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Stochastic version of the linear instability analysis

György Steinbrecher

Association EURATOM-MECT, University of Craiova,
Physics Faculty, A. I. Cuza Street 13, Craiova-200585, Romania

Xavier Garbet

CEA, IRFM, F-13108 Saint Paul Lez Durance, France

Boris Weyssow

Association EURATOM-Etat Belge,
Physique Statistique et Plasmas, Université Libre de Bruxelles,
Campus Plain CP 231, B-1050 Bruxelles, Belgium; and

EFDA-CSU, Boltzmannstrasse 2, D-85748,
Garching bei München, Germany

Abstract

The problem of the stochastic linear stability analysis is treated within the framework of a model of random walk on the complex affine group. The new feature, related to the stochastic aspects of the instability analysis, is the occurrence of the heavy tail of stationary probability density function. We compute the exponent of the heavy tail in the framework of general complex, one- dimensional, slightly sub critical, continuous time random affine multiplicative model. In this model the driving multiplicative noise is complex, whose real part is Gaussian, stationary, with rapid decay of the correlations. The additive noise is complex, nonlinear and is subjected to restrictions of technical nature.

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I. INTRODUCTION

The large time statistical properties of complex deterministic dynamical systems in many situations can be approximated by suitable chosen stochastic processes, that in contrast with the initial non integrable system, it allows an explicit analytic treatment. It is important to observe that because any nonlinear evolution equation can be reformulated as an equivalent linear problem in a higher dimensional functional space, the stochastic version of this linear problem can be approximated as a lower dimensional random walk in some linear Lie group, or according to the terminology used in the physical literature, by random multiplicative processes (RMP) The study of the continuous time RMP is motivated by their connection to the heavy tail (HT) effects [1], by recently found connection with one-dimensional noise driven on-off intermittency (NOOI) models [2] and their interpretation as the simplest possible stochastic versions of the linear stability analysis (SLSA) of dynamical systems (DS).

The first classical examples of the discrete time RMP was the random affine iterated function systems (RAIFS) [3–5], currently used for image compression [5]. The support of their stationary probability distribution is a bounded *self-similar* set of lower fractal dimension [5].

If the multiplicative term in the one-dimensional RAIFS is randomly contracting and dilating with prevailing contractions, then the support of the stationary probability density function (PDF) extends to infinity and has an inverse power law, or heavy tail asymptotic behavior [1, 6–8]. More exactly, the stationary cumulative probability distribution function (CDF) of the solution $x(t)$ of the RMP process has the asymptotic behavior $prob[|x(t)| \geq A] = O(A^{-\beta})$ for $A \to \infty$, i.e. it is *self-similar* at large scales, as observed in self-organized criticality (SOC) related phenomena [9]. The positive exponent β is called the heavy tail exponent (HTE). HT effects were observed in self-organized criticality (SOC) models [9], in experiment in tokamaks [10], condensed matter [1], population dynamics [11, 12], internet traffic fluctuations [13], mathematical finance models [14], electronic circuits [15], power grids [16], Monte-Carlo simulations [17]. Extremely low values for β, close to zero, was observed in tokamak experiments [10] and in some SOC models [9].

There exist different explanations for HT effects, not related to the RMP [18, 19]. Nevertheless HT effects related to instability growths were previously explained by overdamped linear RMP models in discrete time [1, 6, 8, 20], as well as by continuous time models [2, 15, 21, 22]. The RMP model from [22] correlates two experimentally observed effects on DIII-D tokamak: the self-similarity of the fluctuations [24] and the very low value of HT exponent [10].

A rigorous mathematical treatment of a general class discrete time models that include [1, 6, 7], was given in [25]. In the case of the continuous time models with very general driving noise we remark that β depends only on the instability threshold and on the zero frequency component of the additive noise [2, 22]. This dependence is recovered also in the case of models studied in this work, as well as the simple form of β.

Result on the HT effects in particular models, equivalent to higher-dimensional RMP were obtained in [20]. Partial analytic results were also obtained in linear oscillator model with multiplicative noise [26]. Infinite-dimensional linear RMP models occurs naturally in the study of the linear stability study of the solutions of nonlinear partial differential evolution equation with noise terms [27]. Besides mathematical difficulties, the restriction of the our study to one-dimensional, real or complex RMP can be justified by the fact that we can approximate the behavior of the higher-dimensional system by its least stable eigenmode mode, which is dominating in the large time limit.

In the framework of one-dimensional continuous time RMP, explicit form for β was obtained in [22] for a category of linear real SDE, where driving noises are linear combination of independent colored noises. This result was extended to the case of linear real SDE with general stationary Gaussian multiplicative noise and constant additive term in [2], by using rigorous Banach space methods. In this work we generalize the results from [2, 22]. Compared to the previous studies, the RMP from this work will be described by a stochastic differential equation (SDE), having the phase space the whole complex plane. The dominant random linear term contains a complex valued random factor, whose real part is a stationary Gaussian noise. Because usually the RMP models are linear approximations of nonlinear equations, we will study a class of models with nonlinear additive term, subjected to restrictions of technical nature. This generalization will be used in the study of the OOI models and simplified version of SLSA. The possibility to obtain exact result with the nonlinear additive term extends the range of applicability of RMP models, because it is possible to approximate nonlinear equations without Taylor expansion.

The heavy tail effects are only the stationary counterpart of a more complex phenomena related to the noise driven on-of intermittency. Intermittency is a common phenomenon in fluid dynamics [30] and it was observed also in the plasma turbulence related to controlled fusion experiments [31–33]. It was predicted on reduced models of profile relaxations [34] as well by numerical simulations [35–37]. A robust form of the manifestation of the intermittency, different from the generic mechanism given in [38], is the on-off intermittency (OOI) mechanism [39, 40]. It was discovered in numerical simulations [41–43] and latter was identified in many physical models and experiments [44–56].

