form using two tau functions is realised. These tau functions are not tau functions in the rigorous sense as in the Jimbo-Miwa theory [10] but rather holomorphic combinations of the only tau function. However the form is quite convenient and the full discretisation is obtained immediately.

2 The continuous Korteweg de Vries equation

The *soliton* and the *nonlinear integrable system* are the two notions that have revolutionized mathematics and nonlinear physics in the last twenty years.

The soliton was first revealed in 1834 by the ship designer John Scott Russell who observed a solitary, very stabile traveling wave in a shallow water channel in Scotland. That's how he described it in his own words:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation." [11]

Based on his observations he insisted that such waves do occur, but several prominent mathematicians, including G.G. Stokes and G.B. Airy were convinced that a solitary wave with permanent shape could not exist. This gave rise to a big controversy which lasted for more than fifty years.

In 1895 D.J. Korteweg and G. de Vries discovered a nonlinear equation of evolution (called subsequently KdV) which described the propagation of small amplitude waves in shallow waters, explaining this way the phenomenon observed by Russel. Their equation had permanent form wave-like solutions. Also, J.V. Boussinesq (1871) and Lord Rayleigh (1876) found some equations that governed the propagation for these waves.

In spite of all these, concrete physical applications for the KdV equation were not found until the 1960s, when C.S. Gardner and G.K. Morikawa, studied the interaction of free electromagnetic waves and found, in certain conditions, the KdV equation. The physical problem that motivated the recent discoveries concerning the KdV equation was the famous Fermi, Pasta, Ulam's problem (FPU) formulated in 1955. They conducted a numerical experiment on an anharmonic one dimensional nonlinear network. Research on this FPU model was continued by N.J. Zabusky and M.D. Kruskal who have reached in certain asymptotic conditions exactly the Korteweg de Vries equation:

$$u_t + 6uu_x + u_{xxx} = 0$$

The solution of this equation, well known in that time, was the profile of highly localized solitary wave type.

$$u = 2k^2 \sec h^2 k(x - 4k^2 t - x_0)$$

where k and x_0 are constant parameters. Of particular importance was the finding that these solitary waves interact perfectly elastic, the only effect being a phase shift, namely the modification of the position for each wave center (we underline the fact that in general, solitary waves do not interact elastically). This property led Zabusky and Kruskal to the designation of "soliton".

Later it was observed that any physical system that combines a low dispersion with a weak nonlinearity is described asymptotically by a KdV equation. Also it was shown that KdV equation can be written as a compatibility condition of two linear operators which allow solving by means of linear techniques from inverse scattering theory (IST) and singular integral equations. This procedure was launched by Zabusky and Kruskal and was the beginning of the theory of completely integrable nonlinear systems (see [12] for details).

An alternative method much simpler than IST was discovered in the same time by Hirota. For example the generic two-soliton solution for KdV (which is extremely complicated by means of IST) can be written as:

$$u = 2\partial_x^2 \log F$$

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+a_{12}}$$

$$\eta_i = k_i x + \omega_i t + \eta_i^0$$

where for the KdV equation we have specifically the dispersion relation

$$\omega_i = -k_i^3$$

and the phase factor (which represents precisely the interaction term between the soliton i and soliton j)

$$e^{a_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}$$

The complete integrability of KdV equation imposes that the interaction term is factorised by two body terms and accordingly the three-soliton solution has the form:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1+\eta_2+a_{12}} + e^{\eta_1+\eta_3+a_{13}} + e^{\eta_2+\eta_3+a_{23}} + e^{\eta_1+\eta_2+\eta_3+a_{12}+a_{13}+a_{23}}$$

where the dispersion relation and the phase factor are the same as before.

Based on these forms, Hirota finds the N-soliton solution using the induction method.

$$F = \sum_{\mu_{i,j}=0,1} \exp\left(\sum_i^N \mu_i \eta_i + \sum_{1 \le i < j} a_{ij} \mu_i \mu_j\right)$$

3 Differential-difference KdV equation

And because in this paper we will focus on the total discretization of the KdV equation we should specify that the role of discretization has appeared in the study of Volterra's prey-predator type equations.

To illustrate in a physical way what a Volterra system represents we consider an infinite chain of species having a "Prey-predator" dynamics. Each species will be indexed with n and the dynamics of the number of individuals for each species will be given by the equation:

$$\frac{du_n}{dt} = (1 + u_n)(u_{n+1} - u_{n-1})$$

In other words the number of individuals u_n of the n-th species grows through the interaction with those of the $n+1$ species and decreases through the interaction with those of the $n-1$ species. The term $(1 + u_n)$ represents the simplest interaction term between species. It can be taken in a more general form. It turns out that the interaction term u_n^2 will give a completely integrable space-discretisation of the KdV equation [14].

The system under consideration is:

$$\dot{u}_n = \frac{du_n}{dt} = u_n^2(u_{n+1} - u_{n-1}) \tag{1}$$

In [14] it was shown that using the following nonlinear substitution

$$u_n = \frac{f(n+1,t)f(n-1,t)}{f^2(n,t)} \equiv \frac{\overline{f}\underline{f}}{f^2} \tag{2}$$

we can cast the equation in the following form:

$$D_t(f \cdot \overline{f}) = \overline{\overline{f}}\underline{f} - f\overline{f}, \tag{3}$$

where $\overline{f} = f(n+1,t)$, $\underline{f} = f(n-1,t)$ etc. and $D_t^n a \cdot b = \partial_y^n a(t+y)b(t-y)|_{y=0}$ is the usual Hirota operator. This bilinear form allows direct computation of N-soliton solution in the partition function-like form:

