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Linear stability of the stationary solutions of self-organized criticality model of the edge plasma turbulence

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Abstract

Stationary solutions of the self- organized criticality model of the edge plasam turbulence are computed. The properties of the instabilities and the range of parameters for the stability of the stationary solution are studied.

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1 Introduction

Reduced model of the edge plasma turbulence allows to understand the main mechanisms that give rise to the main part of the particle and energy transport. The outcome of this class of models reduces the CPU time for the first principle optimization of the parameters of the large tokamaks. The paradigm of the self organized criticality [1] was previously used in the study of the tokamak edge plasma turbulence [2]-[5]. The class of models we study is part of a long series, based on the general principles of the self -organized criticality. We compute the stationary solutions of the model in physically relevant regimes and explore the range of the parameters when these solutions are linearly stable.

2 The model and its stationary solutions

The equations of the $1d$ model, that describe the joint evolution of the particle density $h(x,t)$ and of the turbulence intensity $\Phi(x,t)$ [see ref. [6]],are

$$\partial_t \Phi = \Phi \left(\gamma_0 \left(Z - z_c \right)_+ - \mu \Phi \right) + S_1 \tag{1}$$

$$\partial_t h = -\partial_x \left(\mu_0 \Phi Z \right) + S_0(x) \tag{2}$$

We denoted the density gradient

$$Z(x,t) = -\partial_x h \tag{3}$$

and the positive part of u by $(u)_+$ ($(u)_+ = u$ for $u > 0$ and equal 0 otherwise) together to the constant positive parameters $\gamma_0,\ z_c,\ \mu,\ \mu_0$. In this section S_1 will be considered independent of time and of the radial coordinate x while S_0 is a given function. Remark also that in the stationary limit our dynamic variables are $Z(x)$ and $\Phi(x)$. Using Eq. (10) from [6], in the limit $\Delta t \to 0$ results that

$$\lim_{t \to 0+} Z(x,t) = 0 \tag{4}$$

By rescaling the previous equation can be reduced to (we use the same notation for rescalled variables)

$$\partial_t \Phi = \Phi \ \left(\left(\ Z - z_c \right)_+ - \Phi \right) + S_1 \tag{5}$$
$$\partial_t h = -\partial_x \left(\Phi Z \right) + S_0(x) \tag{6}$$
$$Z(x,t) = -\partial_x h \tag{7}$$

Remark 1 *In the following all of the variables are dimensionless.*

2.1 The stationary solutions

2.1.1 Constant distributed source

Consider the stationary case

$$\partial_t \Phi = \partial_t h = 0 \tag{8}$$

with $S_0(x) = const.$

From Eqs. (8, 6, and 4) results (the trivial time dependence is omitted)

$$\Phi(x) Z(x) = S_0 x \tag{9}$$

Because $\Phi \geq 0$ and $z_c > 0$ we have

$$\Phi \ \left(\ Z - z_c \right)_+ = z_c \left(\frac{\Phi \ Z}{z_c} - \Phi \right)_+ \tag{10}$$

and using Eqs. (5, 8, 10) we obtain

$$z_c \left(\frac{\Phi \ Z}{z_c} - \Phi \right)_+ = \Phi^2 - S_1 \tag{11}$$

We denote

$$\phi_0 = \frac{S_0}{z_c} \tag{12}$$

If we combine Eqs. (9), (11) and (12) results a piecewise algebraic equation for $\Phi(x)$

$$z_c \left(\phi_0 x - \Phi \right)_+ = \Phi^2 - S_1 \tag{13}$$

From the study of the intersection of the graph of l.h.s and r.h.s from Eq. (13) (with Φ on the abscissa, the l.h.s. respectively the r.h.s. from Eq. (13) on the vertical axis) we obtain the piecewise characterization of the solutions, as follows. First, we denote

$$\Phi_{crit} = \sqrt{S_1} \tag{14}$$
$$x_{crit} = \frac{\Phi_{crit}}{\phi_0} = \frac{z_c \sqrt{S_1}}{S_0} \tag{15}$$