In OOI systems the stationary PDF has a dominating singularity near an invariant manifold. The period when the system is in the close neighborhood of this manifold is interpreted as the "laminar phase", so by assuming ergodicity the strength of the singularity is related to the mean sojourn time in the quiescent state. This singularity structure is clearly visible in a subclass of NOOI models [2, 21, 28, 29]. One of the motivation of the present work is to generalize the previous models to obtain a more general exact result for the singularity exponent λ, because its value give a quantitative characterization of the statistical aspects of the laminar periods, i.e. it is a measure of the intermittency itself. We will see that also in the case of the complex phase space the study of NOOI models can be reduced to complex-valued RMP.

In this article the main technical part is devoted to the large time behavior of the fractional order moments of the solution $Z(t)$ of a very general one-dimensional complex valued RMP process, i.e. the function $\langle|Z(t)|^p\rangle$. Despite the mathematical methods are similar, the full rigorous proof from [2] cannot be extended in this case of complex RMP under the general assumptions. *In the general case we prove that iff the order p is less then some computable critical value, then $\langle|Z(t)|^p\rangle$ is bounded and its infinite time limit is independent from $Z(0)$. This critical value is just the HTE β of the stationary PDF*, if the stationary PDF exists at all. If $p > \beta$ and $|Z(0)|$ is sufficiently large then $\langle|Z(t)|^p\rangle$ diverges exponentially. The very existence and non triviality of the stationary PDF, of the lowest order moments in the large time limit will be proved in a restricted class of complex RMP models.

The structure of this article is as follows:

In Section II the basic assumptions concerning the new RMP model are presented, as well as the main result: the exact formula for HTE and large time asymptotic behavior of the moments $\langle|Z(t)|^p\rangle$, for different values of p. In Section III applications of the results are given. In the Subsection III A the consequences on the stochastic version of the linear stability analysis are exposed. In the Subsection III B we exemplify our results on some solvable models and we give a possible hint for the study of the higher-dimensional cases.

The proof to the main results concerning the bounds on the moments of solution the RMP $\langle |Z(t)|^p \rangle$, is exposed in the Section IV. These results allow computing the HT exponent if the existence of stationary PDF is assumed. For the sake of more mathematically oriented readers we included also an existence proof of the stationary PDF in the Section V. The proof uses additional restrictions on the additive part of the RMP.

II. THE RMP MODEL AND THE RESULTS

A. The complex weakly nonlinear RMP model

In analogy to the previous continuous time RMP models, our model describes the instability growth when the multiplicative term is perturbed by some random, stationary noise. It can be considered as a new step in the modelling of complex DS by studying the response under stochastic perturbations of the stable eigenmode that is most close to the instability threshold. If the perturbations of the eigenmodes have comparable amplitudes, it is expected that the largest response will be provided by this eigenmode and the others can be neglected or approximated by the additive term.

The phase space of the model is the complex plane and the one-dimensional complex valued stochastic process $Z(t)$ specifies the time evolution. It obeys the following one-dimensional complex valued SDE

$$\frac{dZ(t)}{dt} = -(a + \zeta(t))Z(t) + R(t) \tag{1}$$

where $a = a_1 + ia_2$ is constant, whose real part $\mathrm{Re}(a) = a_1$ can be interpreted as an instability threshold. The instability threshold a_1 is assumed positive in the cases of interest, that means that the process is mainly contractive. Nevertheless our intermediate results on the bounds on the moments of PDF still hold for arbitrary a_1. By $R(t)$ we denoted the random additive term

$$R(t) = \sum_{k=1}^{N} \phi_k(t) \tag{2}$$

and in Eqs.(1, 2) by Greek letters we denoted the noises. Real and linear particular versions of the equation Eq.(1) was studied in [2, 22]. Here $\zeta(t)$ is the complex driving multiplicative noise, and $R(t)$ is the driving additive noises. We will suppose that all of the stochastic processes $\zeta(t), R(t)$ are stationary and has continuous realizations, consequently the Itô and Stratonovich interpretations of Eq.(1) are identical [57]. This choice of the SDE allows to extend the main result from [2, 22].

Concerning the driving additive noises, we will suppose only that all of the moments of integer order of $R(t)$ exists.

$$\langle |R(t)|^n \rangle = \nu_n < \infty \tag{3}$$

This property from Eq.(3) holds for instance when the stochastic process $R(t)$ is Gaussian. In addition we suppose that the random processes $\zeta_1(t) = \mathrm{Re}(\zeta_1(t))$ and $R(t)$ are independent.

This category of SDE, with dominating linear term and bounded additive terms allows to construct stochastic one-dimensional complex approximations of higher-dimensional nonlinear dynamical systems with increased domain of validity.

The SDE Eq.(1) is a generalization of the previous continuous time models of RMP processes related to HT effects [22] . Remark that Eq.(1) can be obtained if we approximate the behavior of slightly subcritical linear DS by approximating it by the less stable linear eigenmode. In order that Eq.(1) be a reliable approximation we suppose that in the range of the large amplitudes the dominant term is linear.

In this article we will study the problem of the large $|Z|$ asymptotic behavior of the stationary PDF associated to the SDE Eq.(1).

Remark 1 *The additive terms $R(t)$ are mandatory to introduce in Eq.(1), despite their explicit form do not influence our final results. Indeed, even in the more general case of Eq.(1), when $Z(t)$ is a random vector, a is a linear operator, $\zeta(t)$ is a random linear operator, whose matrix elements are stationary, in the case when $R(t) = 0$ the equation is invariant under the group of dilatations $Z(t) \to \alpha Z(t)$. It follows that in the limit $t \to \infty$ the stationary PDF of $\|Z(t)\|$, if it exists at all, must be self-similar, either of the form $\rho(\|Z\| = const\,\|Z\|^{-\gamma}$, which is non integrable, or trivial, singular, collapsed to the origin: $\rho(\|Z\| = \delta(\|Z\|)$. In the Section V the existence of the large time limit of $\langle |Z(t)|^p \rangle$ is proven, under additional restrictions on the additive noise. We conjecture that the independence of the additive and multiplicative noise is a sufficient condition for the non triviality of the stationary PDF.*

We will denote by $\langle\ \rangle_\zeta$, respectively by $\langle\ \rangle_R$ the expectation value of all of the realizations of the stochastic processes $\zeta(t)$, respectively $R(t)$, where it is necessary to avoid confusions. The symbol $\langle . \rangle$ will denote the averaging with respect to all of the processes $\zeta(t),\ R(t)$.These subscript will be omitted if no confusion appears. We will consider the starting position of the solutions of SDE's fixed.