$$f = \sum_{\mu_{i,j}=0,1} \exp(\sum_i^N \mu_i \eta_i + \sum_{1 \le i < j} a_{ij}\mu_i\mu_j) \tag{4}$$

where $\eta_i = k_i n - 2\sinh k_i t$ and k_i are the free soliton parameters. The most important property of Hirota bilinear operator and of all bilinear equations in general is the so called *gauge-invariance*. It imposes an invariance with respect to the multiplication of the τ function with an exponential of linear terms. More precisely

$$D_t^m \exp(an+bt)f \cdot \exp(an+bt)g = \exp(2an+2bt)D_t^m f \cdot g \tag{5}$$

This gauge-invariance defines uniquely the Hirota bilinear operator [13]. The continuum limit is hit in the multiple scale formalism by the following scalings $x = (n+2t)\epsilon$, $\tau = \epsilon^3 t$, $u(n,t) = 1 + \epsilon^2 U(x,\tau)$ in the form of the usual KdV partial differential equation

$$U_\tau + 2UU_x + \frac{1}{3}U_{xxx} = 0 + \mathcal{O}(\epsilon). \tag{6}$$

All the features specific to integrable systems appear in this equation. It has a Lax pair, Painleve property, Backlund transformations and nonlinear superposition formulas etc. It is one of the very few integrable differential difference equations.

4 Integrable discretisation

The differential-difference KdV equation can be discretised also in time. This is extremely important from the point of view of applications in cellular automata. Also this will give a numerical integrator for continuous KdV. The topic of fully discrete integrable systems is not well understood so far and it seems that is more fundamental than the continuos one. However finding an *integrable discretisation* which retains some of the initial dynamics is an extremely complicated procedure. Usually, an indirect approach is used, for instance Lax pair (which are linear operators and can be discretised easily) or some specific integrability detectors as singularity confinement algebraic entropy or multidimensional consistency. The motivation for finding a fully discrete version is that the discrete integrable systems are fundamental compared to the continuous ones; the dynamics is way richer and moreover they can lead to cellular automaton forms. Here we implement a method used by Hirota, which consists in the following steps:

- starting from a differential discrete equation, one has to write it in the bilinear form,
- discretize the Hirota operator directly by replacing ordinary derivatives with finite differences and then impose gauge invariance to the other terms,
- prove integrability for the new discrete bilinear equation by computing multisoliton solution (or by other methods),
- recovering the nonlinear form (the most difficult step).

The discrete form of KdV equation is quite well known. However it was found by some complicated procedures. Here we show that a specific bilinear form will help in finding immediately the integrable discretisation.

The new bilinear form which we proposed is based on a projective nonlinear substitution involving *two τ functions* namely $u = G(n,t)/F(n,t)$. Introducing in (1) we get the bilinear dispersion relation

$$D_t G \cdot F = \overline{G}\underline{F} - \underline{G}\overline{F} \tag{7}$$

and the soliton-phase constraint

$$G^2 = \overline{F}\underline{F} \tag{8}$$

In order to discretize time we consider $t = \delta m$, where m is the *discrete time* and δ is the small discretisation step. Imposing gauge-invariance and replacing Hirota operator with finite differences we get (with the notation $\tilde{G} = G(n, \delta m + \delta)$)

$$\tilde{G}F - G\tilde{F} = \delta(\tilde{\overline{G}}\underline{F} - \underline{G}\tilde{\overline{F}}) \tag{9}$$

$$G\tilde{G} = \tilde{\overline{F}}\underline{F} \tag{10}$$

Now the main problem here is to prove integrability of this system of bilinear equations. It can be done by showing that it admits multisoliton solution. Indeed if we take $G = \tilde{\overline{f}}\underline{f}$, $F = f\tilde{f}$ then the second equation will become an identity whilst for the first the general N-soliton solution in term of f has the form:

$$f = \sum_{\mu_{i,j}=0,1} \exp(\sum_i^N \mu_i \eta_i + \sum_{i<j} a_{ij}\mu_i\mu_j) \tag{11}$$

with $\eta_i = k_i n + \omega_i m$, $\sinh \omega_i = \delta \sinh k_i$ and the interaction term:

$$\exp a_{ij} = \left(\frac{e^{k_i} - e^{k_j}}{e^{k_i}e^{k_j} - 1} \right)^2 \tag{12}$$

In order to find the nonlinear form, divide the first equation with $F\tilde{F}$

$$\frac{\tilde{G}F - G\tilde{F}}{F\tilde{F}} = \delta \frac{\tilde{\overline{G}}\underline{F} - \underline{G}\tilde{\overline{F}}}{F\tilde{F}}$$

and if $X = G/F$ we get:

$$\tilde{X} - X = \delta \frac{\tilde{\overline{G}}\underline{F} - \underline{G}\tilde{\overline{F}}}{\tilde{\overline{F}}\underline{F}} \frac{\tilde{\overline{F}}\underline{F}}{\tilde{F}F}$$

Eliminating the last factor in the right hand side using the second bilinear equation we get immediately:

$$\tilde{X} - X = \delta(\tilde{\tilde{X}} - \underline{X})X\tilde{X} \tag{13}$$

We have to note that the equation (13) we obtained through a simple direct discretisation of the bilinear equation represents exactly the fully integrable discrete KdV equation obtained by Hirota in [14].

5 Conclusions

In this paper we discussed an alternative and simpler integrable time-discretisation of the differential-difference KdV equation. The main point was an alternative simpler bilinear form obtained by means of two tau functions. Also using the gauge invariance it was discretised immediately the Hirota bilinear equations. Finally the nonlinear form was recovered. The method seems to be powerful enough to be applied to other more complex differential-difference systems. This we intend to do in a future publication.

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