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Fractal dynamics as long range memory modeling technique

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Abstract

The paper shows that long memory effects can be included in mathematical models using fractional calculus. It is analyzed how the memory effects influence the evolution of a particular dynamical system.

Keywords: memory effects, fractional calculus

1 Introduction

The fractional calculus (i.e. calculus of integrals and derivatives of any arbitrary real order) has gained considerable importance during the past three decades, due mainly to its applications in diverse and widespread fields of science and engineering. The simultaneous evolution of the fractal theory and the higher performances of the computers has positively influenced the development of the fractional calculus from a theoretical point and has multiplied its applications.

The concept of fractional calculus is not new. The first question was raised in 1695 by L'Hospital to Gottfried Leibniz ([1], preface) about the derivative of order 1/2. Mention of the fractional derivatives (as an open problem) was made by Euler (1730), Lagrange (1772), Laplace (1812), Lacroix (1819), Fourier (1822). The first systematic results were obtained by J. Liouville in 1832 [2] and continued by B. Riemann [3]. The first work devoted exclusively to the subject of fractional calculus was published in 1974 [4]. From that moment many monographs dedicated to this subject were published (for example [5]-[7]).

The fractional derivatives are defined using integrals, so they are non-local operators. The fractional derivative in time contains information about the function at earlier points, so it possess a memory effect. The fractional derivative in space includes non-local spatial effects, Levy flights for example.

Because of their non-local property fractional derivatives can be used to construct simple material models and unified principles. They have important applications in astrophysics [8], economics [9], fusion plasmas [10]-[13], mechanics [14]-[15], viscoelasticity [16]

In this paper we are interested to show how long memory effects can be incorporated in mathematical models using fractional calculus.

2 Fractional derivatives and memory effect

The fractional calculus is an extension of classical calculus for non-integer order of derivation. More than one way to transfer integer-order operations to the non-integer case was developed. There are at least three important definitions for fractional derivatives. We can speak about Riemann-Liouville, Caputo and Grunwald-Letnikov fractional derivatives of a function. Because we are interested in presenting the influence of the memory effect we will consider only left derivatives of functions depending on time, i.e. $y = y(t)$.

Definition 1 *For $y \in L_1([a,b])$ the left Riemann-Liouville derivative,$D^{\alpha}_{a+}y : [a,b] \to \mathbf{R}$ of order $\alpha \in [n-1,n)$ is defined by*

$$D^{\alpha}_{a+}(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)}\int_a^t \frac{y(x)}{(t-x)^{\alpha-n+1}}dx \tag{1}$$

The left Caputo-derivative ${}^{C}D^{\alpha}_{a+}y : [a,b] \to \mathbf{R}$ of order $\alpha \in (n-1,n]$ is defined by

$$\left({}^{C}D^{\alpha}_{a+}y\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{y^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx \tag{2}$$

The left Grunwald-Letnikov derivative $y^{\alpha}_{a+} : [a,b] \to \mathbf{R}$ of order α is defined by

$$y^{\alpha}_{a+}(t) = \lim_{h\to 0}\frac{\sum_{k=0}^{N_a}(-1)^k \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}y(t-hk)}{h^{\alpha}} \tag{3}$$

where $N_a = \left[\frac{t-a}{h}\right]$.

Some preliminary definitions of what we call now "fractional Riemann-Liouville" derivatives were proposed in the middle of XIX-th century in a sequence of papers of J. Liouville and G. F. B. Riemann.

In 1869 N.Y. Sonin gave a compact form to these definitions extending the Cauchy's integral formula for integer order derivatives to arbitrary $\alpha > 0$ [17], but their actual form (1) was given in 1884 by H. Laurent [18].

The main objections in using this derivative are: the fractional derivative of a constant is not 0 when $\alpha \notin \mathbf{N}$ (which leads to many complications in usual computations) and it is not compatible with the Laplace transform (i.e. the usual formula for $\left(Ly^{(n)}\right)(p)$ is not verified for $\alpha \notin \mathbf{N}$).

In 1967 M. Caputo proposed the definition (2) which avoided these disadvantages [19]. The Caputo fractional derivative is today frequently used in applications, especially for linear differential equations that can be solved using the Laplace transform .