Then the unique non negative solution of Eq. (13) is given by

$$z_c(\phi_0 x - \Phi) = \Phi^2 - S_1,\ for\ x \geq x_{crit} \tag{16}$$
$$\Phi = \Phi_{crit},\ for\ x \leq x_{crit} \tag{17}$$

or in a more explicit form

$$\Phi = \frac{1}{2}\left(\sqrt{z_c^2 + 4S_1 + 4S_0 x} - z_c\right);\quad x \geq x_{crit} \tag{18}$$
$$\Phi = \Phi_{crit},\ for\ x \leq x_{crit} \tag{19}$$

Using Eq. (9) and the notations (14, 15)

$$\Phi = \frac{1}{2}\left(\sqrt{z_c^2 + 4S_1 + 4S_0 x} - z_c\right);\quad x \geq \frac{z_c\sqrt{S_1}}{S_0} \tag{20}$$
$$\Phi = \sqrt{S_1} = \Phi_{crit},\ for\ x \leq \frac{z_c\sqrt{S_1}}{S_0} = x_{crit} \tag{21}$$
$$Z(x) = \frac{2S_0 x}{\sqrt{z_c^2 + 4S_1 + 4S_0 x} - z_c};\ x \geq \frac{z_c\sqrt{S_1}}{S_0} \tag{22}$$
$$Z(x) = \frac{S_0 x}{\sqrt{S_1}} \leq z_c;\quad x \leq \frac{z_c\sqrt{S_1}}{S_0} = x_{crit} \tag{23}$$

In particular $Z(x_{crit}) = z_c$ and can be verified by simple algebra that $Z(x)$ is strictly increasing, without tendency to have a plateau, like in figure 2 and 5 from ref. [6] .This discrepancy between our approximation and simulated results can be explained by the instability induced by the first term, $(Z - z_c)_+$ in Eq. (5), for $x > x_{crit}$.

In the case of the model having linear response to the density gradient, i.e. when we have simply $(Z - z_c)$ instead $(Z - z_c)_+$ in the equations (1 5), the stationary solution is given by all range of the variable x.

$$\Phi = \frac{1}{2}\left(\sqrt{z_c^2 + 4S_1 + 4S_0 x} - z_c\right) \tag{24}$$
$$Z(x) = \frac{2S_0 x}{\sqrt{z_c^2 + 4S_1 + 4S_0 x} - z_c} \tag{25}$$

2.1.2 Concentrated source

Consider now the case when $S_0(x) = \Gamma\delta(x)$, with a constant flux Γ. From Eq.(6) results

$$\Phi(x)Z(x) = \Gamma \tag{26}$$

that combined with Eq. (5), similar to Eq. (11) gives

$$z_c\left(\frac{\Gamma}{z_c} - \Phi\right)_+ = \Phi^2 - S_1 \tag{27}$$

From Eqs. (26, 27) we will obtain spatially homogenous stationary solutions $\{\Phi_0, Z_0\}$, as follows.

1. If $\frac{\Gamma}{z_c} \leq \sqrt{S_1}$ we have

$$\Phi_0 = \sqrt{S_1} \tag{28}$$

$$Z_0 = \frac{\Gamma}{\sqrt{S_1}} \leq z_c \tag{29}$$

2. If $\frac{\Gamma}{z_c} \geq \sqrt{S_1}$ we obtain

$$\Phi_0 = \frac{1}{2}\left(\sqrt{z_c^2 + 4S_1 + 4\Gamma} - z_c\right) \tag{30}$$

$$Z_0 = \frac{2\Gamma}{\sqrt{z_c^2 + 4S_1 + 4\Gamma} - z_c} \geq z_c \tag{31}$$

The last inequality can be verified by simple algebra.