We suppose in the following only that the real part of noise $\operatorname{Re}\zeta(t) = \zeta_1(t)$ is stationary, Gaussian and without loss of generality we consider it centered, i.e. $\langle \zeta_1(t) \rangle \equiv 0$, because a constant term in $\zeta_1(t)$ means a shift of a_1. Suppose that the correlation function $\langle \zeta_1(t)\zeta_1(0) \rangle$ has a rapid decay

$$|\langle \zeta_1(t)\zeta_1(0) \rangle| = O[(1+|t|)^{-2-\varepsilon}] \tag{4}$$

where $\varepsilon > 0$. The power spectrum of $\zeta_1(t)$ is given by

$$P_1(\omega) = \int_{-\infty}^{\infty} \langle \zeta_1(t)\zeta_1(0) \rangle \exp(i\omega t) dt \tag{5}$$

Introduce the random walk $Y(t)$ associated to the real part $\zeta_1(t)$ by the usual definition

$$Y(t) = \int_0^t \zeta_1(t') dt' \tag{6}$$

and denote $\langle Y(t)^2 \rangle /2 = d(t)$. The process $Y(t)$ is a centered Gaussian process with stationary increments. From the Taylor-Kubo formula, for $t \to \infty$ we obtain

$$\left\langle Y(t)^2 \right\rangle = 2d(t) \asymp P_1(0)\ t = 2Dt \tag{7}$$

where we introduced the "diffusion constant" $D = P_1(0)/2$. For technical reasons we will restrict our study to the case when the following *condition hold*: there exists a constant k such that for all $t \geq 0$ we have

$$Dt - k \leq d(t) \leq Dt + k \tag{8}$$

The condition Eq.(8) can be justified easily if we postulate Eq.(4).

Explicit form of D in the case when $\zeta_1(t)$ is a linear combination of independent Ornstein-Uhlenbeck processes is given in [22]. It can be verified that in that case Eq.(8) holds.

B. The results

The main part of this article is concentrated on the calculation of the HT exponent β, that specifies the large $|Z|$ behavior of the stationary PDF associated to SDE from Eq.(1). The exponent β defined in the previous works [2, 22] by the asymptotic form of stationary CDF $prob(|Z(t| \geq x) = O(\ x^{-\beta})$, will be identified by the critical value p_{cr} of the fractional power of the moments $\langle |Z(t)|^p \rangle$, such for $p < p_{cr}$ the moments are bounded and if $p > p_{cr}$ they diverges. If the stationary CDF exits then $p_{cr} = \beta$.

Under the previous assumptions, we will prove the following results.

Theorem 2 *If $p < a_1/D = p_{cr}$ then for $t \to \infty$ the fractional moments $\langle |Z(t)|^p \rangle$ are bounded (thus $\beta \geq a_1/D$). If $p > \mathrm{Re}(a)/D = p_{cr}$, and $|Z(0)|$ sufficiently large, then for $t \to \infty$ the moments $\langle |Z(t)|^p \rangle$ diverges (thus $\beta \leq a_1/D$). If $p > a_1/D$ then we have the asymptotic behavior*

$$\langle |Z(t)|^p \rangle = O\{\exp pt(p - a_1)\} \tag{9}$$

In particular, if $a_1 \leq 0$ then all of the moments diverges.

From this theorem follows the following main Corollary.

Corollary 3 *In the limit $t \to \infty$, the stationary cumulative probability distribution function $P(x) = prob(|Z(t| \geq x)$, when it exists, for $x \to \infty$ has the asymptotic behavior characteristic to HT*

$$prob(|Z(t| \geq x) = O(x^{-\beta}) \tag{10}$$

where the HTE β is given by

$$\beta = a_1/D = 2a_1/P_1(0) \tag{11}$$

The explicit formula for β from [2, 22] are particular cases of the previous Corollary 3. Remark that Eqs.(9, 10, 11) predicts the importance of the low frequency component of the driving noise on the evolution of instability. Similar conclusions on related discrete time NOOI model was reported in [58]. Exactly this low frequency component is amplified and observed in experiments as the dominant noise, e.g. in the tokamak plasma edge fluctuation measurements [59]. The power spectrum dependence from Eq.(11) is a hint that contrary to the linear SDE driven by additive noise, the RMP process filters out the effects of high frequency components of the multiplicative noise at large scales.

III. APPLICATIONS

A. Stochastic instability analysis

In this part we will summarize the implications of the previous result on the problem of modelling the instability growth in the linear approximations of DS exposed to random stationary perturbations. We suppose that starting from the high-dimensional linear approximation of the instability growth we made a further approximation by retaining the mode that has the highest growth rate. The contribution of the rest of the more quickly decaying modes, as well a part of the nonlinear term we model by the nonlinear additive noise. We will see that the results of the stochastic instability analysis, performed in the overdamped approximation [2] remains correct in this more general case.

1. The stochastic linear stability analysis, linear system without saturation term.

In this case the driving equation of the evolution of the instabilities under the effects of the noise is described by the Eq.(1). We consider the following cases:

a. Subcritical linear system perturbed by noise. This system is described by the Eq.(1) with $\mathrm{Re}(a) > 0$. In this case according to Theorem 2 and Corollary 3 the PDF of amplitude of the instabilities in the stationary state has a HT, with exponent $\beta = \mathrm{Re}(a)/D$. The large time evolution of the instabilities can be described as follows: If $0 < p < \beta$ then $\langle |Z(t)|^p \rangle$ remain bounded and for $p > \beta$ we have an exponential blow-up given by Eq.(9). Observe that despite the DS without noise is stable, in the presence of the noises we have large amplitudes with high probability. In conclusion the linear approximation of DS is questionable.

b. Supercritical linear system perturbed by noise. This system is described by the Eq.(1) with $\mathrm{Re}(a) < 0$. In this case we have no stationary state at all and according to Theorem 2 and Eq.(9) for all $p > 0$ $\langle |Z(t)|^p \rangle$ diverges exponentially. In other terms, the solution of the SDE in this case behaves similarly with its deterministic counterpart.