The Caputo fractional derivative is strongly connected to the Riemann-Liouville fractional derivative because

$$\left(D^{\alpha}_{a+}y\right)(t) = \left({}^{C}D^{\alpha}_{a+}y\right)(t) + \sum_{k=0}^{n-1}\frac{1}{\Gamma(1+k-\alpha)}y^{(k)}(a)(t-a)^{k-\alpha} \quad , \quad \alpha \in (n-1,n).$$

It is also interesting to point out that

$$y^{(n)}(t) = \left(D^{n}_{a+}y\right)(t) = \left({}^{C}D^{n}_{a+}y\right)(t)$$

The Grunwald-Letnikov fractional derivatives, introduced in [20] [21] are very important now from numerical point of view. In fact the finite Grunwald-Letnikov derivative

$$\left({}_{F}y^{\alpha}_{a+}\right)(t) = \frac{1}{h^{\alpha}}\sum_{k=0}^{N_a}(-1)^k \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}y(t-hk). \tag{4}$$

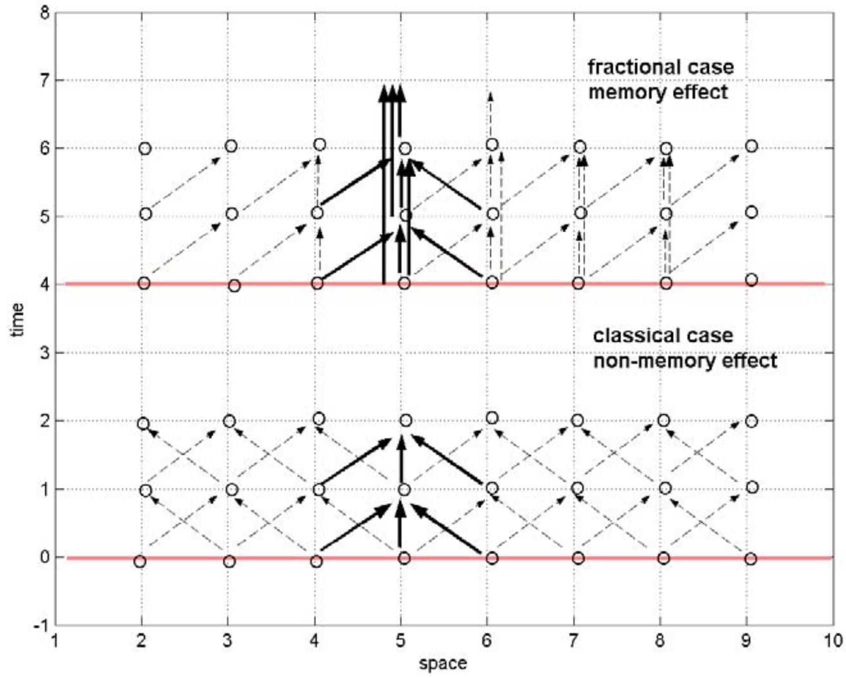

Figure 1: Including long range memory effect through fractional temporal derivatives (top of the figure) and classical short memory effect through classical temporal derivative of order I (bottom of the figure)

is an approximation of the first order of the Riemann-Liouville derivative iff $y(0) = 0$, because [22]

$$\left({}_F y_{a+}^{\alpha}\right)(t) = \left(D_{a+}^{\alpha} y\right)(t) + O(h) + O(y(0)).$$

From the previous relation one can notice that, iff $y(0) = 0$,

$$\left(y_{a+}^{\alpha}\right)(t) = \left(D_{a+}^{\alpha} y\right)(t).$$

Because the analytical computation of Riemann-Liouville or Caputo fractional derivatives is difficult, quite impossible in many cases, their numerical approximation is very useful. For this reason efficient algorithms for numerically solving fractional differential equations were elaborated. Part of them use the Grunwald-Letnikov derivatives [22].

The consideration of memory effect in the computation of fractional derivatives is evident if we look the formula (4). If $\alpha \notin \mathbf{N}$, for a fixed initial moment t_0 and a step time $\Delta t = h$, the fractional G-L derivative $\left({}_F y_{a+}^{\alpha}\right)(t_0 + sh)$ takes into account all the values $y(t_0)$, $y(t_0 + h)$, $y(t_0 + 2h)$, ...,$y(t_0 + (s-1)h)$, $y(t_0 + sh)$. When we compute $y'(t_0 + sh) \approx \frac{y(t_0+sh)-y(t_0+(s-1)h)}{h}$, we take into account only the last two values in the previous sequence, so the memory effect it not present.

This observation is graphically pointed out in figure 1

In computing the fractional G-L derivative the "memory-interval" is important.

For exemplification, we plotted in figure 2 the derivative of order $\alpha = 0.5$ of the function $y(t) = \sin t$, starting from various initial moments t_0. The fractional derivative is defined on the interval $(t_0, 4\pi]$. In figure 2 are presented the graphs of the derivatives staring from $t_0 = 0$, $t_0 = \pi/2$, $t_0 = \pi$, $t_0 = 2\pi$, respectively $t_0 = 3\pi$.

