Note on a class of one-dimensional Hamiltonian systems

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Abstract

In this paper it is proved that for dynamical systems described by Hamiltonians of the type $H_0(qp)$ there exists an equivalent second-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space.

It is well-known that the Euler–Lagrange [1] and Hamilton [2] equations play a central role in theoretical physics. For systems described by non-degenerate Lagrangians the Euler–Lagrange equations and Hamilton equations are equivalent. In the case of constrained (degenerate) systems [3]–[5] the equivalence between the two sets of equations is no longer manifest and must be implemented via the introduction of Lagrange multipliers. Another approach to constrained systems can be found in [6]–[7].

For a non-degenerate system, locally described by the bosonic canonical pair (q,p) and the Hamiltonian $H_0(q,p)$, the Hamilton equations

$$\dot{q} = \frac{\partial H_0}{\partial p},\ \dot{p} = -\frac{\partial H_0}{\partial q}, \tag{1}$$

can be derived from the first-order variational principle based on the action

$$S_0[q,p] = \int_{t_1}^{t_2} dt\,(\dot{q}p - H_0(q,p)). \tag{2}$$

It is easy to see that the Euler–Lagrange equations for the first-order Lagrangian

$$\mathcal{L}(q,p,\dot{q},\dot{p}) = \dot{q}p - H_0(q,p) \tag{3}$$

coincide with the Hamiltonian equations (1). In consequence, given a Hamiltonian formulation of dynamics, we can always construct an equivalent first-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space.

In this paper we prove that for dynamical systems described by Hamiltonians of the type $H_0(qp)$ we can find an equivalent second-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space.

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In this respect we start with a class of Hamiltonians of the form

$$H_0(q,p) = H_0(qp). \tag{4}$$

The corresponding Hamilton equations are given by

$$\dot{q} = \frac{dH_0}{du} q, \tag{5}$$

$$\dot{p} = -\frac{dH_0}{du} p, \tag{6}$$

with

$$u = qp. \tag{7}$$

We choose the initial conditions

$$q(t_0) = q_0, \; p(t_0) = p_0. \tag{8}$$

Now, we take the second-order Lagrangian

$$\bar{L}_0(q,p,\dot{q},\dot{p}) = \dot{q}\dot{p} - \bar{V}(u), \tag{9}$$

where $\bar{V}(u)$ is defined via the relation

$$\frac{d\bar{V}}{du} = -\left(\frac{dH_0}{du}\right)^2. \tag{10}$$

With the help of (9)–(10) we derive the following second-order Euler–Lagrange equations

$$\frac{\delta \bar{L}_0}{\delta q} \equiv -\ddot{p} + \left(\frac{dH_0}{du}\right)^2 p = 0, \tag{11}$$

$$\frac{\delta \bar{L}_0}{\delta p} \equiv -\ddot{q} + \left(\frac{dH_0}{du}\right)^2 q = 0. \tag{12}$$

In order to be able to compare the time evolutions described by the Hamilton equations (5)–(6) and respectively the Euler–Lagrange ones, (11)–(12), we must impose in each formalism some initial conditions that are mutually compatible. This means that given the Hamiltonian initial conditions (8), we must take the Lagrangian ones as

$$q(t_0) = q_0, \; p(t_0) = p_0, \tag{13}$$

$$\dot{q}(t_0) = q_0 \frac{dH_0}{du}(q_0 p_0), \; \dot{p}(t_0) = -p_0 \frac{dH_0}{du}(q_0 p_0). \tag{14}$$

Under these considerations, the next theorem represents our main result.

Theorem 1 *The (second-order) Euler–Lagrange equations(11)–(12), subject to the initial conditions (13)–(14), describe the same dynamics like the Hamilton equations (5)–(6) in the presence of the initial conditions (8), i.e.*

$$\left\{ \begin{array}{c} -\ddot{p} + \left(\frac{dH_0}{du}\right)^2 p = 0, \\ -\ddot{q} + \left(\frac{dH_0}{du}\right)^2 q = 0, \\ q(t_0) = q_0, \; p(t_0) = p_0, \\ \dot{q}(t_0) = q_0 \frac{dH_0}{du}(q_0 p_0), \; \dot{p}(t_0) = -p_0 \frac{dH_0}{du}(q_0 p_0), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} \dot{q} = \frac{dH_0}{du} q, \\ \dot{p} = -\frac{dH_0}{du} p, \\ q(t_0) = q_0, \; p(t_0) = p_0. \end{array} \right. \tag{15}$$

Proof In order to prove the theorem we will explicitly find the solutions to the equations (11)–(12), subject to the initial conditions (13)–(14), and respectively to the equations (5)–(6) in the presence of the initial conditions (8).