In conclusion, due to the multiplicative noise, the linear approximation of complex nonlinear DS is questionable. So in order to study the instability growth we need some mechanism that give rise to saturation.

B. Solvable models and a conjecture

Consider two simple limiting cases of the stochastic process described by SDE Eq.(1), driven by white noises. Particular cases of the first model was already studied in [15, 21]

$$dx(t) = [-a_1 dt + \sigma dw_1(t)] \circ x(t) + h(x) + g(x) dw_2(t) \tag{12}$$

where a_1, σ are constant, $a_1 > 0$ and the functions corresponding to the additive noises $h(x)$, $g(x)$ are bounded. In the first term we used the Stratonovich multiplication [57], denoted by $\circ$. The processes $w_1(t)$ and $w_2(t)$ are independent real standard Brownian motions. According to Eqs.(6 , 8) we have $D = \sigma^2/2$. By direct calculation can be verified that the stationary PDF $\rho(x)$, the solution of the Fokker-Planck equation has the following large x asymptotic form: $\rho(x) \asymp x^{-1-\beta}$ with $\beta = a_1/D$, thus Eq.(10) is verified. Observe that in this limiting case the Stratonovich prescription give the correct value of the HTE.

The complex version of Eq.(12) is

$$dZ(t) = [-a dt + \sigma dW_1(t)] \circ Z(t) + r dW_2(t) \tag{13}$$

where $a = a_1 + i a_2$, $a_1 > 0$ and r is a complex parameter are similar to the case of the Eq.(12). Again $D = \sigma^2/2$. The stochastic processes $W_1(t)$, $W_1(t)$ are standard complex independent Brownian motion (BM), more exactly their real and imaginary parts are standard real independent BM. Remark that in this model the Stratonovich or Itô in the first term of Eq.(13) give the same result. By straightforward calculation we can verify that the stationary Fokker-Planck equation has a symmetric solution $\rho(|Z|)$ with the expected large $|Z|$ asymptotic behavior $\rho(|Z|) \asymp |Z|^{-2-\beta}$, in accord with Eq.(10).

The problem of the generalization of the previous results stated in Proposition 3 and Theorem 2 to the high-dimensional models is still an open problem. A possible generalization of the Eq.(1) is the following

$$\frac{d\mathbf{Z}(t)}{dt} = -(\widehat{\mathbf{a}} + d\widehat{\mathbf{Y}}(t)/dt)\mathbf{Z}(t) + \mathbf{R}(t) \tag{14}$$

where $\mathbf{Z}(t)$ is an N-dimensional vector, $\mathbf{R}(t)$ are N-dimensional vector valued functions, whose components are has bounded moments in analogy to the our previous discussions, $\widehat{\mathbf{a}}$ is a constant complex $N \times N$ matrix, with positive definite Hermitian part and $d\widehat{\mathbf{Y}}(t)/dt$ is a complex $N \times N$ matrix-valued stationary Gaussian process. Remark that even in the case when the SDE Eq.(14) is real and two-dimensional, it cannot be reduced to the previous SDE Eq.(1). Denote by $\widehat{\mathbf{Y}_1}(t)$its Hermitian part in analogy to Eqs.(6, 7), whose matrix elements has the following asymptotic form for large t

$$\langle Y_{i,j}(t) Y_{m,n}(t) \rangle \asymp 2D_{i,i,m,n} t \tag{15}$$

Previous examples suggest a possible route to simplify the study of the asymptotic properties of the solutions of Eq.(14) by the following

Conjecture 4 *The large $\|\mathbf{Z}\|$ asymptotic behavior of the stationary PDF of the SDE Eq.(14) is independent of the nonlinear additive terms. It the same as that of SDE, defined in Stratonovich sense and obtained from Eq.(14) by replacing the matrix valued Gaussian process $\widehat{\mathbf{Y}}(t)$ with stationary increments, by the matrix-valued Brownian motion $\widehat{\mathbf{B}}(t)$. The matrix elements of $\widehat{\mathbf{B}}(t)$ are correlated according to*

$$\langle B_{i,j}(t) B_{m,n}(t') \rangle = 2D_{i,j,m,n} \min(t, t') \tag{16}$$

with identical coefficients $D_{i,j,m,n}$ in Eqs.(15, 16).

The study of higher-dimensional generalizations, applied to discretized versions of linear noisy reaction-diffusion partial differential equations, relation to Anderson localization is outside of the scope of the present work.

IV. THE LARGE TIME ASYMPTOTIC BEHAVIOR OF THE MOMENTS OF PDF

Because the driving noises are classical functions, the integral form of the SDE Eq.(1), with the initial condition $Z(0) = Z_0$ is

$$Z(T) = \int_0^T R(Z,t) \exp\left[-a(T-t) - U(T) + U(t)\right] dt + Z_0 \exp\left[-aT - U(T)\right] \tag{17}$$

where we denoted

$$U(t) = \int\limits_0^t \zeta(t') dt \tag{18}$$

Recall that according to Eq.(8) we have $\mathrm{Re}(U(t)) = Y(t)$. The fractional order moments, whose large time asymptotic behavior will be studied, are defined as

$$M(p,T) = \langle |Z(T)|^p \rangle \tag{19}$$

We denote the random variables

$$A_U(T) = \exp\left[-aT - U(T)\right] \tag{20}$$

$$B_U(T) = \int_0^T R(Z(t),t) \exp\left[-a(T-t) - U(T) + U(t)\right] dt \tag{21}$$

It will be proved that for large T, and for Z_0 sufficiently large, if $p > \mathrm{Re}(a)/D$ then $M(p,T)$ diverges exponentially. If $p < \mathrm{Re}(a)/D$ then $M(p,T)$ is bounded and the limit, when it exists it is independent of the initial point Z_0. Then it follows that the stationary CDF, if it exists, has a HT Eq.(10), with the HTE given by Eq.(11). The results of the following subsection, of technical nature, will be used in the final proof of Eq.(11).

