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# Generalized evolutionary equations and their invariant solutions

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**Abstract**

The paper appplies the Lie symmetry approach to a general $1D$ dynamical system described by a second order pde. A new algorithm is proposed for studying the evolutionary equation with imposed solution. This approach will be exemplified for two particular systems, the Fokker-Planck model and the „backward" Kolmogorov one.

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# 1 Introduction

The Lie group analysis is a mathematical theory that synthesizes symmetries of the differential equations which may be a point, a contact, and a generalized or nonlocal symmetry. Many studies have been devoted to classified differential equations in terms of their symmetry group, in order to identify the set of equations which could be integrated or reduced to lower-order equations by group theoretical algorithms. Group analysis provides two basic ways for construction of exact solutions :group tranformations of known solutions and construction of invariant solutions [1, 2].

The aims of this paper are: $i$) to determine the set of differential equations which is able to determine the Lie symmetry generator and for a particular class of second order pdes, $ii$) to study the same class of dynamical systems with imposed invariant solutions.

The paper is organized as follows: in section 2 the determining equations of the Lie symmetries are obtained for a general $1D$ dynamical system described by a second order pde. The invariant solutions for the evolutionary equation are also defined using the general characteristic method. Section 3 is devoted to analyze dynamical equations with imposed solutions. The general method was applied for the Fokker-Plank model and the „backward" Kolmogorov one. The paper will end with some concluding remarks.

# 2 Lie symmetries and invariant solutions

## 2.1 Determining equations of the Lie symmetries

Let us consider a general class of dynamical systems described by the following evolutionary equations:

$$u_t = A(x,t)u_{2x} + B(x,t)u_x + C(x,t)u \tag{1}$$

with $A(x,t), B(x,t), C(x,t)$ arbitrary functions.

The general expression of the classical Lie operator which leaves (1) invariant is:

$$U(x,t,u)=\varphi(x,t,u)\frac{\partial}{\partial t}+\xi(x,t,u)\frac{\partial}{\partial x}+\eta(x,t,u)\frac{\partial}{\partial u} \tag{2}$$

Following the symmetry theory [3], the invariance condition of (1) is given by the relation:

$$U^{(2)}[u_t-A(x,t)u_{2x}-B(x,t)u_x-C(x,t)u]=0 \tag{3}$$

where $U^{(2)}$ is the second extension of the Lie symmetry generator (2).

By extending the condition (3) an equivalent form is obtained:

$$\begin{aligned}0=&(-\varphi A_t-\xi A_x)u_{2x}+(-\varphi B_t-\xi B_x)u_x-\varphi C_t u-\xi C_x u+\\&-C\eta-B\eta^x+\eta^t-A\eta^{2x}\end{aligned} \tag{4}$$

By particularization of the general expressions [3] of the functions $\eta^x$, $\eta^t$, $\eta^{2x}$, for the analyzed model (1), substituting them into the condition (4) and then equating with zero the coefficient functions of various monomials in derivatives of $u$, the following partial differential system is obtained:

$$\begin{aligned}\varphi_x&=0\\ \varphi_u&=0\\ \xi_u&=0\\ \eta_{2u}&=0\\ \varphi A_t+\xi A_x+A\varphi_t-2A\xi_x&=0\\ -\varphi B_t-\xi B_x+B\xi_x-\xi_t-B\varphi_t-2A\eta_{xu}+A\xi_{2x}&=0\\ -\varphi C_t u-\xi C_x u+C\eta_u u-C\varphi_t u-C\eta-B\eta_x+\eta_t-A\eta_{2x}&=0\end{aligned} \tag{5}$$

For any particular model of type (1), the system (5) contains the determining equations for the coefficient functions $\varphi(x,t,u),\xi(x,t,u),\eta(x,t,y)$ of the symmetry operator (2).

## 2.2 Determination of invariant solutions by the characteristic method

The general system (5) generates the unknown functions $\varphi,\xi,\eta$ of the symmetry operator (2), for any $1D$ model of the type (1). An invariant solution with respect to a symmetry group is a solution which is unalterable under the action of the group's transformations. Invariant solutions can be expressed via the invariants of the symmetry group [4].