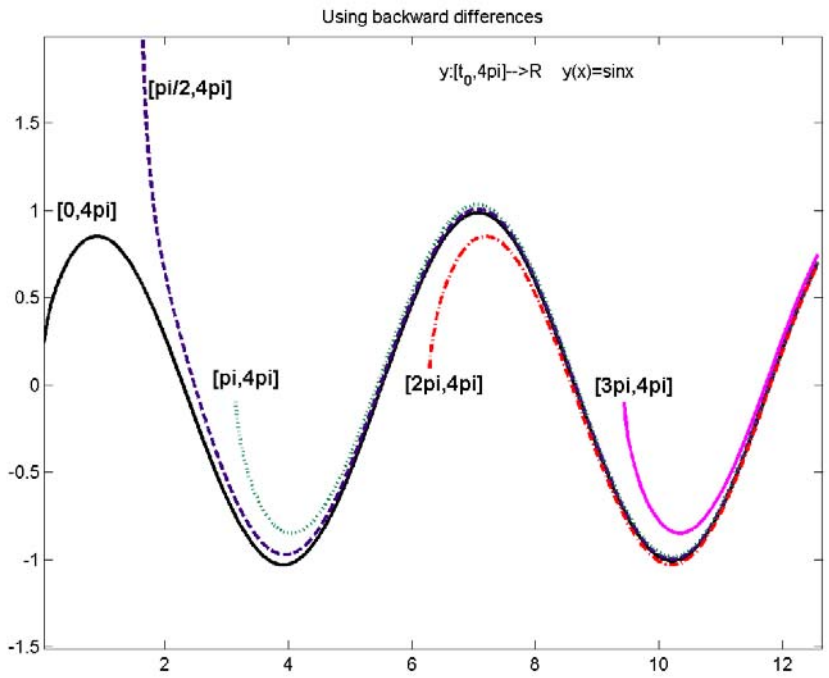

Figure 2: Left Grunwald-Letnikov derivatives of $y(t) = \sin t$ on various memory intervals

In figure 3 we draw fractional derivatives y^{α}_{0+} for $y(t) = \sin t$, $t_0 = 0$ and $\alpha \in \{0, 0.1, 0.2, ..., 0.9, 1\}$. The graph of $y^{0}_{0+} \equiv y$ is the continuous black line, the two dashed black lines correspond to $y^{0.5}_{0+}$, respectively y^{1}_{0+} and the other lines are the graphs of fractional derivatives for intermediate values of α. In this case Riemann-Liouville, Caputo and Grunwald-Letnikov derivatives of the same order coincide because $y(0) = 0$.

3 Fractional models including memory effects

In order to observe the influence of the memory effect on the equation of forced pendulum with friction we consider the problem

$$y''(t) + y'(t) + y(t) = 8, \quad y(0) = 0, \quad y'(0) = 5 \tag{5}$$

We can consider the memory effects by changing the previous equation with its "fractional version" which is generated supposing that the friction only is memory-dependent:

$$y''(t) + \left(D^{\alpha}_{0+}y\right)(t) + y(t) = 8, \quad y(0) = 0, \quad y'(0) = 5. \tag{6}$$

The first derivative $y'(t)$ from (5) was replaced by the derivative of order α, with $\alpha \in (0, 1)$.

Both problems (5) and (6) may be solved using Laplace transform. In (6) we observe that the Riemann-Liouville derivative coincide with the Caputo and with the Grunwald-Letnikov derivatives, because $\alpha \in (0, 1)$ and $y(0) = 0$.