A. The main lemma: bounds on the moments of $A_U(T)$ and $B_U(T)$

In order to study the large time behavior of $\langle |Z(t)|^p \rangle$ we will study first separately the moments of the terms from Eq.(17). Denote

$$M_A(p,T) = \langle |A_U(T)|^p \rangle \tag{22}$$
$$M_B(p,T) = \langle |B_U(T)|^p \rangle \tag{23}$$

Observe that $\langle |A_U(T)|^p \rangle = \exp(-pa_1 T) \langle \exp(-pY(T)) \rangle$ and

$$c_1 \exp\left[-pT(a_1 - pD)\right] \leq M_A(p,T) \leq c_2 \exp\left[-pT(a_1 - pD)\right] \tag{24}$$

for some fixed constants c_1, c_2 that are independent of T. Such constants, independent of T, will be denoted in the following by $c_3,\ c_4, ...$ Their exact values are unimportant. Consequently we obtain for $p < p_{cr} = a_1/D$ the large time behavior

$$M_A(p,T) = O\left\{\exp\left[-pT(a_1 - pD)\right]\right\} \to 0 \tag{25}$$

From this result we will obtain that for $p < \mathrm{Re}(a)/D$ the limiting value of $\langle |Z(t)|^p \rangle$ is independent of $Z(0)$.

The difficult part of the proof is to obtain bounds on $M_B(p,T)$. The following main lemma will be proved.

Lemma 5 *The fractional moments $M_B(p,T)$, defined by Eqs.(23, 21) are bounded by*

$$M_B(p,T) = \langle |B_U(T)|^p \rangle \leq c \left|\exp\left[-pT(a_1 - pD)\right] - 1\right| \tag{26}$$

for some constant c independent of T.

In order to prove this lemma, the first step is to bound all the complex numbers by their absolute values in Eq.(21). We use the notation

$$|R(t)| \overset{Not.}{=} f(t) \tag{27}$$

The stochastic process $f(t)$ introduced in Eq.(27) is stationary, independent relative to $\zeta(t)$ and has finite fractional moments [see [23]]. By using in Eqs.(23, 21,) the obvious fact $|\sum_i a_i|^p \leq (\sum_i |a_i|)^p$ and Eq.(27) then we obtain a more easily tractable form

$$M_B(p,T) \leq \langle (B_{Y,f}(T))^p \rangle \overset{Not.}{=} N_B(p,T) \tag{28}$$

where the following notations were introduced

$$B_{Y,f}(T) = \int_0^T f(t) E_Y(t,T) dt \tag{29}$$
$$E_Y(t,T) = \exp\left[-a_1(T-t) - Y(T) + Y(t)\right] \tag{30}$$

Remark 6 . *Because the couple of stochastic processes $\{Y(t),\ f(t')\}$ are independent and $Y(t)$ has stationary increments, it follows that in the mean value calculations, like $\langle (B_{Y,f}(T))^p \rangle$ from Eqs.(28, 29, 30) we can substitute everywhere $Y(T) - Y(t)$ by $Y(T-t)$. In this case the terminology "$Y(T) - Y(t)$ is equal in distribution with $Y(T-t)$" and the notation $Y(T) - Y(t) \overset{d}{=} Y(T-t)$ is used [57].*

By applying this rule in Eq.(30) and denoting

$$H_Y(t) = \exp\left[-a_1 t - Y(t)\right] \tag{31}$$

from Eqs.(28, 29, 30, 31) we obtain

$$N_B(p,T) = \left\langle \left[\int_0^T f(t) H_Y(T-t) dt \right]^p \right\rangle_{f,Y}$$

By the change of the variable $t \to T - t$ results

$$M_B(p,T) \leq N_B(p,T) = \left\langle \left[\int_0^T f(T-t) H_Y(t) dt \right]^p \right\rangle_{f,Y} \tag{32}$$

In order to study the asymptotic behavior of the term $N_B(p,T)$ we will study several ranges of values p separately.

1. Case 1≤ p

When $p \geq 1$ observe that $M_A(p,T)$ and $M_B(p,T)$ can be expressed by the Lebesgue space norms. Define, for any functional $F(W,f)$, depending on some stochastic processes $W(t), f(t)$, the L^p Banach space norm [61]

$$\left\| F\left[W\left(.\right), f(.)\right] \right\|_p \overset{Not.}{=} \left\langle \left| F\left[W\left(.\right), f(.)\right] \right|^p \right\rangle_{W,f}^{1/p} \tag{33}$$

In our case $W(t) = U(t)$ or $W(t) = Y(t)$.

According to Eq.(32) and Eq.(33) we have

$$\left[M_B(p,T)\right]^{1/p} \leq \left\| \int_0^T f(T-t) H_Y(t) dt \right\|_p \tag{34}$$

as well as the simple L^p bound

$$\left[M_B(p,T)\right]^{1/p} \leq \int_0^T \left\| f(T-t) H_Y(t) \right\|_p dt \tag{35}$$

Because the processes $f(.)$ and $Y(.)$ are independent, the L^p norm from 35 factorizes

$$\left\| f(T-t) H_Y(t) \right\|_p^p = \left\langle f(T-t)^p H_Y(t)^p \right\rangle_{Y,f} = \left\langle f(T-t)^p \right\rangle_f \left\langle H_Y(t)^p \right\rangle_Y \tag{36}$$

Because the stochastic process $f(t)$ has finite moments μ_p from 36 results $\left\| f(T-t) H_Y(t) \right\|_p^p \leq \mu_p \left\langle H_Y(t)^p \right\rangle_Y$, which implies, together with Eq.(35) yield

$$M_B(p,T)^{1/p} \leq \mu_p^{1/p} \int_0^T \left\langle H_Y(t)^p \right\rangle_Y^{1/p} dt \tag{37}$$

