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Classical and Non-classical Symmetries for the $2D$-Kuramoto-Sivashinsky Model

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Abstract

In this paper we perform the Lie symmetry analysis for the $2D$- Kuramoto-Sivashinsky model. The Lie invariants associated to the symmetry generators and some similarity solutions are pointed out. The non-classical symmetries for the model are also investigated.

Keywords: Kuramoto-Sivashinsky model, Lie symmetries, smilarity solutions.

1 Introduction

The $2D$ Kuramoto-Sivashinsky (KS) model represents a nonlinear dynamical system defined in a two dimensional space $\{x, y\}$, the dependent variable $h = h(x, y, t)$ satisfying a fourth order partial derivative equation of the form:

$$h_t = -\nabla^4 h - \nabla^2 h + (\nabla h)^2,\ \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{1}$$

This evolution equation can be written in the equivalent form:

$$h_t = -h_{4x} - 2h_{(2x)(2y)} - h_{4y} - h_{2x} - h_{2y} + h_x^2 + h_y^2 \tag{2}$$

The KS equation has been derived e.g. in the context of chemical turbulence [1]. It is also important because it can describe the flow of a falling fluid film [2]. In this last context, the combination of the $\nabla^2 h$ and $(\nabla h)^2$ terms models the effect of particles being knocked out at the interface by the bombarding ions. It is important to note that in the form (1) of the evolutionary equation the two terms $\nabla^2 h$ and $(\nabla h)^2$ have the coefficients ± 1, fact which correspond to a rotational symmetry. This is true for normal incidence. For other than normal incidence, however, separate coefficients are needed in front of each term contained within $\nabla^2 h$ and $(\nabla h)^2$. The $-\nabla^4 h$ term models the effect of surface diffusion [3].

Some important results concerning the model are already known [4]. For example, fractal tracer distributions of particles advected in the velocity field of the $2D$-Kuramoto-Sivashinsky equation was found. The measured dimensions of attractors are in good agreement with calculated Lyapunov dimensions. We will try now to analyze the problem of the symmetries of the model. In the next section, the classical Lie type symmetries will be computed. They will be associated with some invariant quantities and with solutions of the reduced evolution equation obtained by applying the similarity procedure. Section 3 is devoted to the non-classical symmetries of the model, symmetries generated when a supplementary condition, the invariance surface condition, is imposed. Some concluding remarks will end the paper.

2 Classical symmetries for the model

2.1 Lie symmetry operators and associated Lie Algebra

In this section the *classical symmetry method* (CSM) will be applied [5]. The general expression of the classical Lie operator which leaves (2) invariant is:

$$\begin{aligned} U(x,y,t,h) &= \varphi(x,y,t,h)\frac{\partial}{\partial t}+\xi(x,y,t,h)\frac{\partial}{\partial x} \\ &\quad +\eta(x,y,t,h)\frac{\partial}{\partial y}+\phi(x,y,t,h)\frac{\partial}{\partial h} \end{aligned} \tag{3}$$

Without loss of generality we can impose $\varphi = c_0 = const.$

Because (2) is a fourth order partial differential equation, the invariance condition for the $2D$ KS model is given by the relation:

$$U^{(4)}[h_t + h_{4x} + 2h_{(2x)(2y)} + h_{4y} + h_{2x} + h_{2y} - h_x^2 - h_y^2] = 0 \tag{4}$$

where $U^{(4)}$ is the extension of forth order of the Lie operator (3). It has the general expression:

$$\begin{aligned} U^{(4)} &= U+\phi^x\frac{\partial}{\partial h_x}+\phi^y\frac{\partial}{\partial h_y}+\phi^t\frac{\partial}{\partial h_t}+\phi^{2x}\frac{\partial}{\partial h_{2x}}+ \\ &\quad \phi^{xy}\frac{\partial}{\partial h_{xy}}+\phi^{xt}\frac{\partial}{\partial h_{xt}}+\phi^{2y}\frac{\partial}{\partial h_{2y}}+\phi^{yt}\frac{\partial}{\partial h_{yt}}+ \\ &\quad \phi^{2t}\frac{\partial}{\partial h_{2t}}+\phi^{3x}\frac{\partial}{\partial h_{3x}}+...+\phi^{4x}\frac{\partial}{\partial h_{4x}}+\phi^{4y}\frac{\partial}{\partial h_{4y}}+... \end{aligned} \tag{5}$$