The characteristic equations have the forms:

$$\frac{dt}{\varphi}=\frac{dx}{\xi}=\frac{du}{\eta} \tag{6}$$

By integrating the characteristic system of ordinary differential equations (6) the invariants $I_1$, $I_2$ of the analyzed dynamical system could be found. The invariant solution can be expressed in the form:

$$I_2=\phi(I_1) \tag{7}$$

Taking into account the concrete expressions of the invariants for any particular model, the relation (7) generates the invariant solution $u(x,t)$.

# 3 Evolutionary equations with imposed solutions

## 3.1 The general approach

Let us impose for (1) a general solution of the form:

$$u(x,t) = \phi(f(x,t))g(x,t) \tag{8}$$

with the property:

$$\ddot{\phi}(z) = \frac{d^2\phi(z)}{dz^2} = 0,\ \ z = f(x,t) \tag{9}$$

By choosing the invariants associated to the dynamical system (1) to be exactly the functions $f(x,t)$ and $\phi(f(x,t))$, (9) will show the linear dependence between these invariants.

Some of the derivative forms of the general solution (8) have the expressions:

$$u_t = \dot{\phi} f_t g + \frac{u}{g} g_t \tag{10}$$

$$u_x = \dot{\phi} f_x g + \frac{u}{g} g_x \tag{11}$$

$$u_{2x} = \ddot{\phi} f_x^2 g + \dot{\phi} f_{2x} g + 2\dot{\phi} f_x g_x + \frac{u}{g} g_{2x} \tag{12}$$

By substituting $\dot{\phi}$ from (10) in (11) and (12) the following differential system will result:

$$u_x = \left[\frac{g_x}{g} - \frac{g_t}{g}\frac{f_x}{f_t}\right] u + \frac{f_x}{f_t} u_t \tag{13}$$

$$u_{2x} = \left[\frac{g_{2x}}{g} - \frac{g_t}{g}\frac{f_{2x}}{f_t} - 2\frac{g_t g_x}{g^2}\frac{f_x}{f_t}\right] u + \frac{g_x}{g}\frac{f_x}{f_t} u_t + \frac{g_x}{g}\frac{f_x}{f_t} u_t \tag{14}$$

By substituting the product $g\ f_x u_t$ from (13) only in one of the last two terms of (14), an equivalent evolution equation is obtained:

$$u_t = \frac{g}{g_x}\frac{f_t}{f_x} u_{2x} - \frac{f_t}{f_x} u_x - \left[\frac{g_{2x}}{g_x}\frac{f_t}{f_x} - \frac{g_t}{g_x}\frac{f_{2x}}{f_x} - \frac{g_t}{g} - \frac{g_x}{g}\frac{f_t}{f_x}\right] u \tag{15}$$

Now, by comparing the dynamical equations (1) and (15), it results that the coefficient functions take the forms:

$$A(x,t) = \frac{g}{g_x}\frac{f_t}{f_x},\ \ B(x,t) = -\frac{f_t}{f_x},$$
$$C(x,t) = -\frac{g_{2x}}{g_x}\frac{f_t}{f_x} + \frac{g_t}{g_x}\frac{f_{2x}}{f_x} + \frac{g_t}{g} + \frac{g_x}{g}\frac{f_t}{f_x} \tag{16}$$

## 3.2 Some applications

*i*) The first model we have considered is the general Fokker-Planck equation of the form [5]:

$$u_t = [-\partial_x A(x) + \partial_{2x} B(x)]u \tag{17}$$

where $A(x)$ and $B(x)$ are called diffusion and drift coefficients, such that $B(x) \rangle 0$.