The quantity from the r.h.s. of Eq.(37) now can be calculated exactly in a straightforward manner. From Eq.(31) we have

$$\int_0^T \left\langle H_Y(t)^p \right\rangle_Y^{1/p} dt = \int_0^T \exp\left[-a_1 t\right] \left\langle \exp\left[-pY(t)\right] \right\rangle_Y^{1/p} dt \tag{38}$$

By using simple Gaussian integration it results $\left\langle \exp\left[-pY(t)\right] \right\rangle = \exp\left[p^2 d(t)\right] \leq K_2 \exp(p^2 D t)$ for some constant K_2 Combined with Eqs.(37 and 38) we obtain the sought for bound

$$M_B(p,T)^{1/p} \leq c_3 \left| \exp\left[-T(a_1 - pD)\right] - 1 \right| / \left| a_1 - pD \right| \tag{39}$$

for some constant c_3, independent of T. This last inequality prove the lemma for $p \geq 1$.

2. Case $0 < p \leq 1$

In order to obtain bounds on $N_B(p,T)$ for large T, we will use a method similar to [2]. Consider $T = n\tau$, with $\tau > 0$ and n large. Observe that from Eq.(32) results

$$N_B(p,n\tau) = \left\langle \left[\sum_{k=1}^{n} \int_{(k-1)\tau}^{k\tau} f(n\tau - t) H_Y(t) dt \right]^p \right\rangle \tag{40}$$

From Eq.(40) we obtain the first inequality

$$N_B(p,n\tau) \leq \sum_{k=1}^{n} \left\langle \left[\int_{(k-1)\tau}^{k\tau} f(n\tau - t) H_Y(t) dt \right]^p \right\rangle \tag{41}$$

In analogy to [2] denote

$$b_k(t') = \left\langle \left[\int_{(k-1)\tau}^{k\tau} f(t' - t) H_Y(t) dt \right]^p \right\rangle \tag{42}$$

and rewrite Eq.(41) like

$$N_B(p,n\tau) \leq \sum_{k=1}^{n} b_k(n\tau) \tag{43}$$

At this stage we use the Proposition [17] respectively the subsequent inequality [12] from the our work [23]. By summation, for fixed p, a, D, from Eq.(43) we have

$$N_B(p,n\tau) \leq c_4 \left| \exp\left[(n+1)\tau p(pD - a\right] - 1 \right| \tag{44}$$

From this inequality and Eq.(32) the lemma for $0 < p \leq 1$ is proved.

B. Determination of the heavy tail exponent β

We emphasize that in this general setting we will not prove the very existence of the stationary PDF. We will suppose that at least for some initial condition it exists. A particular case, when existence of the stationary PDF can be proven, is studied in the Section V. In the general case the HTE β defined in Eq.(10) will be determined from the following Conditions (7 and 8), on the moments of the solution $Z(t)$ of Eq.(1) or Eq.(17).

Condition 7 *If $p < \beta$ then for all initial condition the mean value $\langle |Z(T)|^p \rangle$ are bounded by a constant independent of T and of the initial condition $Z(0)$. If the large T limit of $\langle |Z(T)|^p \rangle$ exists than it is also independent of $Z(0)$.*

Condition 8 *If $p > \beta$ then for $|Z(0)|$ sufficiently large we have $\langle |Z(T)|^p \rangle \to \infty$.*

We will see that the critical value of p from Conditions(78) is $\mathrm{Re}(a)/D$, so we will obtain $\beta = \mathrm{Re}(a)/D$. If we assume that Corollary (3) and Eq.(10) holds then these Conditions can be verified easily. In the following we will prove that such a value of β exists and is given by Eq.(11).

According to Eqs.(17, 20, 21)

$$\langle |Z(T)|^p \rangle = \langle |Z_0 \, A_U(T) + B_U(T)|^p \rangle \tag{45}$$

By using Conditions (7, 8) we will prove separately the inequalities $\beta \geq \mathrm{Re}(a)/D$ respectively $\beta \leq \mathrm{Re}(a)/D$, which imply our main result Eq.(11).

1. *Demonstration of the inequality $\beta \geq \mathrm{Re}(a)/D$*

Suppose $p < \mathrm{Re}(a)/D$. We will prove that $\langle |Z(T)|^p \rangle$ are bounded for $T \to \infty$. Then results that Condition (7) holds and $\beta \geq \mathrm{Re}(a)/D$.

In the case when $p \leq 1$, from Eq.(45) we obtain $\langle |Z(T)|^p \rangle = \langle |Z_0 A_U(T) + B_U(T)|^p \rangle \leq \langle [|Z_0 A_U(T)| + |B_U(T)|]^p \rangle$ and from main lemma from [23] results

$$\langle |Z(T)|^p \rangle \leq |Z_0|^p \langle |A_U(T)|^p \rangle + \langle |B_U(T)|^p \rangle \tag{46}$$

From Eq.(24)follows that the first term in Eq.(46) vanishes for large T. From this inequality and our previous bounds Eqs.(24 and 26) it follows that there exists a constant c, independent of Z_0 and T such that $\langle |Z(T)|^p \rangle < c$.

In a similar manner, in the case $p \geq 1$ we will use Eq.(45) and the Minkowski inequality. We obtain

$$\langle |Z(T)|^p \rangle^{1/p} = \|Z(T)\|_p \leq |Z_0| \, \|A_U(T)\|_p + \|B_U(T)\|_p \tag{47}$$

Then again from Eq.(24) we obtain that the first term in Eq.(47) goes to zero for large $T \to \infty$. From Eq.(26) we obtain for large T the uniform boundedness $\langle |Z(T)|^p \rangle < c$, with c independent of Z_0 and T, whenever $p < \mathrm{Re}(a)/D$.

It follows at this stage, that independent of the initial conditions, in both cases, if $p < \mathrm{Re}(a)/D$, then the fractional moments $\langle |Z(T)|^p \rangle$ are bounded by some some constant that is independent of T and of the initial condition. According to Eq.(7) if the limiting stationary PDF exists then the HTE is bounded by $\beta \geq \mathrm{Re}(a)/D$.