The invariance condition (4) is equivalent with the equation:

$$\phi^t - 2\phi^x h_x - 2\phi^y h_y + \phi^{2x} + \phi^{2y} + \phi^{4x} + \phi^{4y} + 2\phi^{(2x)(2y)} = 0 \tag{6}$$

where the functions $\phi^x, \phi^y, \phi^t, \phi^{2x}, \phi^{2y}, \phi^{4x}, \phi^{4y}$, $\phi^{(2x)(2y)}$ can be written using the expressions listed in [5]:

$$\begin{aligned} &\phi^x = D_x\phi - (D_x\xi)h_x - (D_x\eta)h_y,\ \phi^y = D_y\phi - (D_y\xi)h_x - (D_y\eta)h_y, \\ &\phi^t = D_t\phi - (D_t\xi)h_x - (D_t\eta)h_y, \\ &\phi^{4x} = D_{4x}\phi - (D_{4x}\xi)h_x - (D_{4x}\eta)h_y,\ \phi^{4y} = D_{4y}\phi - (D_{4y}\xi)h_x - (D_{4y}\eta)h_y \\ &\phi^{(2x)(2y)} = D_{(2x)(2y)}\phi - [D_{(2x)(2y)}\xi]h_x - [D_{(2x)(2y)}\eta]h_y \end{aligned}$$

$$\phi^{2x} = D_{2x}\phi - (D_{2x}\xi)h_x - (D_{2x}\eta)h_y,\ \phi^{2y} = D_{2y}\phi - (D_{2y}\xi)h_x - (D_{2y}\eta)h_y \tag{7}$$

where D is the total derivative operator.

Extending the relations (7), substituting them into the condition (6) and then imposing the condition that all the coefficient functions of various monomials in partial derivatives of h vanish, we find a system of 22 partial differential equations. We solved this system with Maple and we obtained the following solutions:

$$\begin{aligned} \xi &= -2c_1 t + \frac{1}{2}c_2 x + c_5 y + c_6 \\ \eta &= -2c_4 t - c_5 x + \frac{1}{2}c_2 y + c_7 \\ \phi &= c_1 x + c_4 y + c_2 h + c_3 \end{aligned} \tag{8}$$

where $c_i, i = \overline{1,7}$ are arbitrary constants.

The independent symmetry operators associated with the solution (8) are:

$$
\begin{aligned}
U_0 &= \partial_t,\ U_1 = -2t\partial_x + x\partial_h,\ U_2 = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + h\partial_h,\ U_3 = \partial_h, \\
U_4 &= -2t\partial_y + y\partial_h,\ U_5 = y\partial_x - x\partial_y,\ U_6 = \partial_x,\ U_7 = \partial_y
\end{aligned} \tag{9}
$$

When the Lie algebra of these 8 operators is computed, the only nonvanishing relations are obtained in the form:

$$
\begin{aligned}
&[U_0, U_1] = -2U_6, [U_0, U_4] = -2U_7, [U_1, U_2] = \frac{U_1}{2}, \\
&[U_1, U_5] = -U_4, [U_1, U_6] = -U_3, [U_1, U_0] = 2U_6 \\
&[U_2, U_1] = -\frac{U_1}{2}, [U_2, U_3] = -U_3, [U_2, U_4] = -\frac{U_4}{2}, \\
&[U_2, U_6] = -\frac{U_6}{2}, [U_2, U_7] = -\frac{U_7}{2}, [U_3, U_2] = U_3, \\
&[U_5, U_6] = U_7, [U_5, U_7] = -U_6, [U_6, U_1] = U_3, \\
&[U_6, U_2] = \frac{U_6}{2}, [U_6, U_5] = -U_7, [U_7, U_2] = \frac{U_7}{2}, \\
&[U_7, U_4] = U_3,\ \ [U_7, U_5] = U_6,
\end{aligned}
$$

$$
\begin{aligned}
&[U_4, U_2] = \frac{U_4}{2}, [U_4, U_5] = U_1, [U_4, U_7] = -U_3, \\
&[U_4, U_0] = 2U_7, [U_5, U_1] = U_4, [U_5, U_4] = -U_1,
\end{aligned} \tag{10}
$$