Equation (17) is an equation of motion for the distribution function $u(x,t)$ [6]. We restrict our application to the particular case:

$$A(x) = -x,\ B(x) = 1 \tag{18}$$

For this particular dynamical model, the general system (16) becomes:

$$\frac{g}{g_x}\frac{f_t}{f_x} = 1 \tag{19}$$

$$-\frac{f_t}{f_x} = x \tag{20}$$

$$-\frac{g_{2x}}{g_x}\frac{f_t}{f_x} + \frac{g_t}{g_x}\frac{f_{2x}}{f_x} + \frac{g_t}{g} + \frac{g_x}{g}\frac{f_t}{f_x} = 1 \tag{21}$$

The previous differential system has the solution:

$$\begin{aligned} &g(x,t) = me^{-x^2/2}, \forall\ m = const. \\ &f(x,t) = \rho(\ln x - t),\ \forall\ \rho \Rightarrow e^{f(x,t)} = xe^{-t} \end{aligned} \tag{22}$$

It is important to remark that the solution (22) generates the two invariants of the $1D$ Fokker-Planck model.

*ii*) The second application we considered is represented by the equation of „backward" Kolmogorov type [7]:

$$u_t = u_{2x} + (x - \frac{1}{x})u_x \tag{23}$$

For this particular model we must impose $C(x,t) = 0$ in (16). This condition imposes the existence of the following differential system:

$$g_x^2 - gg_{2x} = 0 \tag{24}$$

$$gf_{2x} + g_x f_x = 0 \tag{25}$$

which has solutions of the forms:

$$f(x,t) = f_3(t) + f_4(t)e^{-f_1(t)x},\ \forall f_1(t),\ f_3(t),\ f_4(t) \tag{26}$$

$$g(x,t) = f_2(t)e^{f_1(t)x},\ \forall f_2(t) \tag{27}$$

Thereby, the invariants of the Kolmogorov type model are respectively the function (26) and $\frac{u}{g(x,t)}$, where function $g(x,t)$ has the expression (27).

# 4 Conclusions

The results of this paper can be synthesized as follows: ($i$) the general determining equations (5) for the Lie symmetries of evolutionary equations of the type (1) have been obtained. It is important to remark that any Lie symmetry generator for any dynamical system described by an differential equation of class (1), could be obtained directly by solving the system (5) particularized for the analyzed system. Some examples of systems of type (1) could be the Fokker -Planck equation used in various fields of natural sciences such as quantum optics, solid-state physics, chemical physics, theoretical biology [8], the Black-Scholes model applied to financial problems [9] etc.($ii$) the characteristic method used for obtaining invariant solutions for a general dynamical system is explained $iii$) an algorithm able to express the coefficient functions respective $A(x,t), B(x,t), C(x,t)$ of general differential equation (1) in terms of the invariants of the dynamical system is proposed. It has been applied in two particular cases: the Fokker-Planck model and the „backward" Kolmogorov one. Following the results of this algorithm, the invariants of these concrete dynamical systems have been pointed out.

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## References

[1] N.H. Ibragimov, „Handbook of Lie Group Analysis of Differential Equations", Volume1,2,3 CRC Press, Boca Raton, Ann Arbor, London, Tokyo, (1994,1995,1996).

[2] G.Baumann, „Symmetry Analysis of Differential Equations with Mathematica", Telos, Springer Verlag, New York (2000).

[3] P. J. Olver, Applications of Lie Groups to Diferential Equations, Grad. Texts in Math. 107, Springer-Verlag, New York, (1986).

[4] G.W. Bluman, S.C.Anco, „Symmetry and Integtation Methods for Differential Equations", Springer-Verlag, New York (2002).

[5] H. Risken, „The Fokker–Planck Equation: Methods of Solutions and Applications", 2nd edition, Springer Series in Synergetics, Springer, ISBN 3-540-61530-X, (1989).

[6] M.Tatari, M. Dehghan, M. Razzaghi, Mathematical and Computer Modelling.**45**, (2007), 639-650.

[7] J.Biazar, Int. Math. Forum, 3, no 19, (2008), 945-954.

[8] G. I. Burde, Proceedings of Institute of Mathematics of NAS of Ukraine,Vol. 43, Part 1, (2002), 93–101.

[9] R.K.Gazizov and N.H. Ibragimov, Nonlinear Dynamics **17,** (1998), 387-407.