2. *Demonstration of the inequality $\beta \leq \mathrm{Re}(a)/D$*

If $p \leq 1$ then from Eq.(45) results

$$|Z_0|^p \langle |A_U(T)|^p \rangle - \langle |B_U(T)|^p \rangle \leq \langle |Z(T)|^p \rangle \tag{48}$$

From this inequality and our previous bounds Eq.(24) and Eq.(26) it follows that if $p > \mathrm{Re}(a)/D$, and the initial condition Z_0 is selected such that

$$|Z_0| > (c/c_1)^{1/p} \tag{49}$$

then for large T we have an exponential blow-up of the moments

$$\langle |Z(T)|^p \rangle \geq c_5 \exp\left[Tp(pD - a_1)\right] \tag{50}$$

with $c_5 = |Z_0|^p \, c_1 - c$, for $p \leq 1$.

In a similar manner, if $p > 1$ we will use Eq.(45) and the Minkowski inequality to obtain

$$|Z_0| \, \|A_U(T)\|_p - \|B_U(T)\|_p \leq \|Z(T)\|_p = \langle |Z(T)|^p \rangle^{1/p} \tag{51}$$

From this inequality and our previous bounds Eq.(24) and Eq.(26) it follows that if $p > \mathrm{Re}(a)/D$ and Z_0 is selected according to Eq.(49) then for large T and we have again an exponential blow-up of the moments according to Eq.(50). This time we have $c_5 = [|Z_0| \ c_1^{1/p} - c^{1/p}]^p$. It follows that there exists initial conditions in both cases, such that if $p > \mathrm{Re}(a)/D$, then the fractional moments $\langle |Z(T)|^p \rangle$ has an exponential blow-up. According to Eq.(8) if the limiting stationary PDF exists then the HTE is bounded by $\beta \leq \mathrm{Re}(a)/D$.

Finally, from these inequalities for β results $\beta = \mathrm{Re}(a)/D$.

V. EXISTENCE OF THE LIMITS

For the convenience of mathematically oriented readers we will give a short proof of the existence of the limit of the moments, but under more restrictive conditions on additive noise term in Eq.(1). Consider a particular case of the SDE Eq.(1), with initial condition $Z(0) = Z_0$

$$\frac{dZ(t)}{dt} = -(a + \zeta(t))Z(t) + \phi(t) \tag{52}$$

where $\mathrm{Re}(a) > 0$, the process $\zeta(t)$ is stationary and Gaussian with the properties Eqs.(6, 7, 8). Suppose that the processes $\zeta(t)$ and $\phi(t)$ are independent, $\phi(t)$ is stationary and symmetric, i.e. the processes $\phi(t)$ respectively the time reversed version $\psi(t) \equiv \phi(-t)$ has the same correlation functions. In other terms for all t_0 we have

$$\phi(t) \overset{d}{=} \phi(t_0 + t) \tag{53}$$

$$\phi(t) \overset{d}{=} \phi(-t) \tag{54}$$

Moreover we suppose that for all $p > 0$ we have

$$\langle |\phi(t)|^p \rangle = \mu_p < \infty \tag{55}$$

Remark that any stationary Gaussian noise satisfy Eqs.(53, 54, 55). We will prove the following

Proposition 9 *Under above conditions, if $p < \beta = \mathrm{Re}(a)/D$ and $\tau > 0$, then the sequence $\langle |Z(n\tau)|^p \rangle$ has the Cauchy property (consequently exists its limit for $n \to \infty$ and it is finite).*

In order to prove Proposition Eq.(9) we observe that from stationarity. and symmetry of $\phi(t)$ it follows that for a fixed constant t_0 we have the equality in distribution [see also [57] and Remark(6)]

$$\phi(t) \overset{d}{=} \phi(t_0 - t) \tag{56}$$

Because $\phi(t)$ is independent of the multiplicative noise, Eq.(56) means that in all of the mean value calculation we can made the simultaneous "change of variables " $\phi(t) -> \phi(t_0 - t)$.

Introduce again the function $U(t)$ by Eq.(18). In analogy with Eq.(17), taking into account Eq.(56), the fact that $\phi(t)$ and $U(t)$ are independent and $U(0) = 0$, we obtain the following integral form of Eq.(52)

$$Z(T) = \int_0^T \phi(T - t) \exp\left[-a(T - t) - U(T) + U(t)\right] dt \tag{57}$$

$$+ Z_0 \exp\left[-aT - U(T) + U(0)\right] \tag{58}$$

By using the independence of the couple $\{\phi(t),\ U(t)\}$ and the stationarity. of the increments of $U(t)$, i.e. $U(T) - U(t) \overset{d}{=} U(T - t)$, the Eq.(57) can be rewritten in the form $Z(T) \overset{d}{=} \int_0^T \phi(T - t) \exp\left[-at - U(T - t)\right] dt + Z_0 \exp\left[-aT - U(T - 0)\right]$. By using Eq.(56) and by performing change of the variable $t \mapsto T - t$ in the previous equation, we obtain a more manageable form

$$Z(T) \overset{d}{=} \int_0^T \phi(t) \exp\left[-at - U(t)\right] dt + Z_0 \exp\left[-aT - U(T)\right] \tag{59}$$

In the case $1 \leq p < \mathrm{Re}(a)/D$, then the previous Banach space method can be used in almost straightforward manner. In this case we are interested in the large time behavior of $\langle |Z(n\tau)|^p \rangle^{1/p} = \|Z(n\tau)\|_p$. Consider the integers $n > m$ and large. From Eq.(59) results

$$\left| \|Z(n\tau)\|_p - \|Z(m\tau)\|_p \right| \leq \|Z(m\tau) - Z(n\tau)\|_p \tag{60}$$