2.1.3 Linear stability study

In the case of the more concentrated source it is straightforward to obtain linear stability results. In the Eqs. (5-7) we perform the linearization as follows. Let $\Phi(x,t) = \Phi_0 + \delta\Phi(x,t)$, $Z = Z_0 + \delta Z$, $h = h_0 + \delta h(x,t)$. Denoting

$$\delta\Psi = \partial_x \delta\Phi \tag{32}$$

$$A = -2\Phi_0 + (Z_0 - z_c)_+ \tag{33}$$

$$B = \Phi_0 \Theta(Z_0 - z_c) \tag{34}$$

we obtain

$$\partial_t \delta\Psi = A\delta\Psi - B{\partial_x}^2 \delta h \tag{35}$$

$$\partial_t \delta h = -Z_0 \delta\Psi + \Phi_0 {\partial_x}^2 \delta h. \tag{36}$$

The boundary conditions for $\delta h(x,t)$ are

$$\delta h(x,t)|_{x=L} = 0 \tag{37}$$

$$\partial_x \delta h(x,t)|_{x=0} = 0 \tag{38}$$

The Liapunov eigenvalue spectrum can be obtained from the set of solutions of Eqs. (35, 36) in the form

$$\delta\Psi = \delta\Psi_0 \exp(\lambda t) \cos(k_n x) \tag{39}$$

$$\delta h = \delta h_0 \exp(\lambda t) \cos(k_n x)$$

$$k_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L};\ n \in \mathcal{Z}_+ \tag{40}$$

The following simple eigenvalue equations results:

$$\lambda \delta\Psi_0 = A\delta\Psi_0 + Bk_n^2 \delta h_0$$

$$\lambda \delta h_0 = -Z_0 \delta_0 \Psi - \Phi_0 k_n^2 \delta h_0.$$

or, more explicitly

$$\lambda^2 - S_n \lambda + D_n = 0 \tag{41}$$

with the following coefficients

$$S_n = (Z_0 - z_c)_+ - (k_n^2 + 2)\Phi_0 \tag{42}$$

$$D_n = k_n^2 \Phi_0 \left[2\Phi_0 + z_c \Theta(Z_0 - z_c)\right] \tag{43}$$

We consider separately the following domains of parameters.

Low flux case $\frac{\Gamma}{z_c} \le \sqrt{S_1}$ In this case the solutions of the Eq. (41) are

$$\lambda_{\pm,n} = -\Phi_0 \left(k_n^2 + 2 \pm \left|k_n^2 - 2\right|\right)/2,$$

so in this low flux regime, or high value of the critical gradient, the stationary solution is always stable.

High flux case $\frac{\Gamma}{z_c} > \sqrt{S_1}$ In this case $Z_0 > z_c$. We denote in the following

$$\alpha = 1 + \frac{z_c}{2\Phi_0} > 1 \tag{44}$$

$$\beta = \frac{Z_0 - z_c}{2\Phi_0} > 0 \tag{45}$$

and we obtain from Eqs. (42, 43)

$$S_n = -(k_n^2 + 2)\Phi_0 \left(1 - \frac{\beta}{k_n^2 + 2)}\right) \tag{46}$$

$$D_n = \frac{2k_n^2}{(k_n^2 + 2)}\alpha \tag{47}$$

The eigenvalues are given by

$$\lambda_{\pm,n} = -\frac{(k_n^2 + 2)\Phi_0}{2}\left[1 - \frac{\beta}{k_n^2 + 2)} \pm \sqrt{\left(1 - \frac{\beta}{k_n^2 + 2)}\right)^2 - 8\alpha\frac{k_n^2}{k_n^2 + 2)}}\right] \tag{48}$$

It is clear from Eq. (48) that we will have only a finite number of unstable modes. Indeed, for $n \to \infty$, we have the asymptotic estimate

$$-\frac{(k_n^2 + 2)\Phi_0}{2}\left[1 \pm i\sqrt{8\alpha - 1}\right]\left(1 + O(\frac{1}{n^2})\right)$$

From Eq. (48) it is clear that we will have at least an unstable mode whenever $S_n > 0$ for some n. It follows from Eq. (47) that there exists a minimal value of the parameter $\beta_{\min}$ such that in the case

$$\beta = \frac{Z_0 - z_c}{2\Phi_0} < \beta_{\min} \tag{49}$$

the stationary solution is still stable, in the continuation to the previous case $Z_0 \le z_c$, that corresponds to $\beta \le 0$. This minimal value, is given by

$$\beta_{\min} = 2 + \left(\frac{\pi}{2L}\right)^2 \tag{50}$$

This result completes the study of stability criteria.

3 Conclusions

Analytic form of the stationary solution of the self organized criticality model exists in relevant physical regimes. Stable regimes were found in the case of small particle flux Γ.

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