In this case one can use the relations

$$L\left(D^{\alpha}_{0+}y\right)(s) = s^{\alpha}(Ly)(s)$$

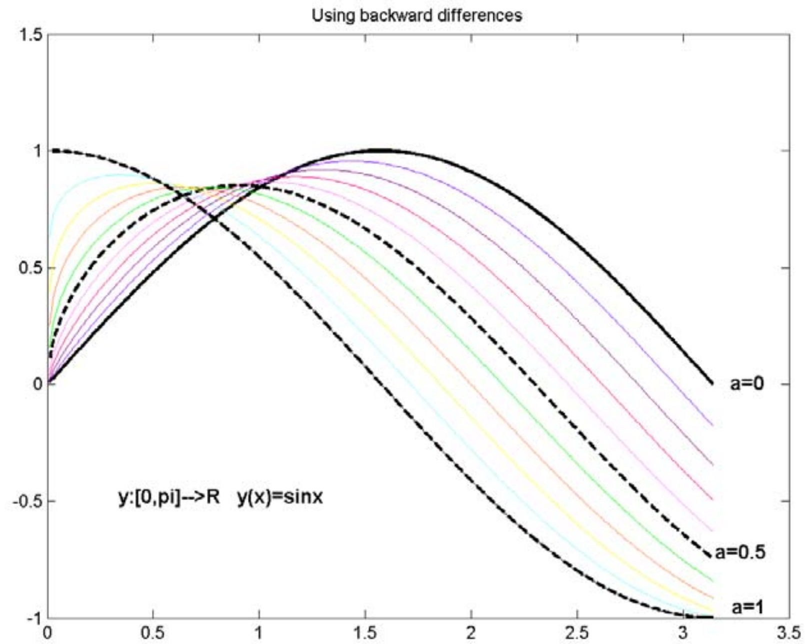

Figure 3: Left Grunwald-Letnikov derivatives of different orders for $y : [0, \pi]$, $y(x) = \sin x$

$$\left(L^{-1}\frac{s^{\alpha-1}}{s^{\alpha}-1}\right)(t) = E_{\alpha,1} = \sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$$

but the analytical expression of the solution is very complicate, involving Mittag-Lefler functions.

In this situation the numerical approximation of the solution is appropriate. For this purpose we used the numerical method described in [23], based on finite Grunwald-Letnikov derivative.

The graphs of the solutions on the interval $[0, 10]$ are plotted in figure4 for $\alpha = 0.5$ (continuous line), $\alpha = 0.99$ (dashed line), respectively $\alpha = 1$ (dot-dashed line).

Numerical simulations show that, for short period of initial time, $t \in [1, 10]$, the solution of (6) with different orders of fractional derivatives does not converge to the solution of (5), that is for $\alpha \to 1$. It means that the evolution of the system is deeply affected by the memory effects.

4 Conclusions

In this paper we analyzed the memory effects induced to a dynamical system when an evolution described through a fractional differential equation is considered. By the general definition (3), the fractional derivative supposes memory effects because of its dependence on many time moments.

For the particular example of $1D$ forced pendulum with friction, model described by (6), the influence of the memory effects on the dynamics were pointed out. It was shown that the evolution of the system is different in the first decade of time, depending on the order of the fractional derivative. In *Figure 4* three different evolutions are represented. Two of

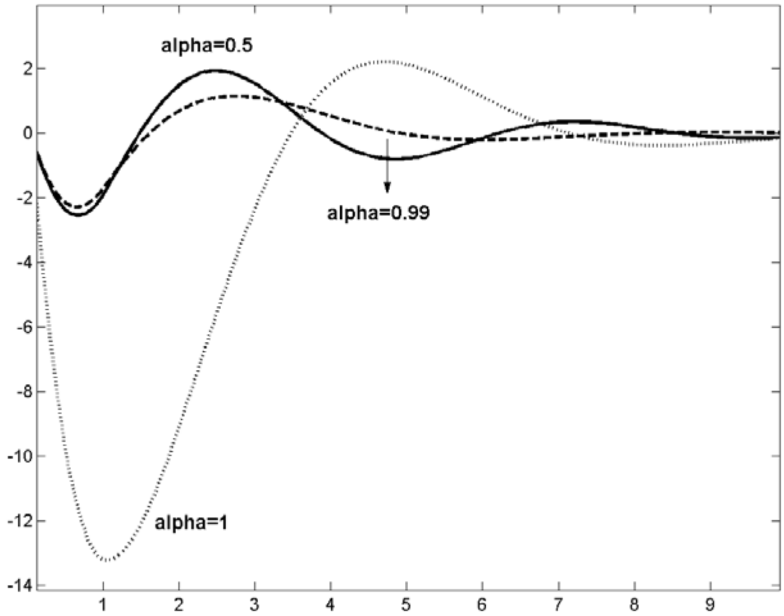

Figure 4: Solution of (6) for $\alpha = 0.5$, $\alpha = 0.99$, $\alpha = 1$

them correspond to the fractional derivatives of order $\alpha = 0,5$ respectively $\alpha = 0,99$. The third graph in the same figure, corresponding to the value $\alpha = 1$, describes the evolution of the system under the action of a first order derivative friction force. It is clear that this last graph does not represent the limit $\alpha \to 1$ of the first two graphs. These different looking oscillations are known as memory effects. After a longer time, the evolutions of the two systems become identical.

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