We observe that after transforming $Z(T)$ to obtain 59, the r.h.s. from Eq.(60) is more accessible for estimations. From Eqs.(60, 59) and the Minkowski inequality we obtain

$$\left| \|Z(n\tau)\|_p - \|Z(m\tau)\|_p \right| \leq |Z_0| \left(A_m + A_n\right) + B_{n,m} \tag{61}$$

where we denote

$$A_k = \exp(-a_1 k\tau) \left\langle |\exp\left[-U(k\tau)\right]|^p \right\rangle^{1/p} \tag{62}$$

$$B_{n,m} = \left\| \int_{m\tau}^{n\tau} \phi(t) \exp(-a_1 t) \exp\left[-U(t)\right] dt \right\|_p \tag{63}$$

Recall that $\mathrm{Re}[U(t)[= Y(t)$ and it is Gaussian process. Then the first terms in Eq.(61) can be rewritten as $A_k = \exp(-a_1 k\tau) \left\langle \exp\left[-pY(k\tau)\right] \right\rangle^{1/p}$ and according to Eqs.(20, 22, 24) we have

$$A_k \leq c_5 \exp(k\tau D(p - \beta)) \tag{64}$$

The term $B_{n,m}$ can be bounded as follows

$$B_{n,m} \leq \int_{m\tau}^{n\tau} \|\phi(t)\|_p \exp(-a_1 t) \left\|\exp\left[-Y(t)\right]\right\|_p dt$$

and the rest of the calculations are standard. Indeed, according to Eq.(55) the quantity $\|\phi(t)\|_p$ is bounded. The norm $\|\exp\left[-Y(t)\right]\|_p = \left\langle |\exp\left[-Y(t)\right]|^p \right\rangle^{1/p}$ can be bounded by simple Gaussian integration. By straightforward calculations it results that $\left| \|Z(n\tau)\|_p - \|Z(m\tau)\|_p \right|$ can be made arbitrary small for m, n sufficiently large, that proves that $\|Z(n\tau)\|_p$ is a Cauchy sequence and consequently has finite limit.

In a similar manner, if we suppose that $0 < p \leq 1$ and $p < \mathrm{Re}(a)/D$, then from Eq. (59) results

$$\left| \langle |Z(n\tau)|^p \rangle - \langle |Z(m\tau)|^p \rangle \right| \leq |Z_0|^p \left(M_A(p, n\tau) + M_A(p, m\tau)\right) + V_{n,m} \tag{65}$$

In the previous formula the notation $M_A(p,t)$ is defined by Eq.(22) and

$$V_{n,m} = \left\langle \left| \int_{m\tau}^{n\tau} \phi(t) H_Y(t) dt \right|^p \right\rangle \tag{66}$$

with $H_Y(t)$ defined in Eq.(31).

We will prove that for m, n sufficiently large all of the terms in the r.h.s. of the inequality Eq.(65) can be made arbitrary small, and consequently $\langle |Z(n\tau)|^p \rangle$ is a Cauchy sequence and has a limit. By using Eq.(25) it result that

$$M_A(p, k\tau) \leq K_2 \exp(-pk\tau(a_1 - pD)) \tag{67}$$

$M_A(p, n\tau) + M_A(p, m\tau)$ vanishes for m, n large. Be using the previously introduced notation for $b_k(t)$ in Eq.(42) we obtain

$$|V_{n,m}| \leq \sum_{k=m+1}^{n} b_k(0) \tag{68}$$

From the exponential bounds for $b_k(0)$, proved in [23] and Eq.(68) we obtain that for large m, n the term $V_{n,m}$ can be made arbitrary small, if $p < a_1/D$, because

$$|V_{n,m}| \leq C \exp[-pm\tau(a_1 - pD)] \tag{69}$$

Recollecting the previous estimates Eqs.(69,68,67,65) we find that the moments $\langle |Z(n\tau)|^p \rangle$ generate a Cauchy sequence and has a finite limit for $n \to \infty$.

So we proved that if $p < a_1/D$ then the moments $\langle |Z(n\tau)|^p \rangle$ has finite limits. By analytic continuation arguments, similar to [2], it follows that also exists a limiting cumulative probability distribution function of $|Z(n\tau)|$. If it is nontrivial, then the PDF has a HTE given by Proposition (3) and Eq.(11) .

Heuristically, the problem of non-triviality of the stationary PDF in this case can be deduced from the fact that near $Z = 0$ the additive term is dominant, consequently the behavior of the stationary PDF near origin is governed by the statistical properties of the additive noise.

VI. CONCLUSIONS

Continuous time, slightly sub critical random multiplicative processes, whose phase space is the complex plane, was studied. The main part of the work is devoted to the study of the statistical aspects of the large time and large amplitude excursions of the solution of a large class of stochastic differential equations. This class of equations has a dominating random linear part and a nonlinear random additive part subjected to boundedness conditions of technical nature. It arises naturally in the study of the linear stability when a dynamical systems is perturbed by noise. The one-dimensional approximation used in this work is admissible whenever we can approximate the study of a full linear system by the least stable eigenmode.

This process is a natural, nonlinear and complex generalization of the overdamped random multiplicative affine processes, used previously in the study of the heavy tail effects. Under very general assumptions on the driving noises, we obtain a generalized formula for the heavy tail exponent as well as large time asymptotic results for the fractional order moments of the solution.

We prove that the study of the singularity of stationary probability density function of the noisy relaxation oscillations, can be reduced to this model. We computed the exponent of the singularity of the stationary probability density function on this class of noise driven on–off intermittent models of the relaxation oscillations.

Remarkable robustness of both of the exponents was proven. They depend only on the instability threshold and the zero frequency power spectrum of the real part of the driving noise, and they are not influenced by the nonlinear components. This independence generates a conjecture that if will be proven, then it simplifies the study of the higher dimension random multiplicative processes. The fact that the statistical properties of the large amplitude fluctuations depends only on the low frequency component explains qualitatively some experimental data on the edge plasma intermittency in tokamaks.

By assuming ergodicity, from our formula it follows that for on-off intermittency system driven by noise of fixed, moderate intensity but with increasing correlation times, the system will spend much of time in the laminar phase. The robustness of this mechanism of the noise-induced stabilization suggests the existence of similar, computable singularity structure of the PDF in more complicated intermittent dynamical systems.

The consequences of these results on the stochastic analogue of the linear stability analysis are discussed.

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