2.2 Lie invariants and similarity solutions

We can now compute the invariants associated with the symmetry operators (9). They can be obtained by integrating the characteristic equations. For the first operator, $U_1 = -2t\partial_x + x\partial_h$,this means:

$$
\frac{dt}{0} = \frac{dy}{0} = \frac{dx}{-2t} = \frac{dh}{x} \tag{11}
$$

The corresponding invariants have the form:

$$
I_1 = t,\ I_2 = y,\ I_3 = h + \frac{x^2}{4t} \tag{12}
$$

Taking into account the last invariant, we assume a similarity solution of the form:

$$
h = f(t, y) - \frac{x^2}{4t} \tag{13}
$$

and we substitute it into (2) to determine the form of the function $f(t, y)$. We obtain that $f(t, y)$ has to be a solution of the following differential equation:

$$
f_t + f_{4y} - \frac{1}{2t} + f_{2y} - f_y^2 = 0 \tag{14}
$$

The equation (14) has a general solution of the form:

$$
f(t, y) = f(t, y) + a_1 t + B(y) + a_2,\ a_{1,2} = const \tag{15}
$$

where $B(y)$ is, at it turn, solution of the equation:

$$
\frac{d^4B(y)}{dy^4} + \frac{d^2B(y)}{dy^2} - \left(\frac{dB(y)}{dy}\right)^2 + a_1 = 0 \tag{16}
$$

Thereby the similarity solution, in this case, has the expression:

$$h(t,x,y) = \frac{1}{2}\ln t - \frac{x^2}{4t} + a_1 t + B(y) + a_2 \tag{17}$$

in which $B(y)$ satisfy (16).

Because the equation (2) is symmetric in x and y, it has a similarity solution, associated to the operator $U_4 = -2t\partial_y + y\partial_h$,of the form:

$$h(t,y) = \frac{1}{2}\ln t - \frac{y^2}{4t} + a_3 t + A(x) + a_4,\ a_{3,4} = const \tag{18}$$

with the requirement that $A(x)$ has to verify an equation similar with (16), that is:

$$\frac{d^4A(x)}{dx^4} + \frac{d^2A(x)}{dx^2} - \left(\frac{dA(x)}{dx}\right)^2 + a_3 = 0 \tag{19}$$

By similar arguments, the invariants generated by the operators $U_5 = y\partial_x - x\partial_y$ and $U_2 = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + h\partial_h$ are h, t, $\frac{x^2}{2} + \frac{y^2}{2}$, respectively t, $\frac{h}{x^2}$, $\frac{h}{y^2}$.

For the operators $U_0 = \partial_t$, $U_3 = \partial_h$, $U_6 = \partial_x$, $U_7 = \partial_y$, the invariants are respectively arbitrary functions $u(x,y,h)$, $v(t,x,y)$, $w(t,y,h)$, $p(t,x,h)$.

3 Non-classical symmetries

In this section we will apply the so called nonclassical *symmetry method* (NSM) [6]. The NSM has become the focus of a lot of research and many applications to physically important partial differential equations as in [7, 8, 9, 10, 11]. The NSM consists in adding the invariance surface condition to the given equation, and then in applying the CSM. As a general rule, NSM gives much more symmetries than the classical one (any classical symmetry is a nonclassical one, but not conversely).

The invariance surface condition associated to the general symmetry operator (3) is given by the relation:

$$\phi - h_t - \xi h_x - \eta h_y = 0 \tag{20}$$

Without loss of generality we choose $\varphi = 1$.

The following step consists in the substitution of the derivative h_t from (20) and of the maxim order derivatives $h_{4x} + h_{4y} + 2h_{(2x)(2y)}$ from (2) into the invariance condition (6). In the equivalent equation we can ask for the vanishing of the coefficient functions of various monomials in the derivatives of the variable h.The system we obtained by this procedure was solved with Maple. By doing that, he only solutions we found were exactly the solution (8) obtained through the classical symmetry approach. This means that no supplementary symmetries, of non-classical type, are specific for our model.

4 Conclusions

The paper studied the problem of determining the largest possible set of symmetries for an important example of nonlinear dynamical system: the Kuramoto-Sivashinsky model in two spatial and one temporal dimensions. The model is important not only because it is connected with important evolutionary physical phenomena, but also by itself, as a fourth order partial derivative equation. The main results we have reported here can be expressed as follows:

- by applying the CSM for the $2D$ KS model, we have found 8 classical symmetry operators $\{U_j, j = \overline{0,7}\}$. These operators which generate invariance of the evolution equation (2) correspond to the following transformations: U_0 generate temporal translation, U_1 and U_4 Galilean boosts, U_2 spatial dilatation and gauge transformation, U_3 a gauge transformation, U_5 a spatial rotation, U_6 and U_7 spatial translations. The interesting Lie Algebra (10) of these symmetry generators was obtained;

- The Lie operators are responsible for the existence of interesting invariant quantities, from the simplest (the coordinates themselves) to quite arbitrary functions. Using these invariants and imposing the similarity condition, we were able to obtain 2 similarity solutions, (17) and (18), of the model;
- By applying the NSM for the $2D$ KS model we concluded that the analyzed model do not admit supplementary, nonclassical type, symmetries. Using this procedure, the classical Lie operators only were generated.

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