From (11)–(12) we find the relations

$$
\begin{aligned}
q\frac{\delta \bar{L}_0}{\delta q}+p\frac{\delta \bar{L}_0}{\delta p} &= 2\left(\dot{q}\dot{p}+\left(\frac{dH_0}{du}\right)^2 u\right)-\ddot{u}, && (16)\\
\dot{q}\frac{\delta \bar{L}_0}{\delta q}+\dot{p}\frac{\delta \bar{L}_0}{\delta p} &= -\frac{d}{dt}\left(\dot{q}\dot{p}+\bar{V}\left(u\right)\right), && (17)
\end{aligned}
$$

which prove that if $(q(t), p(t))$ are solutions of equations (11)–(12), subject to the initial conditions (13)–(14), then they are also solutions of equations

$$\dot{q}\dot{p}+\bar{V}\left(u\right)=-q_0p_0\left(\frac{dH_0}{du}\left(q_0p_0\right)\right)^2+\bar{V}\left(q_0p_0\right), \tag{18}$$

$$\ddot{u}-2\left(\dot{q}\dot{p}+\left(\frac{dH_0}{du}\right)^2 u\right)=0 \tag{19}$$

in the presence of the same initial conditions. Substituting (18) in (19) we infer the equation

$$\ddot{u}-2\left(\left(\frac{dH_0}{du}\right)^2 u-\bar{V}\left(u\right)-q_0p_0\left(\frac{dH_0}{du}\left(q_0p_0\right)\right)^2+\bar{V}\left(q_0p_0\right)\right)=0. \tag{20}$$

The initial conditions (13)–(14) further imply that

$$u\left(t_0\right)=q_0p_0,\ \ \dot{u}\left(t_0\right)=0. \tag{21}$$

The solution of equation (20) can be written in the form

$$u=u_h+u_n, \tag{22}$$

where u_h is the general solution of the homogeneous equation associated with (20), while u_n is a particular solution of (20). It is easy to see that

$$u_n=q_0p_0. \tag{23}$$

In consequence, u_h is the solution of the following Cauchy problem

$$\begin{cases} \ddot{u}-2\left(\left(\frac{dH_0}{du}\right)^2 u-\bar{V}\left(u\right)\right)=0, \\ u\left(t_0\right)=0,\ \ \dot{u}\left(t_0\right)=0. \end{cases} \tag{24}$$

The uniqueness of the solution to the above problem ensures us that

$$u_h\equiv 0, \tag{25}$$

from which we obtain

$$u=q_0p_0. \tag{26}$$

On behalf of (26), we easily derive that the solutions of equations (11)–(12), subject to the initial conditions (13)–(14), are given by

$$q = q_0 \exp\left(\left(\frac{dH_0}{du}(q_0p_0)\right)(t-t_0)\right), \tag{27}$$

$$p = p_0 \exp\left(-\left(\frac{dH_0}{du}(q_0p_0)\right)(t-t_0)\right). \tag{28}$$

On the other hand, from the Hamilton equations (5)–(6) we find the relation

$$\frac{d}{dt}(qp) = 0, \tag{29}$$

which further leads, taking into account (8), to

$$qp = q_0p_0. \tag{30}$$

By virtue of the last formula we easily deduce that the solutions of the Hamilton equations (5)–(6) in the presence of the initial conditions (8) are expressed precisely by (27)–(28). This proves the theorem. ■

In the context of a class of Hamiltonians of the type (4) the above theorem emphasizes a new relationship between Lagrangian and Hamiltonian formalisms.

To conclude with, in this paper we have proved that that for dynamical systems described by Hamiltonians of the type $H_0(qp)$ we can find an equivalent second-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space.

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